Frost Penetration Below Highway And Airfield Pavements

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Fundamentals related to the penetration of below freezing temperatures into the ground are outlined and a rational method of computing the maximum depth of frost penetration below highway and airfield pavements is presented.

Methods for making quick preliminary predictions of frost penetration are presented as well as "design" procedures based on thermal properties of the soil and weather conditions at the site. Of particular interest is the modified Berggren formula which was derived for the Corps of Engineers. Furthermore, reference to refined procedures, numerical and analogue computer solutions, are given for those cases where economic and engineering considerations dictate an "exact" solution.

Status of current research related to heat transfer between the air-pavement interface by radiation and convection-conduction is discussed.

● A NEWS item in the Christian Science Monitor on February 24, 1954 read: "Early warm weather in New Hampshire has made is necessary to put more than 100 highway trucks and crews to work repairing frost-buckled pavements. Further, the Highway Department says, unless cold weather comes soon, it may be necessary to keep trucks heavier than 20,000 pounds off New Hampshire roads."

In 1948 New Hampshire spent over $500,000 repairing roads damaged by frost action. Kentucky spent over $5,000,000 resurfacing roads that were damaged after the freezing and quick thawing that same year.

Frost action below pavements has assumed increased importance as wheel loadings, traffic frequency, and cost of pavement construction and maintenance have increased. Thermal problems in soil are receiving additional attention as a result of construction activity in permafrost areas. Furthermore, the potential of using the ground as a heat source or sink for heating and air-conditioning by heat pumps adds importance to this phase of soil engineering.

It is generally recognized that three requirements must be met simultaneously for significant ice-segregation and frost heave to occur in a foundation soil: (1) most obviously, below freezing temperatures must penetrate into the ground, (2) the soil must meet certain grain-size requirements to be "frost susceptible" and (3) a source of water must be available. This paper deals with the first requirement only, namely the prediction of the maximum depth of frost penetration with special reference to highway and airfield pavements. It is written primarily for practicing engineers and the author's soil engineering students.

After a presentation of the thermal properties of soil and their fundamental relationships, frost penetration formulas based on idealized assumptions are presented. The derivation of a rational formula, given the name modified Berggren formula, is presented. The effect of varying surface temperature, heat transfer between pavement surface and air, non-uniform soil, ice segregation and a depressed freezing point on the depth of frost penetration is discussed. The status of current research utilizing analogue computers to improve prediction techniques is presented.

The Arctic Construction and Frost Effects Laboratory of the New England Division, Corps of Engineers, U.S. Army, has been largely responsible for sponsoring the work described in this paper. Research is continuing in all areas of the problem, particularly frost penetration through non-uniform soil and air-pavement temperature relationships which are being studied with the aid of a hydraulic analogue computer.

NOTATION

In this paper the following symbols are used throughout:

NOTATION

a = thermal diffusivity ($=k/C$), in sq ft per hr;
A = area, in sq ft;
c = specific heat, in Btu per lb per deg F;
C = volumetric heat, in Btu per cu ft deg F;
d = thickness of soil layer in non-uniform, profile, ft;
F = surface freezing index, deg days F;
i = thermal gradient, deg F per ft
k = thermal conductivity, Btu per hr per ft per deg F;
l = length, ft;
L = latent heat, in Btu per cu ft;
q = rate of heat flow per unit area, in Btu per hr per sq ft;
Q = Rate of heat flow, Btu per hr;
t = time, hr;
t = duration of freezing period, days;
u = thermal energy, Btu per cu ft;
v = temperature, deg F;
v_o = temperature by which mean annual temperature exceeds freezing point of soil moisture, deg F;
v_s = temperature by which effective surface temperature is less than freezing point of soil moisture during freezing period (= F/t), in deg F;
w = water content, percent
x = coordinate direction measured vertically downward from ground surface, ft;
X = depth of frost penetration, in ft;
∂ = thermal ratio, dimensionless;
γ_d = dry density, lb per cu ft;
λ = correction coefficient, dimensionless; and
μ = fusion parameter, dimensionless.

Subscripts:

u = unfrozen soil; and
f = frozen soil.

FUNDAMENTAL PRINCIPLES

Certain thermal properties of soil and fundamental relationships among these properties are common to frost penetration problems. Indeed, these concepts are fundamental to all heat transfer problems.

Thermal Properties of Soil

Thermal conductivity, volumetric heat and latent heat are common names for the three thermal properties of soil. They are primarily dependent on the dry unit weight (dry density) and water content of the soil.

Thermal conductivity, k, expresses the rate of heat flow through a unit of area under a unit thermal gradient. Thus, the units of k are Btu per hr per sq ft per deg F per ft or Btu per hr per ft per deg F. Typical values of thermal conductivity in these units[1] are:

k_{air}	= 0.014	k_{water}	= 0.35
k_{shale}	= 0.9	k_{ice}	= 1.30
$k_{granite}$	= 1.6	(k_{copper}	= 225)

Since inorganic soil is composed of mineral particles, water and air, we would expect

[1] For conversion from the British system to the physical system of units; 1 Btu per hr per ft per deg F = 41.34×10^{-4} cal per sec per cm per deg C.

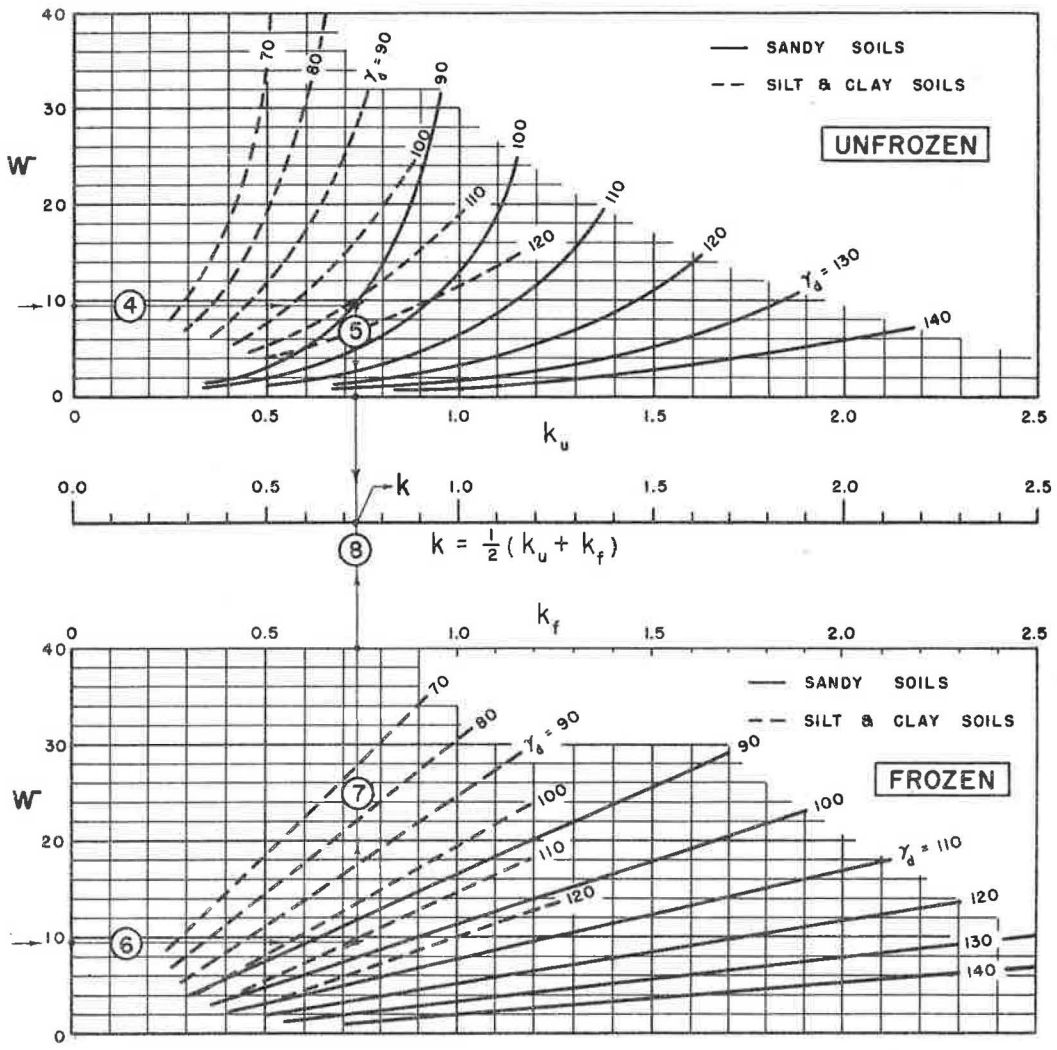

Figure 1. Thermal conductivity of soil after Kersten (1).

the thermal conductivity of soil to be about 1.0 Btu per hr per ft per deg F. Indeed, this would be a reasonable assumption for a preliminary investigation.

Kersten (1) conducted extensive thermal conductivity tests on inorganic cohesionless and cohesive soils. Results of these tests, conducted on both frozen and unfrozen soils, are summarized in Figure 1. Kersten states that "judicious use of the equations (essentially equivalent to the Figure 1 charts) with an understanding of their limitations will give conductivity values not more than 25 percent in error." For sandy soils having a relatively high silt and clay content, Kersten recommends the use of conductivity values intermediate between those given for sandy soils and silt and clay soils.

In the range of water contents (5-10 percent) and dry densities (125-135 lb per cu ft) commonly encountered in base-courses below pavements, thermal conductivity is very sensitive to moisture content and soil type. Scattered data indicate that the thermal conductivity of clean well-graded granular base-course materials lies about midway between values for sandy soils and silt and clay soils given in Figure 1.

Table 1 gives additional values of thermal conductivity which may be needed in depth of frost penetration computations.

Volumetric Heat C, expresses the change in thermal energy[2] in a unit volume of soil per unit change in temperature. Thus, the units of C are Btu per cu ft per deg F.[3] Volumetric heat is derived from the specific heat which expresses the change in thermal energy per unit weight per unit temperature change. Typical values of specific heat c in Btu per lb per deg F are:

$$c_{water} = 1.0 \qquad c_{rock} = 0.17 \text{ (soil minerals)}$$
$$c_{ice} = 0.5$$

at temperatures around the freezing point. Using these values of specific heat we may derive the following relationships for the volumetric heat, C, in Btu per cu ft per deg F for an inorganic soil:

For unfrozen soil:

$$C_u = \gamma_d\left(0.17 + \frac{w}{100}\right) \tag{1}$$

For frozen soil:

$$C_f = \gamma_d\left(0.17 + \frac{0.5w}{100}\right)$$

where:

γ_d = dry density (dry unit weight) of the soil, in pcf.

w = water content (moisture content) of the soil expressed, in percent of dry weight.

