APPLICATION OF MATHEMATICS TO RIGHT-OF-WAY APPRAISAL

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Any consideration of real estate values must be preceded by a conclusion as to the future use the property, under considera- tion, may reasonably be devoted to. "Value is the present worth of the future benefits that accrue by reason of ownership." The benefits, accruing by reason of ownership, of course are directly associated with the use the property is put to. While the use a property may reasonably be put to is governed by a nearly endless number of conditions--its soil qualities, mineral content, climate, environment, etc.--the chief factor governing value and use, in urban and suburban property, is the population density of the area in which it lies. In considering the use to which a property may in the future be devoted, it is helpful to have as realistic a notion of the future growth pattern of the area as possible.

The most recent and thorough study of population growth is that of Professors Pearl and Reed of Johns Hopkins University. They give the general equation:

$$y = d + \frac{k}{Me\ a_1 x + a_2 x^2 + a_3 x^3}$$

where: y = population; x = time; k, M, and a constants, d the total attained in previous growth, and e the old pal that turns up in the oddest places; 2.718, etc., the basis of natural logs.

While this equation may be simplified for a single growth cycle to:

$$y = \frac{k}{1 + Me\ a_1 x},$$

one glance at it will make the average engineer or real estate man shake his head in dismay.

Getting away from the abracadabra of the mathematician and down to words we all understand, this equation probably is, by far, the best yet devised for predicting population increments. It is too academic for most of us to attempt to use.

The curve of the equation is a reflex curve and exactly symmetric around the inflection point. The first part of the curve must be smooth and regular and fit or nearly fit the statistical curve of the past population data. If the inflection point can be located on the adjusted statistical curve, then all we have to do is to fold the paper at the inflection point along both the x and y axis, and the adjusted statistical portion of the curve--the lower left--becomes the prediction part; the upper right and the curve is completed. Locating the inflection point is a major project in pure mathematics. Again looking for the easy way--the table of difference of a symmetric reflex curve, when plotted, is a symmetric type J or mound curve with the inflection point of the reflex at the apex of the J. The apex of a type J curve is obvious on inspection when the data considered include more than 1/2 of the complete curve. It is generally agreed that most, if not all, American towns have more than reached the midpoint of their growth.

Figure 1 is an illustration of the application of the method.

The calculation of the curve, by the Pearl method, requires considerable time and knowledge of mathematics. The graphic method requires about 20 minutes, a french curve and a piece of tracing paper. No claim can be made of equal accuracy, but population increment in detail usually is whimsical. What we require is a reasonable approximation of the pattern, not the exact number of people living in the area 20 years hence.

The practical application of the information, derived from the growth prediction, is more or less a matter of common sense

Census Data		Adjusted	Difference
1870	3,788	4,000	1,500
1880	5,347	5,500	2,000
1890	7,382	7,500	2,500
1900	9,643	10,000	4,000
1910	13,502	14,000	7,000
1920	20,620	21,000	7,400
1930	28,451	28,400	4,100
1940	30,167	32,500	

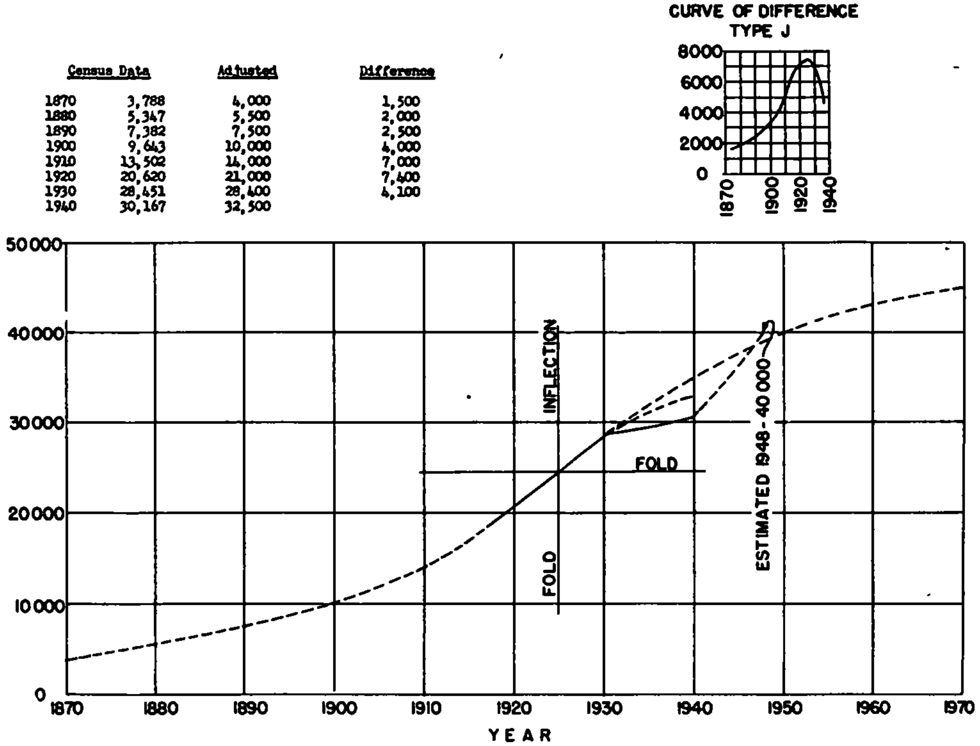

Figure 1.