A typical value for the volumetric heat of soil would be 30 Btu per cu ft per deg F. Values of volumetric heat for other materials are given in Table 1.

Latent Heat L, expresses the change in thermal energy in a unit volume of soil when the soil moisture freezes or thaws at constant temperature. Thus, the units of L are Btu per cu ft. Latent heat depends only on the amount of water in a unit volume of soil. Since one pound of water gives off 143.4 Btu as it freezes, we may derive:

$$L = 1.434\ w\ \gamma_d \tag{2}$$

A dense base-course below a pavement with a water content of 5 percent would have a latent heat of approximately 1000 Btu per cu ft.

TABLE 1

THERMAL PROPERTIES OF COMMON MATERIALS[a]

At Temperatures Around the Freezing Point

Material	Thermal Conductivity, k, Btu/hr/ft/deg F[b]	Volumetric Heat, C, Btu/cu ft/ deg F[c]
Air	0.014	0.019
Water	0.35	62.4
Ice	1.30	28
Snow, loose	0.06	-
compact	0.20	-
Bituminous concrete	0.84	28
Portland-cement concrete	0.54[d]	30

[a] See Figure 1 for soil. See References (2), and (3) for good tabulations of other materials.
[b] 1 Btu per hr per ft per deg F = 41.34×10^{-4} cal per sec per cm per deg C.
[c] 1 Btu per cu ft per deg F = 160×10^{-4} cal per cc per deg C.
[d] Varies considerably with composition, moisture content, and density.

Heat Storage

The relationships among the thermal properties defined above may be described by considering the following physical explanation of heat storage.

All physical objects can be considered to contain heat. A portion of this thermal energy (stored heat) is released when the object cools. If we consider a cubic foot of soil containing water within at least a portion of its voids, a graph representing the idealized thermal energy versus temperature characteristics of the soil is shown in Figure 2.

If the unfrozen soil is cooled, having started at a point represented by A, its thermal energy decreases by an amount C_u for each degree of temperature drop. When the temperature of the soil reaches the freezing point (point B) the soil moisture begins to freeze.

[2] Also referred to as heat content, internal energy, stored heat or enthalpy.
[3] 1 Btu per cu ft per deg F = 160×10^{-4} Cal per cu cm per deg C.

As the water freezes a thermal energy equal to L is released while the temperature of the soil remains nearly constant. After all water in the soil voids has frozen (point C) the temperature falls below freezing as heat is removed from the soil at the rate of C_f Btu per degree change in temperature.

In the absence of freezing or thawing the basic heat storage equation representing the change of thermal energy is simply:

$$u_1 - u_2 = C\ (v_1 - v_2) \qquad (3)$$

where

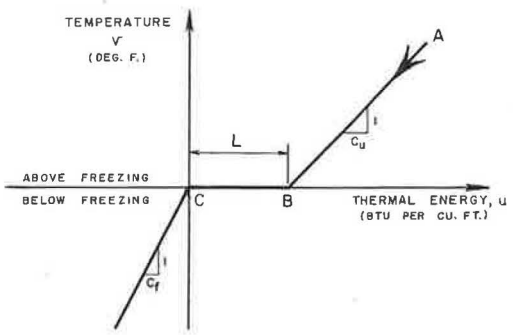

Figure 2. Idealized thermal energy diagram for a unit volume of soil.

u = thermal energy, in Btu per cu ft
v = temperature in deg F

The time rate at which the thermal energy changes is a function of the rate at which heat is transferred which in turn depends on the thermal conductivity of the soil.

Heat Transfer

There are three rather distinct methods by which heat may be transferred within an object or from one object to another. These methods are called conduction, convection and radiation. While convection and radiation play a dominant role in transferring heat between pavement surfaces and the air above, conduction plays a lone role in transferring heat within a soil mass unless water moves within the soil voids. For the time being we shall consider thermal conduction only.

Thermal Conduction. Atomically speaking, heat conduction is considered as a transfer of kinetic energy from the molecules of a warm part of a body to those in a cooler part. The rate at which heat is transferred by conduction is customarily given in the following form:

$$Q = k\,i\,A = k\,\frac{v_1 - v_2}{l}\,A \qquad (4)$$

where:

Q = rate of heat flow, in Btu per hr
i = thermal gradient in deg F per ft
= temperature difference $(v_1 - v_2)$ divided by length l
A = area in sq ft

This relationship is known as the Fourier equation.

A more fundamental expression for Fourier's equation is:

$$q = -\,k\,\frac{\partial v}{\partial x} \qquad (5)$$

where q is the heat conducted per unit area per unit time in the x-direction. The minus sign follows from the second law of thermodynamics according to which heat flows from points of higher temperature to those at a lower temperature.

In this paper, thermal conduction will be considered in one direction only, namely vertical, where the positive x-direction is measured vertically downward from ground surface.

Thermal Continuity. In the absence of freezing or thawing the time rate at which the thermal energy of an element of soil changes plus the net rate of heat transfer into the element must equal zero. This follows from the conservation of thermal energy and may be expressed:

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad (6)$$

The heat storage equation, shown initially in Equation 3, may be written more generally:

$$\frac{\partial u}{\partial t} = C \frac{\partial v}{\partial t} \tag{7}$$

Substitute Equation 7 in Equation 6 and introduce Equation 5 to obtain:

$$C \frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2}$$

or

$$\frac{\partial v}{\partial t} = a \frac{\partial^2 v}{\partial x^2} \tag{8}$$

where "a" is the diffusivity of the soil in sq ft per hr.

Equation 8 is known as the one-dimensional diffusion equation for heat.

At the interface between the frozen and unfrozen soil, $x = X$, the equation of continuity which must be satisfied is:

$$L \frac{dX}{dt} = q_u - q_f$$

where $(q_u - q_f)$ is the net rate of heat flow away from the interface and X is the depth of frost penetration in ft. More specifically:

$$L \frac{dX}{dt} = k_f \frac{\partial v_f}{\partial x} - k_u \frac{\partial v_u}{\partial x} \tag{9}$$

Equations 8 and 9 are the basic differential equations which must be solved for given initial conditions and boundary conditions to yield an expression for the depth of frost penetration. A solution to this problem is discussed later under a section on frost penetration formulas.

TABLE 2

PHYSICAL ANALOGIES IN CONDUCTION

Name	Fluid Flow Through Soil	Head Conduction	Current Conduction
1. Potential	Total head, h (ft)	Temperature, v ($^{\circ}$F)	Voltage, V (volts)
2. Storage	Fluid volume, W (cu ft per cu ft)	Thermal energy, u (Btu per cu ft)	Charge, Q (Coulombs)
3. Conductivity	Coefficient of permeability, k (ft per sec)	Thermal conductivity, k (Btu/$^{\circ}$F/ft/hr)	Electrical conductivity, o [Coulombs per (sec) (meter) (volt)]
4. Flow	Rate of flow, Q (cu ft per sec)	Rate of flow, Q (Btu per hr)	Current, i (Coulombs per sec = amperes)
5. Negative Gradient Along x	Hydraulic gradient, i $i = -\frac{\partial h}{\partial x}$ (ft per ft)	Thermal gradient, i $i = -\frac{\partial v}{\partial x}$ (deg F per ft)	Electric Intensity, E $E = -\frac{\partial V}{\partial x}$ (volts per meter)
6. Conduction Along x	Darcy's Law $Q = -k_x \frac{\partial h}{\partial x} A$	Fourier's Law $Q = -k_x \frac{\partial v}{\partial x} A$	Ohm's Law $i = -o_x \frac{\partial V}{\partial x}$ $A = \frac{V}{R}$
7. Capacitance	Coef of volume change, M (sq ft per cu ft) $M = \frac{dW}{dh} = \frac{\gamma_w a_v}{1+e} = \frac{k}{c_v}$ one dimensional	Volumetric heat, C (Btu/deg F/cu ft) $C = \frac{du}{dv}$	Capacitance, C (Coulombs per volt = farads) $C = \frac{dQ}{dV}$
8. Continuity (General Case)	$\frac{\partial W}{\partial t} + \nabla Q = O$	$\frac{\partial u}{\partial t} + \nabla Q = O$	$\frac{\partial Q}{\partial t} + \nabla i = O$
9. Continuity (Steady conduction)	$\nabla Q = O$	$\nabla Q = O$	$\nabla i = O$
(Laplace equation if material homogeneous and isotropic)	$\nabla^2 h = O$	$\nabla^2 v = O$	$\nabla^2 v = O$
10. Diffusion (One dimension)	$\frac{\partial h}{\partial t} = \frac{k}{M} \frac{\partial^2 h}{\partial x^2}$ ($\frac{k}{M} = c_v$, Coef. of consolidation)	$\frac{\partial v}{\partial t} = \frac{k}{C} \frac{\partial^2 v}{\partial x^2}$ ($\frac{k}{C} = a$, Diffusivity)	$\frac{\partial V}{\partial t} = \frac{o}{C} \frac{\partial^2 V}{\partial x^2}$

Analogies. The equations which have been developed in preceding paragraphs are similar to those encountered in many other branches of physics. For example, soil engineers will recognize that Equation 4 is analogous to Darcy's Law which governs the flow of water through soil. Furthermore, Equation 8 is analogous to the equation representing one-dimensional consolidation of a fine-grained soil.

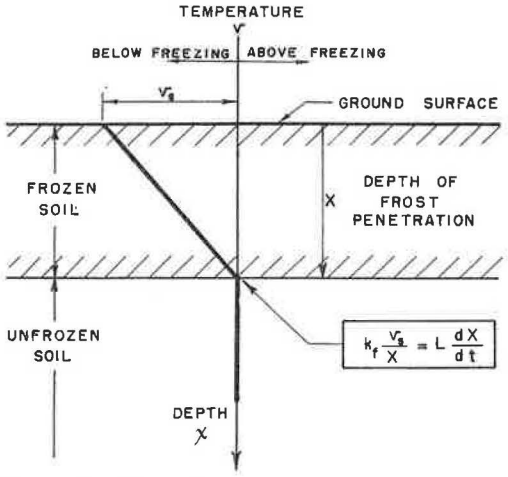

Figure 3. Thermal conditions assumed for Stefan Formula.

Table 2 has been prepared to show the formal correspondence between terms and equations found in three important branches of physics. Knowledge of these analogous relationships and those in magnetism, elastic equilibrium and others, give engineers and physicists a common language for discussion and for solving engineering problems. Often an explicit solution to an actual problem in one branch of physics will be available in analogous form in the literature of another branch.

The author has found analogs extremely useful in classroom instruction as well as in the solution of actual thermal and fluid flow problems.

DEPTH OF FROST PENETRATION

We shall confine our thermal problem to predicting the maximum depth of frost penetration below a pavement surface maintained relatively free of snow. A solution to this problem can be achieved in a variety of ways dependent largely on the objectives of the investigation and use of the result. One objective may be to obtain a single solution for a given set of design soil and weather conditions. Another problem may involve studying the effects of certain variables on the depth of frost penetration. Furthermore, the method of solution will depend on the precision desired in the result and the reliability of the given data.

Methods of Solution

Depth of frost penetration predictions can be obtained by (1) rational and empirical formulas and charts; (2) prototype tests and laboratory experiments; (3) "hand" solutions (graphical and numerical) based on finite difference approximations to the differential equations; (4) "machine" computation, for example automatic computation on IBM equipment; and (5) analogs such as the hydraulic and electronic analog computers which have been built for the New England Division, Corps of Engineers.

By far the most common method of predicting frost penetration is by formulas and charts based on analytic solutions to the fundamental equations and modified by assumptions and observed data. This method will be considered in some detail in the following paragraphs. In particular, a rational formula, given the name modified Berggren formula, will be presented.

Frost Penetration Formulas[4]

Of the numerous formulas which have been developed for predicting the depth of frost penetration, only two will be presented here. They are the J. Stefan formula and the modified Berggren equation. We shall assume initially that the soil is homogeneous, in other words, that it is uniform with depth (non-stratified).

Stefan Formula. Thermal conditions assumed in the derivation of the Stefan Equation, the simplest of the frost penetration formulas, are shown in Figure 3. It is assumed

[4] Reference (2) represents the most thorough theoretical treatment of frost penetration which has been written in English. Reference (4) is also excellent.

that the latent heat of soil moisture is the only heat which must be removed when freezing the soil. Thus, the thermal energy which is stored in the form of volumetric heat and released as the soil temperatures drop to and below the freezing point is not considered. This assumption is equivalent to shifting the sloping lines in Figure 2 to vertical positions.

Under these assumptions the diffusion equations in the frozen and unfrozen soil, Equation 8, do not exist and Equation 9 reduces to:

$$L \frac{dX}{dt} = k_f \frac{v_s}{X} \tag{10}$$

where v_s is the difference between the ground surface temperature and freezing temperature of soil moisture, Figure 3, at any time. In physical language, Equation 10 states that the latent heat supplied by the soil moisture as it freezes a depth dX in time dt is equal to the rate at which heat is conducted to ground surface.

When Equation 10 is integrated:

$$X = \sqrt{\frac{2k_f \int v_s \, dt}{L}} \tag{11}$$

where $\int v_s \, dt$ in deg hr is known as surface freezing index, F. Usually F is expressed in deg days Fahrenheit in which case:

$$X = \sqrt{\frac{48 \, k_f \, F}{L}} \tag{12}$$

which is the common form used to present the Stefan formula.

Since the Stefan equation neglects the volumetric heat of the frozen and unfrozen soil, it will give depths of frost penetration which are always too large. The degree to which Equation 12 overpredicts depends obviously on the relative importance of volumetric heat to latent heat which in turn depends on climate at the site in question and water content of the soil. These factors can be discussed to better advantage after the modified Berggren formula is presented.

Modified Berggren Formula. A detailed derivation of a rational formula, initially presented by the author and Dr. Henry M. Paynter[5] in a report to the Corps of Engineers (5), is given in the Appendix. After this formula was developed, it was discovered to be essentially identical to a formula published by W. P. Berggren (6). Since the general form of the solution differed from that of Berggren, the author and Dr. Paynter have named their solution the modified Berggren formula.

This solution is based on the earlier work of Neumann, as reported in Carslaw and Jaeger (7), dealing with melting and freezing problems in still water. It is assumed, as shown in the Appendix that the soil is a semi-infinite mass with uniform properties and existing initially at a uniform temperature v_o deg. It is further assumed that the surface temperature is suddenly changed from its initial value v_o deg above freezing to a temperature v_s deg below freezing. Equations 8 and 9 govern the penetration of frost. These and other thermal conditions during frost penetration are summarized in Figure 4.

It is indeed fortunate that the results of this rather complex development can be written with very little sacrifice in accuracy in the following simple form which may be compared with the Stefan Equation 11:

$$X = \lambda \sqrt{\frac{2 \, k \, v_s \, t}{L}} \tag{13}$$

[5] Assistant Professor of Mechanical Engineering, M. I. T., and President, Pi-Square Engineering Co., Inc., Boston, Mass.

where

X = depth of frost penetration in ft
k = thermal conductivity of soil, in Btu per hr per ft per deg F generally taken as an average of values in the frozen and unfrozen state.
L = latent heat of soil, in Btu per cu ft
t = time in hr

and

λ = dimensionless correction coefficient which is given in Figure 5 as a function of two important dimensionless parameters, α and μ:

$$\text{Thermal ratio, } \alpha = \frac{v_o}{v_s} \tag{14}$$

$$\text{Fusion parameter, } \mu = \frac{C}{L} v_s$$

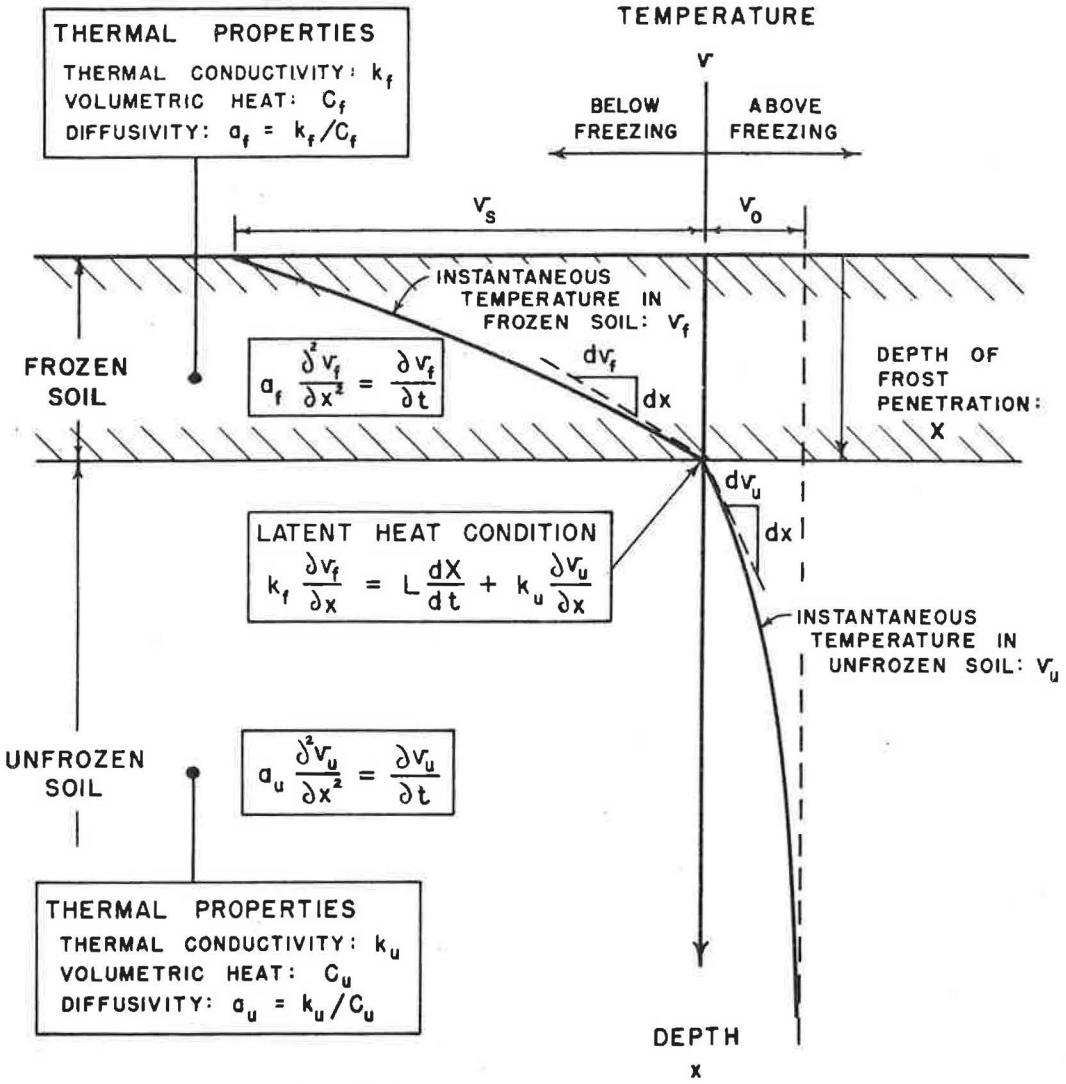

Figure 4. Thermal conditions assumed for modified Berggren formula.

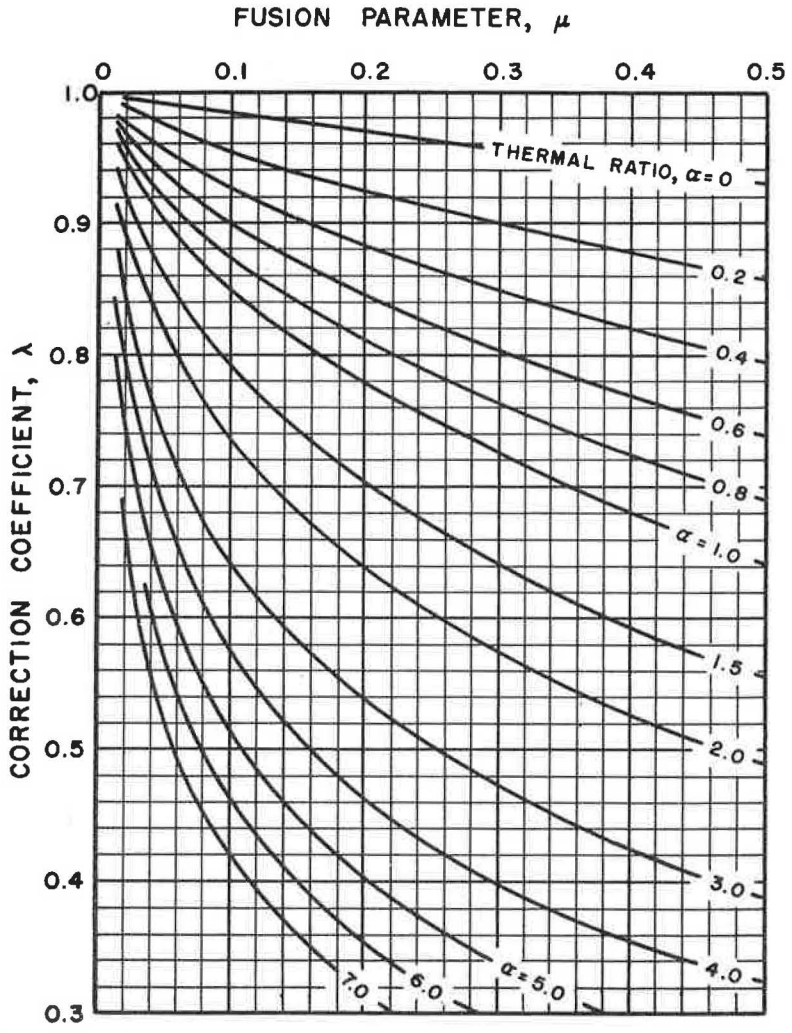

Figure 5. Correction coefficient in the modified Berggren formula.

The correction coefficient λ in equation 13 is essentially a term which corrects the Stefan formula for the effects of volumetric heat which it neglected. We noted that the Stefan equation always predicts too deep. Thus λ is always less than unity.

In the foregoing discussion of frost penetration formulas a number of assumptions were stated or implied in order to develop a rational mathematical approach for determining the maximum depth of frost penetration. While this approach is valuable as an aid to understanding many of the variables affecting the problem, we can no longer ignore complexities introduced by reality since we seek a useful solution to a real problem. The randomness of the weather and its effect on pavement temperatures and the usual heterogeneous nature of the soil are examples of important factors which must now be given consideration.

Effect of Surface Temperature Variations During the Freezing Period. In the development of the modified Berggren formula the surface temperature was assumed to change suddenly from v_o degrees above freezing to v_s degrees below freezing where it remained constant. Thus, the relationship among the dimensionless parameters, Figure 5, is strictly applicable only to this case. In reality the pavement surface experiences daily as well as seasonal temperature fluctuations through local variations in air temperature, wind velocity, precipitation, solar radiation, etc. There can be little doubt that these

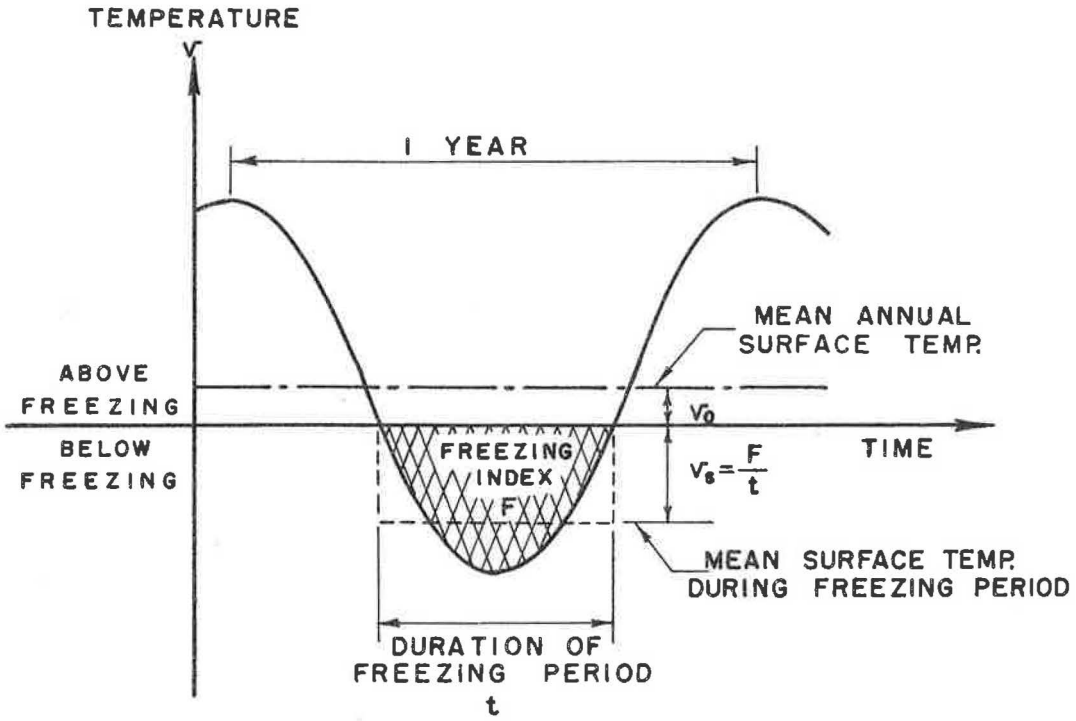

Figure 6. Sine curve for assumed annual variation in surface temperature.

variations have a marked effect on the rate at which freezing temperatures penetrate into the soil. However, the maximum depth of frost penetration is affected little if v_o and v_s are defined as shown in Figure 6.

In conjunction with studies for the Arctic Construction and Frost Effects Laboratory of the New England Division, Corps of Engineers, the author with Dr. Paynter obtained a series of frost penetration solutions assuming that surface temperature varies sinusoidally during the year, Figure 6. These computations, performed by the IBM Company on an automatic computer, gave values for the correction coefficient λ which varied less than 8 percent from those for the step change in surface temperature assumed in the modified Berggren formula. The IBM results and comparisons are given in Table 3. In view of this result and the results of statistical studies on observed depths of frost penetration, the following development from Equation 13 is indicated.

Assume that v_s, Figure 6, represents the time-average difference in temperature between ground surface and the freezing point during the freezing period. Therefore, the term $v_s t$ in Equation 13 becomes the surface freezing index, F. If F is expressed in deg days Fahrenheit, the modified Berggren formula becomes:

$$X = \lambda \sqrt{\frac{48\, k\, F}{L}} \tag{15}$$

which may be compared with the Stefan Equation 12. The correction coefficient λ may be obtained from Figure 5[6] as a function of α and μ which can now be expanded from equation (14) as follows:

$$\text{Thermal ratio } \alpha = \frac{v_o}{v_s} = \frac{v_o\, t}{F}$$
$$\text{Fusion parameter, } \mu = \frac{C}{L} v_s = \frac{CF}{Lt} \tag{16}$$

[6] If desired, a slight correction for seasonal variation in surface temperature can be made by selecting a λ value differing from Figure 5 consistent with IBM results given in Table 3.

where v_o is the deg F by which the mean annual temperature exceeds the freezing point of soil moisture, Figure 6.

We are prepared now to demonstrate the degree to which the Stefan equation over-predicts by observing the magnitude of λ for the different climates and soil conditions. The importance of this demonstration as an aid to a fundamental understanding of frost penetration cannot be over-emphasized. Reference is made to Figure 5:

1. For soils having high water contents the volumetric heat C is small compared to latent heat L in which case the fusion parameter μ is small. Therefore, the correction coefficient λ approaches unity and the Stefan equation may be expected to give reasonable results.
2. For northern climates where the mean annual temperature approaches the freezing point of soil moisture, the thermal ratio α approaches zero and the correction coefficient is greater than 0.9. Thus, the Stefan equation will yield depths of frost penetration not more than 10 percent greater than the actual.[7]

In general then, for northern climates represented by North Dakota, Canada and Alaska and for soils of high water content the correction coefficient may often be assumed equal to unity.[8] However, in more temperate climates represented, for example, by Kansas and Nebraska and for relatively dry or well drained soils, the correction coefficient may be as low as 0.5 in which case the Stefan formula would predict twice the actual depth.

Relationship Between Pavement Surface Temperature and Air Temperature. The depth of frost penetration and indeed all subsurface temperatures are governed by variations in the ground surface temperature. Surface temperature has been used, therefore, in all developments which have been presented thus far. Unfortunately, pavement surface temperature variations are generally unknown while abundant data on air temperatures are usually available at or near a given site. Pavement temperature and air temperature are related but this relationship is one of the most complex and fundamentally important problems remaining to be solved in the frost penetration prediction.

The author has frequently used the following example as a vivid demonstration of the important air-pavement temperature relationship. At an air force base in Greenland in the early summer of 1953, thawing occurred to a depth of about 4 ft below a bituminous concrete pavement before the mean daily air temperature rose above 32 deg F. In other words, the pavement surface was enough warmer than the air to thaw 4 ft of base course before thawing would normally be expected to begin.

A fundamental investigation of factors affecting heat transfer at the air-ground interface has been the objective of recent research at MIT sponsored by the Arctic Construction and Frost Effects Laboratory, New England Division, Corps of Engineers. A report based on the work of Scott (9), is in preparation as Reference 10. Current research is directed toward reducing Dr. Scott's theoretical studies to design criteria. In this paper, the author will describe briefly the important physical phenomena only.

TABLE 3

EFFECT OF SINUSOIDALLY VARYING SURFACE TEMPERATURE ON CORRECTION COEFFICIENT

Thermal Ratio, α	Fusion Parameter, μ	Correction Coefficient λ [a]	Correction Coefficient λ [b]
0	0.191	0.97	0.998
1.025	0.195	0.78	0.724
2.040	0.196	0.64	0.595
0	0.382	0.94	0.998
0.519	0.386	0.80	0.804

[a] Given in Figure 5, modified Berggren formula, for step change in surface temperature.
[b] Determined from results of studies on IBM-701 computer for surface temperature varying sinusoidally.

Transfer of heat between the pavement surface and air is effected through moisture evaporation and condenstion, snow and ice melt, and more important through direct and diffuse solar radiation, net longwave radiation between pavement and sky, and convection-conduction. The latter terms are shown graphically in Figure 7 as they could occur on a sunny day. The principal variables affecting the magnitude of these factors during a winter season may be summarized as follows:

Solar and longwave radiation:

1. Latitude (for sun's altitude) and elevation of the site,

[7] Indeed, the IBM results, Table 3, indicate a correction coefficient equal to unity for the $\alpha = 0$ case. Thus, the Stefan equation should yield a nearly exact solution.
[8] Carlson and Kersten (8) report successful use of the Stefan formula in predicting the depth of freezing and thawing under pavements in Alaska.

2. Atmospheric vapor pressure (for solar absorption and scattering),
3. Cloud cover and type (for equivalent percent sunshine),
4. Atmospheric conditions (whether clear or industrial),
5. Type of surface (color and texture for absorbtivity to radiation).

Convection - conduction:

1. Wind velocity,
2. Type of surface (surface roughness),
3. Topography and vegetation in near vicinity.

Unfortunately, the magnitude of the radiation and convection-conduction heat transfer depends also on the unknown surface temperature itself. Therefore, the solution must be evolved through a successive approximation procedure.

Figure 8 has been prepared to represent idealized temperature curves during the winter season for various surfaces. The sine curves, all drawn about the same vertical axis, are presented for rough qualitative comparisons only. For example it is generally true that a pavement surface, even though maintained relatively free of snow, will exist at a temperature higher than the mean air temperature throughout the year. Furthermore, natural surfaces are warmer still in winter, resulting in part from insulating effects of snow, but during the summer are cooler than the surrounding air. An important factor affecting the latter is evaporation.

From the idealized curves in Figure 8 we can generate the following statements which may be important to the practicing engineer:

1. Estimates of frost penetration depths below a proposed pavement which are based

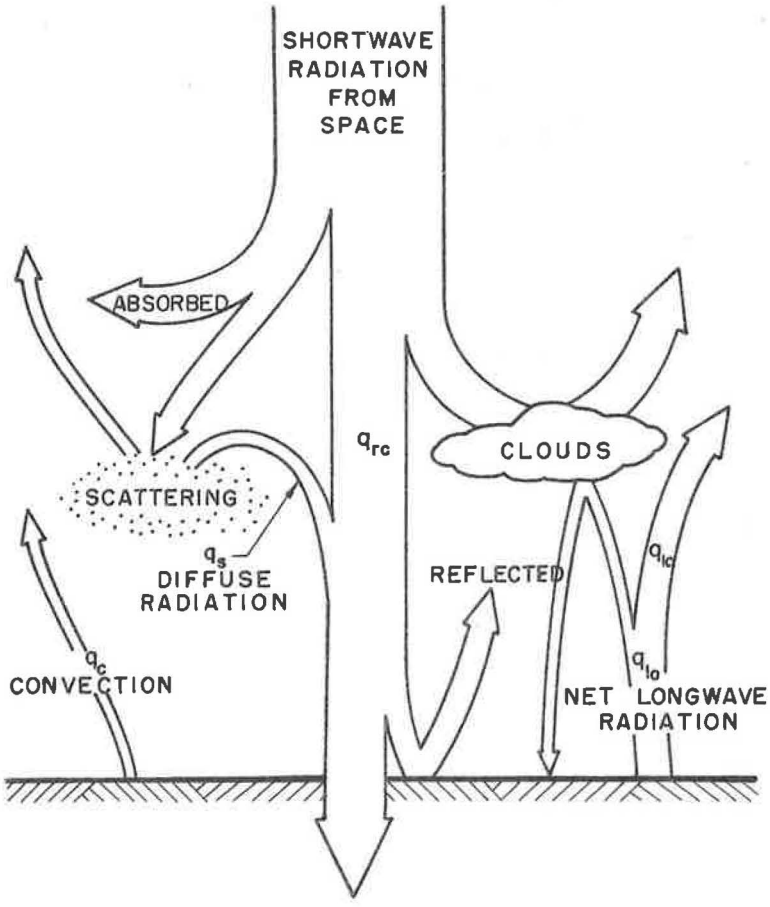

Figure 7. Heat-transfer between ground surface and air on a sunny day.

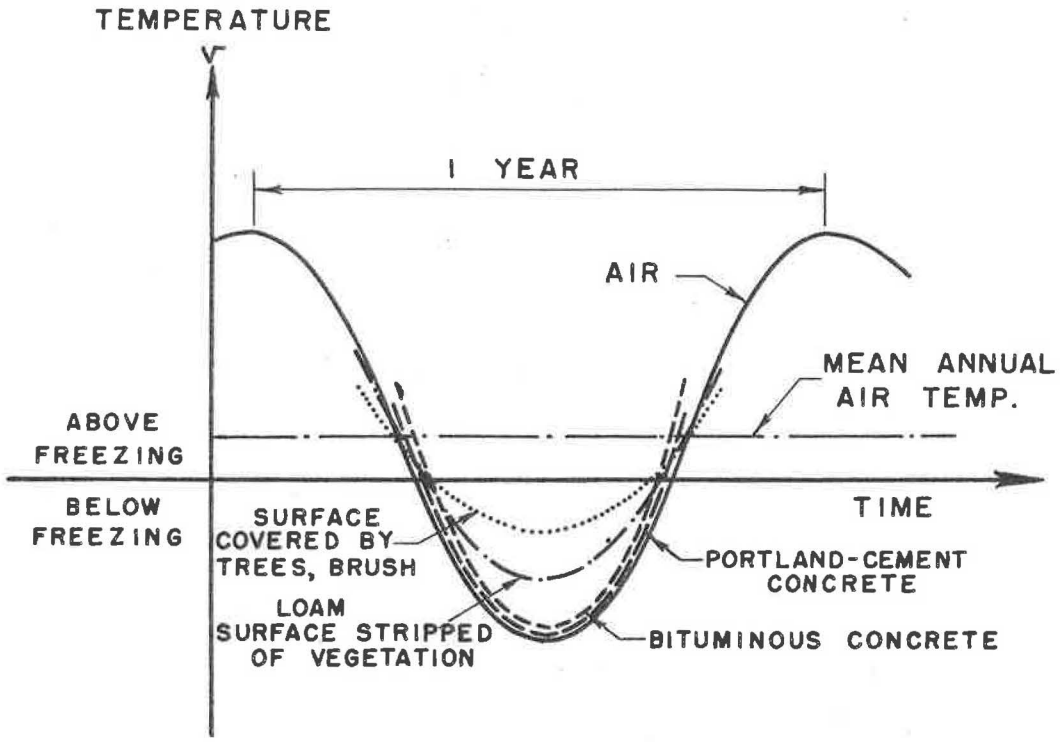

Figure 8. Idealized Temperature curves for various surfaces.

on records of observed frost depths in a given locality are likely to be unsafe for two reasons: (a) the pavement surface temperature will probably be colder than surrounding natural surfaces during the winter months; (b) if the pavement base-course is well drained it is likely to exist at a lower water content than natural soil in the surrounding area. Again, the frost depth will be greater below the pavement because, from Equation 15, we observe:

$$X \sim \sqrt{\frac{1}{L}}$$

where the latent heat L depends directly on the amount of moisture in the soil.[9]

2. In areas of permanently frozen ground (permafrost areas) major engineering problems arise because of changes in the subsurface thermal regime as a result of new surface conditions caused by construction. For example, it can be reasoned from Figure 8 that the seasonal depth of thawing will be greatly increased when an area is stripped of its vegetation and a pavement surface is constructed.

We have defined surface freezing index as the area under the temperature curve during the freezing period, Figure 6. It is clear that the pavement freezing index which we seek for the depth of frost penetration computation is less than the air freezing index which is generally known. The ratio of these indices has been called a surface correction factor n or C_f. It may be determined for a given pavement surface from direct observations of pavement and air temperatures (11). It is evident that the value of n which is applicable to one locality may be very different from that at another site even under identical pavement conditions. Indeed, a limiting value would be zero in a temperate climate with no pavement freezing index while the ratio approaches unity for colder climates.

[9] It is true that other factors in the equation vary in addition to F and L, notably the thermal conductivity k which is lower for the well-drained soil. However, this variation is more than offset by the latent heat.

A general rational approach to predicting pavement freezing index from air temperature data or from the air freezing index would be to base the computation on a surface temperature elevated a fixed amount relative to air temperature rather than on a fixed ratio of freezing indices. The author is frank to admit, however, that the more conservative approach is to base the prediction on air temperature. Indeed, in computations for the seasonal maximum depth of frost penetration below pavements this approach is recommended, at least until results of current studies are completed. Consistent with this assumption is that of basing v_o on the mean annual air temperature rather than surface temperature.

Effect of Ice-Segregation During Freezing. It is well known that distinct layers or lenses of ice will form during freezing in frost-susceptible soils when certain moisture conditions are present. The thickness of individual ice layers is inversely proportional to the rate of frost penetration. However, the rate at which the ground surface heaves is essentially independent of the rate of frost penetration in a frost-susceptible soil. While a discussion of the ice-segregation phenomenon is outside the scope of this paper,

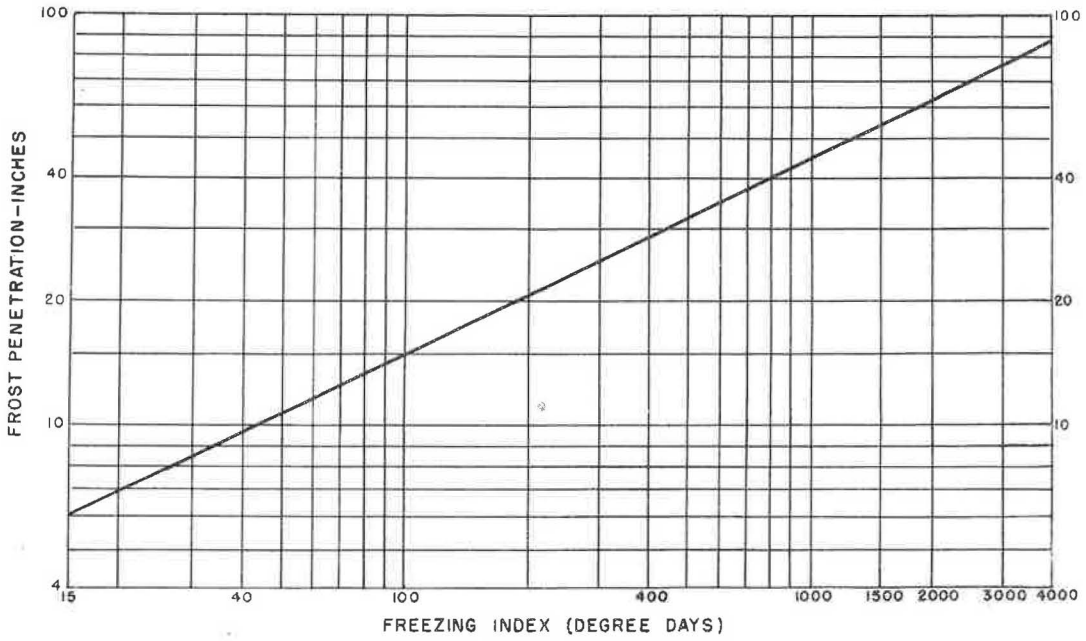

Figure 9. Relationship between frost penetration and freezing index in well-drained, granular, non-frost susceptible base course material. (Courtesy of the Arctic Construction and Frost Effects Laboratory, New England Division, Corps of Engineers (13) and (14).

its effect on the maximum depth of frost penetration will be treated briefly.

In most highway and airfield pavement design today, the thickness of pavement and base is selected to prevent entirely or substantially freezing in a frost-susceptible subgrade. Therefore, even though frost penetration formulas have been proposed, (2) and (4), to account directly for ice segregation, the need for this refinement in practice is not common. The author believes that the following approximate treatment based on the modified Berggren formula should satisfy most requirements.

In the approximate treatment as well as in those formulas accounting for ice-segregation, the magnitude or rate of ice-lense formation must be known or assumed. Once the increase in moisture in the ice-segregated layer is assumed, its effect may be accounted for in the depth of frost penetration computation by altering thermal properties of the layer. The change in thermal conductivity and volumetric heat resulting from an increase in moisture content will be small compared to the latent heat effect. Therefore, in computations based on the modified Berggren or Stefan formulas, the depth of

frost penetration below the frost heaved pavement surface may be computed using a latent heat L based on the equivalent moisture content and dry density of the ice-segregated zone.

As an example, assume that frost penetrates into a frost-susceptible subgrade having a water content of 18 percent and dry density of 113 pcf in the unfrozen state. If the subgrade is assumed to expand 25 percent in thickness from ice lense growth during freezing, the equivalent moisture content is about 30 percent and dry density 95 pcf. In this state the latent heat is about 4000 Btu per cu ft compared to 3000 in the initial state. Thus, in the computation for the depth of frost penetration, L is assumed to be equal to 4000 Btu per cu ft in the subgrade subject to ice-segregation.

It is evident that significant ice-segregation reduces the maximum depth of frost penetration by increasing the latent heat term.

Freezing Point of Soil Moisture. Water normally freezes at 32 F. However, it has been clearly demonstrated (12) that the freezing point of soil moisture in fine-grained soils is generally below 32 F. The author has avoided the use of 32 F for the freezing point in order to maintain flexibility in allowing for a depressed freezing point. Thus, in the frost penetration prediction, the terms F, t, v_o and v_s in Equations 15 and 16 and in Figure 6 may be determined on the basis of any assumed or known freezing point of soil moisture.

In the prediction of frost penetration through granular soils below highway and airfield pavements the freezing point should be taken at 32 F.

Deviations From the One-Dimensional Assumption. The frost penetration prediction has been formulated as a one-dimensional problem. This assumption implies that ground surface temperature at a given time is everywhere uniform over the surface area and that no variation in soil conditions or subsurface temperature exists in a horizontal direction.

The one-dimensional assumption is more nearly satisfied in frost penetration below pavements than below most natural surfaces. This is especially true when the depth of frost penetration is small compared to the pavement width. Indeed, we could not question the assumption when considering frost penetration below airfield pavements.

Frost penetration below the edge of typical highway pavements is generally less than that below the center due largely to the insulating effects of snow on the shoulders. Thus, the one-dimensional assumption would be conservative, yielding depths slightly greater than the actual. One should not assume, however, that the frost problem is less serious below the edges. Ice-segregation, surface heave, and eventual subgrade weakening is frequently more pronounced at the pavement edge.

Preliminary Estimates of Frost Penetration. It is frequently necessary to obtain a quick estimate of the depth of frost penetration below a pavement surface before the data required in a comprehensive analysis are available. Further, we shall see that an analysis presented under the following heading for stratified soil requires in itself a first estimate of frost penetration depth. Thus, it seems appropriate to introduce the following results.

An empirical curve (Figure 9)[10] gives an approximate relationship between depth of frost penetration and air freezing index in non-frost susceptible soil beneath airfield pavements. Thus, the chart must be considered to apply only to pavement surfaces kept relatively free of snow and underlain by well-drained granular base-course materials. Figure 9 was determined from frost penetration data gathered during the winters 1944-45 and 1945-46 at airfields having both bituminous concrete and Portland-cement concrete pavements (14). The observed depths deviated a maximum of about 50 percent from the value given in Figure 9 while half the observations were within plus or minus 10 percent.

A preliminary estimate of the frost penetration depth below a proposed pavement surface may be obtained from Figure 9 if the air freezing index is known. The latter may be determined from weather bureau data (daily mean air temperatures) for the locality

[10] Figure 9 as well as Figures 10 and 11 are reproduced through the courtesy of the Arctic Construction and Frost Effects Laboratory, New England Division, Corps of Engineers. They are published in References 13 and 14.

or estimated from a freezing index map such as that shown in Figure 10 which gives mean values.[11]

Frost Penetration Through Non-Uniform (Stratified) Soil. All practical analytic solutions for the depth of frost penetration are derived for uniform soil. The modified Berggren equation is no exception. Therefore, for the non-uniform or multilayer system which occurs typically below highway and airfield pavements, approximate computation techniques based on the mathematical solutions must be used. Since no exact solution for the multilayer case exists, there is no possibility of verifying, in a simple way, the reliability of the approximate procedure itself. It is true that a numerical approach or an analogue computer will give an "exact" solution. Some progress has been made in this direction (5), (9), and (15).

In computing the maximum depth of frost penetration below a typical highway or airfield pavement the following semi-empirical adaptation of the modified Berggren formula is suggested. We shall assume that the pavement profile is given as are values of water content and dry density for each layer. The layers making up the profile will be numbered consecutively downward starting with the Portland-cement concrete or bituminous concrete layer as No. 1.

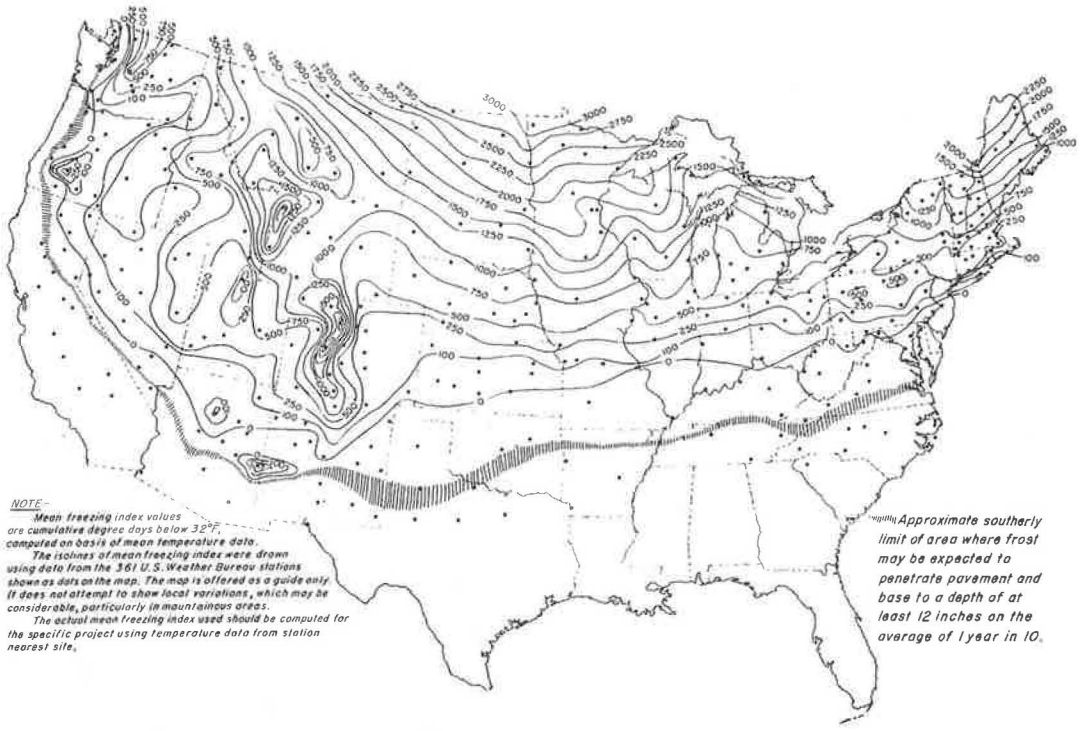

Figure 10. Distribution of mean freezing index values in continental United States. (Courtesy of the Arctic Construction and Frost Effects Laboratory, New England Division, Corps of Engineers (13) and (14).

[11] Linell (13) has recommended the use in design of a one-year-in-ten freezing index rather than the mean, for example, the average of the three coldest years in a 30 year cycle. The one-year-in-ten index which differs little from the one-year-in-five, is approximately equal to the mean index plus 500 deg days. Linell points out that the freezing index to be used in design involves an economic balance between cumulative loss from frost damage over the life of the pavement and cost of protective measures to prevent frost damage.

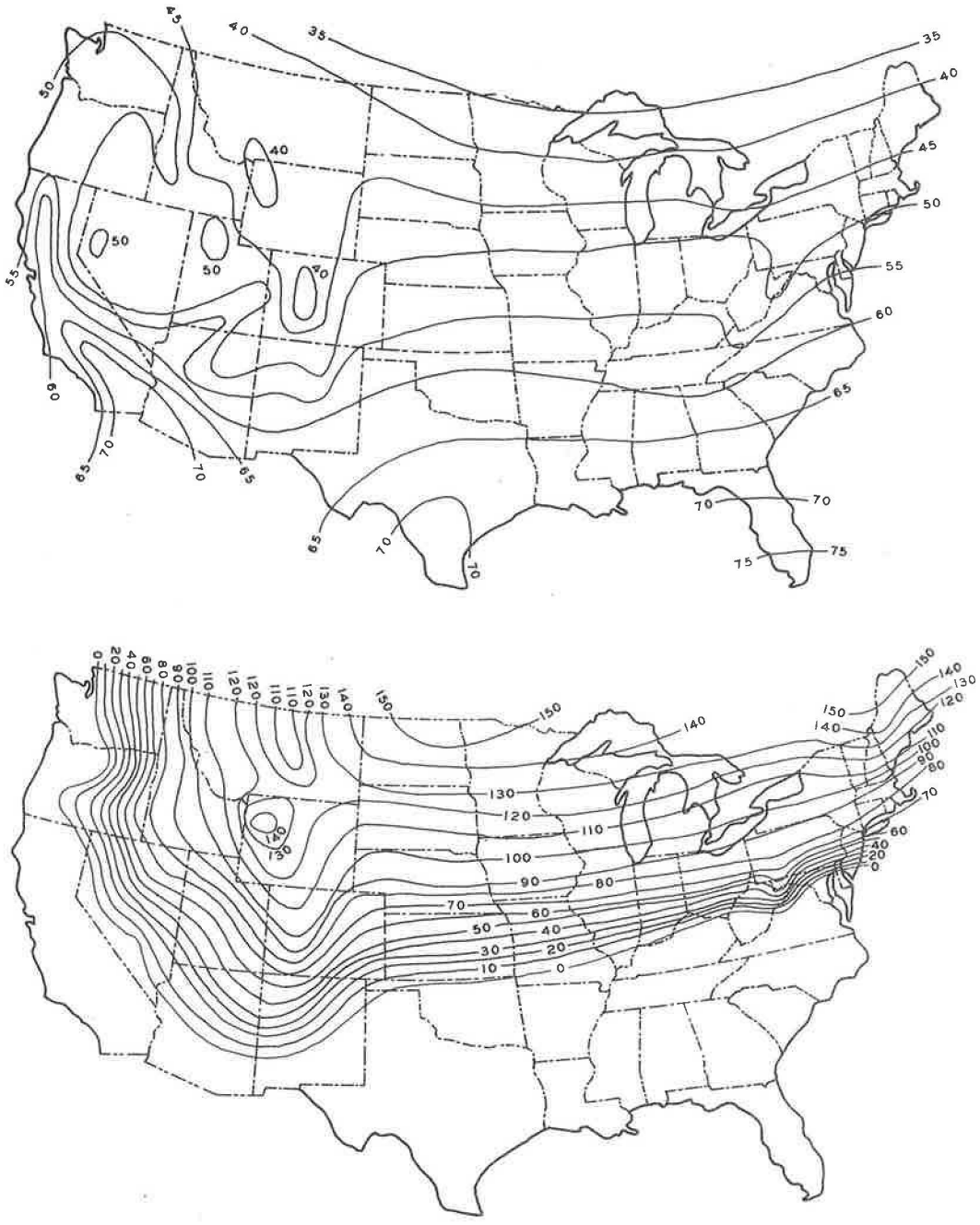

Figure 11. Mean annual air temperature, in degrees F (upper); and duration of normal freezing index, in days (lower). (Courtesy of the Arctic Construction and Frost Effects Laboratory, New England Division, Corps of Engineers (13) and (14).

Step 1. Determine the pavement freezing index F, which may be conservatively assumed to be equal to the air freezing index.

Step 2. Determine the duration of the freezing period t, and the mean annual air temperature from weather bureau data for the locality or from maps such as those shown in Figure 11.

Then:

$$v_o = \text{mean annual temperature minus } 32$$

Step 3. From the given w and γ_d for each layer, determine the thermal properties k, C and L for each stratum within the estimated depth of frost penetration:

$$k = \frac{k_f + k_u}{2} \quad \text{from Figure 1 and Table 1}$$

$$C = \frac{C_f + C_u}{2} \quad \text{from Equation 1}$$

$$L = 1.434\ w\ \gamma_d$$

Step 4. Compute an effective $\frac{L}{k}$ from the following equation [12]:

$$\left(\frac{L}{k}\right)_{eff.} = \frac{2}{X^2}\left[\frac{d_1}{k_1}\left(\frac{L_1 d_1}{2} + L_2 d_2 + \ldots . L_n d_n\right) + \frac{d_2}{k_2}\left(\frac{L_2 d_2}{2} + L_3 d_3 + \ldots . L_n d_n\right) + \ldots\ldots + \frac{d_n}{k_n}\left(\frac{L_n d_n}{2}\right)\right]$$

where X is the estimated depth of frost penetration in feet and d is the thickness of a layer within the depth. Thus:

$$X = d_1 + d_2 + \ldots . d_n$$

Step 5. Compute weighted values of C and L within the estimated depth of frost penetration from:

$$C_{wt} = \frac{C_1 d_1 + C_2 d_2 + \ldots . C_n d_n}{X}$$

$$L_{wt} = \frac{L_1 d_1 + L_2 d_2 + \ldots . L_n d_n}{X}$$

Step 6. Compute the effective values of α and μ from Equation 16:

$$\alpha = \frac{v_o t}{F} \qquad \mu = \frac{C_{wt} F}{L_{wt} t}$$

Step 7. Determine the correction coefficient λ from Figure 5.

Step 8. Compute the depth of frost penetration from Equation 15:

$$X = \lambda \sqrt{\frac{48 F}{\left(\frac{L}{k}\right)_{eff}}}$$

If the computed depth differs appreciably from the assumed depth, steps 4 through 8 may be repeated. Seldom are more than two cycles necessary.

EXAMPLE

As an example of the foregoing procedure, we select the following pavement profile with given thermal properties:

[12] This equation may be derived from developments presented in Reference 5, pages 43 and 44. It is determined by considering the partial freezing indices required to freeze each layer of soil.

Given

Layer	k Btu per hr per ft per deg F	C Btu per cu ft per deg F	L Btu per cu ft
3 in. bit.concrete	0.8	28	0
6 in. base-course	1.00	23	850
21.5 in. subbase	1.30	25	1200
subgrade	1.70	27	2900

Mean annual temperature: 37 F (v_o = 5 F)
Surface freezing index F: 1568 deg days F
Duration of freezing period t: 157.5 days

Solution

Results from Steps 1, 2 and 3 are given.

Step 4.

The estimated depth of frost penetration X, is 5.7 ft

Thus:

$$\left(\frac{L}{k}\right)_{eff} = \frac{2}{5.7^2}\left[\frac{0.25}{0.8}\left(0 + \overset{425}{(850)}(0.5) + \overset{2150}{(1200)}(1.79) + \overset{9160}{(2900)}(3.16)\right)\right.$$
$$+ \frac{0.5}{1.00}\left(212 + 2150 + 9160\right)$$
$$\left. + \frac{1.79}{1.30}\left(1075 + 9160\right) + \frac{3.16}{1.70}\left(4580\right)\right]$$

$$\left(\frac{L}{k}\right)_{eff} = 1970$$

Step 5.

$$C_{wt} = \frac{(28)(0.25) + (23)(0.5) + (25)(1.79) + (27)(3.16)}{5.7} = 26.0$$

$$L_{wt} = \frac{0 + (850)(0.5) + (1200)(1.79) + (2900) + (3.16)}{5.7} = 2060$$

Step 6.

$$\alpha = \frac{v_o t}{F} = \frac{(5)(157.5)}{1568} = 0.503$$

$$\mu = \frac{C_{wt} F}{L_{wt} t} \qquad \frac{(26.0)(1568)}{(2060)(157.5)} = 0.126$$

Step 7.

From Figure 5, $\lambda = 0.90$

Step 8.

$$X = \lambda \sqrt{\frac{48\ F}{\left(\frac{L}{k}\right)_{eff}}} = 0.90 \sqrt{\frac{(48)(1568)}{1970}}$$

X = 5.6 ft

Nordal (15) obtained solutions to this problem with a hydraulic analogue computer assuming in one case a step change in surface temperature and in another case a surface temperature varying sinusoidally to accumulate a freezing index of 1568 deg days F in 157.5 days. Depth of frost penetration from these solutions are 5.7 ft and 5.8 ft respectively, which compare favorably with the semi-empirical computational procedure

based on the modified Berggren formula. Other solutions obtained by Nordal also agree favorably with computed depths.

We now have then a computational procedure known to give reliable predictions for the depth of frost penetration if reliable data are given. Thus, it would appear that our ability to predict the actual depth of frost penetration below a given pavement depends primarily on the reliability of thermal properties and surface temperature used in the computation. The need for continued research on these factors and in other areas related to frost-action in soil are very ably presented by Johnson and Lovell (16).

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Appendix

DERIVATION OF A RATIONAL FORMULA FOR THE PREDICTION OF FROST PENETRATION

By Harl P. Aldrich, Jr. and
Henry M. Paynter

BASIC ASSUMPTIONS

A solution is sought to the thermal problem of a semi-infinite soil mass of uniform properties and at a uniform initial temperature subjected to changes of temperature at the surface, which are assumed uniform over the surface extent to yield a one-dimensional problem.

Figure 4 illustrates the nomenclature and significant variables in this situation. Further assumptions are best listed in the form of specific conditions as follows:

Condition I - At the Ground Surface

It is assumed that the surface temperature is suddenly changed from an initial temperature v_o above freezing to a temperature v_s below freezing. This temperature value is then maintained constant and uniform over the entire surface.

Condition II - In the Frozen Soil

It is assumed that the diffusion equation:

$$a_f \frac{\partial^2 v_f}{\partial x^2} = \frac{\partial v_f}{\partial t}$$

with $a_f = k_f / C_f$ measuring the diffusivity of the frozen soil is satisfied throughout the frozen soil mass, subject to the surface temperature condition (I) and the latent heat condition (III) at its boundaries.

Condition III - At the Moving Frost Interface

It is assumed that at the interface between the frozen soil (above) and the unfrozen soil (below) the temperature remains constant at the freezing point of the soil moisture. It is further assumed that the heat flow upward just above the interface in the frozen soil is equal to the sum of the heat flow just below the interface in the unfrozen soil plus the heat flow due to the removal of the latent heat of fusion of the soil moisture as it freezes.

Condition IV - In the Unfrozen Soil

It is assumed that the diffusion equation:

$$a_u \frac{\partial^2 v_u}{\partial x^2} = \frac{\partial v_u}{\partial t}$$

with $a_u = k_u / C_u$ measuring the diffusivity of the unfrozen soil is satisfied throughout the unfrozen region, subject to the latent heat condition (III) at the frost interface (the upper boundary of the unfrozen soil) and to the lower boundary condition that, at all times, the temperature approaches the initial temperature for sufficiently great depths.

GENERAL SOLUTION

The solution for the temperature in the frozen soil v_f which satisfied conditions I and II may be written:

$$v_f = -v_s + A \operatorname{erf}\left(\frac{x}{2\sqrt{a_f t}}\right) \qquad (A\text{-}1)$$

while the solution for v_u satisfying condition IV may be written:

$$v_u = v_o + B\left[1 - \operatorname{erf}\left(\frac{x}{2\sqrt{a_u t}}\right)\right] \tag{A-2}$$

At the frost interface where x = X, in order to satisfy condition III it is necessary that:

$$v_f = v_u = 0 \tag{A-3}$$

from which:

$$A \operatorname{erf}\left(\frac{X}{2\sqrt{a_f t}}\right) = v_s \tag{A-4}$$

$$B\left[1 - \operatorname{erf}\left(\frac{X}{2\sqrt{a_u t}}\right)\right] = -v_o \tag{A-5}$$

Since these last conditions must be satisfied for all values of time,

$$\frac{X}{\sqrt{t}} = \text{constant}, \gamma \tag{A-6}$$

or

$$X = \gamma\sqrt{t} \tag{A-7}$$

Thus, the constant γ depends upon v_s, v_o, and the thermal coefficients. The manner of this dependence may be found by considering the latent heat requirement of condition III. This thermal continuity condition relates the temperature gradients each side of the interface to the rate of movement of the interface in the form:

$$L\frac{dX}{dt} = k_f \frac{\partial v_f}{\partial x} = k_u \frac{\partial v_u}{\partial x} \tag{A-8}$$

which becomes, upon substituting Equations A-4, A-5, and A-7, performing appropriate differentiation, and simplifying:

$$\frac{v_s k_f}{\sqrt{\pi a_f}} \cdot \frac{e^{-\frac{\gamma^2}{4a_f}}}{\operatorname{erf}\left(\frac{\gamma}{2\sqrt{a_f}}\right)} - \frac{v_o k_u}{\sqrt{\pi a_u}} \cdot \frac{e^{-\frac{\gamma^2}{4a_u}}}{\left[1 - \operatorname{erf}\left(\frac{\gamma}{2\sqrt{a_u}}\right)\right]} = \frac{L}{2}\gamma \tag{A-9}$$

Furthermore, by making the substitutions:

$$\alpha \equiv \frac{v_o C_u}{v_s C_f} \tag{A-10}$$

$$\delta \equiv \sqrt{\frac{a_f}{a_u}} \tag{A-11}$$

$$\mu \equiv \frac{v_s C_f}{L} \tag{A-12}$$

$$Z \equiv \frac{\gamma}{2\sqrt{a_f}} \tag{A-13}$$

it is possible to rewrite Equation A-9 in a simplified non-dimensional form; namely,

$$\mu\left[\frac{e^{-Z^2}}{\operatorname{erf} Z} - \frac{\alpha}{\delta}\frac{e^{-\delta^2 Z^2}}{(1 - \operatorname{erf}\delta Z)}\right] = \sqrt{\pi Z} \tag{A-14}$$

If all the terms in Z are then carried to the right-hand side, one has a direct relation between μ , α , δ , and Z in the form:

$$\mu = \frac{\sqrt{\pi}\, Z}{\left[\frac{e^{-Z^2}}{\text{erf } Z} - \frac{\alpha}{\delta} \frac{e^{-\delta^2 Z^2}}{(1 - \text{erf } \delta Z)} \right]} \qquad \text{(A-15)}$$

This last may then be inverted to obtain a transcendental relation for Z in terms of the parameters α , δ , and μ, which may be solved graphically to give, in symbolic form:

$$Z = \int (\alpha , \delta , \mu) \qquad \text{(A-16)}$$

From Equation A-13, one may obtain

$$\gamma = 2\sqrt{a_f}\, Z \qquad \text{(A-17)}$$

and combining with Equation A-7 for the frost penetration depth:

$$X = \gamma \sqrt{t} = 2 Z \sqrt{a_f t} \qquad \text{(A-18)}$$

It is found convenient to rearrange this last expression so that the radical is expressed in terms of μ in the form:

$$X = \frac{2 Z}{\sqrt{2\mu}} \sqrt{2\mu} \sqrt{a_f t} \qquad \text{(A-19)}$$

By defining the new dimensionless correction coefficient λ as :

$$\lambda = \sqrt{\frac{2 Z^2}{\mu}} \qquad \text{(A-20)}$$

and by substituting under the radical in Equation A-19 for the physical variables, one finally obtains for X, the general expression:

$$X \quad \lambda \sqrt{\frac{2 k_f v_s t}{L}} \qquad \text{(A-21)}$$

where the correction coefficient λ is a function of the three dimensionless parameters α , μ , δ .

PHYSICAL SIGNIFICANCE OF PARAMETERS

(a) Thermal Ratio (α): The thermal ratio α , defined as:

$$\alpha = \frac{v_o C_u}{v_s C_f} \qquad \text{(A-22)}$$

measures the ratio of heat stored initially in the unfrozen soil to the heat loss in the frozen soil. If it may be assumed, as with many existing formulas, that the difference between the volumetric heats C_u and C_f is not usually significant, the thermal ratio α may be written:

$$\alpha = \frac{v_o}{v_s} \qquad \text{(A-23)}$$

which is the ratio of the initial ground temperature above the freezing point to the as sumed constant surface temperature below freezing during the freezing period.

(b) Diffusivity Ratio (δ): The (root) diffusivity ratio δ , defined as:

$$\delta = \sqrt{\frac{a_f}{a_u}} \qquad \text{(A-24)}$$

measures the relative values of diffusivity a = k/C in the frozen and unfrozen soils. It is clear that for most soils of low moisture content δ is approximately unity. Tabulated herewith are typical values of δ for representative soil types and moisture contents.

TYPICAL VALUES OF THE DIFFUSIVITY RATIO[1]

$$\delta = \sqrt{\frac{k_f C_u}{k_u C_f}}$$

Assumed Dry Density = 110 pcf

Moisture Content	Values of δ	
	Silt or Clay	Sand
%		
0	1.00	1.00
5	1.07	0.92
10	1.15	1.14
15	1.22	1.31
20	1.30	1.50

The effect of variations in δ may be found directly from the λ , α , μ graph, Figure 5, by making use of an equivalent α -value given by the expression:

$$\alpha_\delta = \frac{\alpha}{\delta} \left[\frac{1 - \operatorname{erf} \lambda \sqrt{\frac{\mu}{2}}}{1 - \operatorname{erf} \delta \lambda \sqrt{\frac{\mu}{2}}} \right] e^{-(\delta^2 - 1)\frac{\lambda^2}{2}\mu} \qquad \text{(A-25)}$$

which can be approximated by the empirical equation:

$$\alpha_\delta \approx \frac{\alpha}{\delta} \left[1 + \frac{\delta - 1}{2\sqrt{1 + \alpha}} \right] \qquad \text{(A-26)}$$

A comparison of exact values of λ determined from Equation A-15 and A-20 with estimated values using Figure 5 and Equation A-26 shows only negligible differences for all practical values of δ .

The following table indicates the relationship between α , δ and γ_δ as given by Equation A-26.

EQUIVALENT THERMAL RATIO (α_δ)

$$\text{Values of } \alpha_\delta = \frac{\alpha}{\delta} \left[1 + \frac{\delta - 1}{2\sqrt{1 + \alpha}} \right]$$

Values of	Values of δ						
α	0.9	1.0	1.1	1.2	1.3	1.4	1.5
0	0	0	0	0	0	0	0
1	1.07	1.00	0.94	0.89	0.85	0.82	0.79
2	2.16	2.00	1.87	1.76	1.67	1.59	1.53
3	3.25	3.00	2.80	2.62	2.48	2.36	2.25
4	4.35	4.00	3.72	3.47	3.28	3.11	2.97

[1] Source of data for k and C: "Final Report, Laboratory Research for the Determination of the Thermal Properties of Soils", Corps of Engineers, St. Paul District, June 1949.

Since it has been demonstrated that a 1 percent error in water content produces only about a 1.5 percent change in the value of δ, the above tabulation in conjunction with Figure 5 would indicate less than a 1 percent change in the computed depth of penetration due to this variation. Moreover, typical soils have values of δ of the order of 1.15, which would indicate an effective value of α_δ which is roughly 15 percent smaller than that computed assuming $\delta = 1.0$, yet this effect can produce a variation in λ and therefore a change in predicted depth of less than 5 percent for typical thermal conditions.

Since inclusion of a term involving δ would increase the complexity of representation of the formula in graphical form, and would further increase the tendency to predict depths of penetration which are too deep, it seems reasonable to base at least preliminary calculations on the assumption that $\delta = 1.0$.

Consistent, then, with the previous assumption that $C=C_u=C_f$ and $\delta = \sqrt{(\frac{k_f}{k_u})(\frac{C_u}{C_f})} = 1.0$, there would follow then the necessary condition that $k = k_f = k_u$. In practical application, this would suggest the use of average values of C and k.

(c) Fusion Parameter (μ): The fusion parameter μ, defined as:

$$\mu = \frac{v_s C_f}{L} \qquad \text{(A-27)}$$

measures the heat removed in the frozen soil (below the freezing point) compared to the latent heat of the soil moisture.

When $\mu = o$, the only significant soil properties affecting the depth of frost penetration are the latent heat L, and the thermal conductivity, k_f; on the other hand, as μ becomes large, the stored heat in the soil volume becomes proportionately , more significant.

(d) Correction Coefficient (λ): From Equation A-21, it can be seen that the correction coefficient λ is a correction to the calculated depth of frost penetration accounting for latent heat only.

The value of λ may be found from Figure 5 which shows the relationship between α, μ and λ for the case of $\delta = 1$. Data for this figure were obtained by assuming values of z and solving Equation A-15 for μ. Once μ was obtained λ followed directly from Equation A-20.