and experience. Having the size of a town at any particular time and the normal growth expectancy, it is relatively certain what the demand for new residential structures will be. Similarly, the number of stores, garages, gas stations, etc., the area will support can be estimated. With the growth rate in mind, a correlation of the existing home and business structures with the present population usually suggests the future land-use pattern of the undeveloped portions of a community, and the transitions in use of the downtown properties.

Having estimated the present and future highest and best use of a property, we come to the problem of expressing it in dollars. Here social and economic forces of an extremely complex nature manifest themselves. The problem is to estimate how many dollars an informed and willing, but not compelled, customer would probably part with in order to obtain the property under consideration. Usually, what people will do is best guessed by referring to what they have done under the circumstances being considered. In other words, the comparative method.

For the valuation engineer interested in reasonably exact results, this means work. The first thing is to gather the comparative sales data. Figure 2 shows the form we use to record the sales information. We usually visit the land record office, and take from the records the transfers of real properties similar to those under consideration that have taken place within a reasonable time. The number of sales taken is largely controlled by the type of property and the size of the town. It is desirable to get a comprehensive sample. Often, because of lack of comparableness or infrequency of sales, something less than a comprehensive sample is all that is available.

A field study of the sales data is then made. We check the consideration paid, obtain a record of apparent age, condition, size and construction details of the buildings, and the quantity and quality of the land involved in the sale. This is probably a standard procedure throughout the coun-

FIGURE 2 - COMPARATIVE SALES INFORMATION

Date:

By:

Town:

Sales No.

Vol. Page

Location of Property

Grantor Witnesses

Address

Grantee

Address

Date of Sale Recorded

Price $ Terms Cash$ Mortgage$

Stated by Date I. R. Stamps $

ASSESSMENT

List of 19__

Buildings:

House — Rooms____ Bath____ Floors____________ Heat______________

Basement Floor_____________ Foundation________________

Ceilings__________________ Walls__________________

Water Supply_____________ General Condition_____________

Barns__________ Garage_____________ Other Buildings______________

Land:________ Acres Classified as follows:

Grade A.______Acres $______ Grade B.______Acres $______

Grade C.______Acres $______ Pasture______Acres $______

Sproutland____Acres $______Swampland______Acres $______

Woodland______Acres $______ Other________Acres $______

Improvements and Changes Since Purchase:

Utilities: Pavement______ Water______ Sewer______ Gas______

Electricity______Sidewalk______Public Transportation_______

Replacement Cost of Structures:

Estimated Break-down of Prices:

Surroundings:

Special Features:

(See Back of Sheet for Additional Data)

Figure 2.

try. Unfortunately, all the sales disclose is the gross price for the property. The reduction of the sales data to usable form offers some difficulty. What we want is units of value to apply to the property under appraisement; land by the front foot or acre; buildings by the cubic or square foot. Now, if the various lands and buildings can be expressed in terms common to all, or approximately so, there are possibilities for the mathematical breakdown into unit price.

A first-year high school boy would have little difficulty with the solution for the price of each bird of, "A man bought 3 ducks and 4 geese for $15, and later 3 ducks and 2 geese for $9." Except for the fact that the prices actually paid for property usually are not consistent, the same rule the boy used on the ducks and geese would apply to the breaking down of the price for land and building. They are simple simultaneous equations. The rub comes because the price paid for real estate is not usually consistent. The treatment of inconsistent equations should be no novelty to an engineer although its use in land appraisal is not common.

If we write so many units of land--plus so many units of buildings--minus the price paid--plus or minus the variance from the normal price equals zero, we have the problem in a form we should recognize. Using a to represent the number of units of land; b, the number of units of buildings; p, the price paid; x, the unit price of land; y, the unit price of buildings; and v, the departure of this particular sale from the normal for the entire data. This is the equation:

$$ax + by - p = v_1$$

Going back to your school days. Does not this strike a chord of recollection? Did you never have to do a problem in least square adjustments?

By the Legendre principle--when the sum of the squares of the variances are a minimum, we have the most probable value for the unknown quantities. In a short article of this kind, it is not appropriate to go into the complete detail of the solution of inconsistent equations. For the people whose recollection needs only refreshing, the form of the equation should be sufficient. For a complete text, I suggest "Wright and Hayford's Adjustment of Observations." The book is available at, or can be ordered through, any stationery store.

To the readers who have been so long away from pure mathematics as to have completely lost the recollection of the solution of inconsistent equations, it is possible to get a sort of synthetic answer by the cut and try method. Write up your data in the form given, then guess as well as you can the price per unit of the land, and express your buildings as a percentage of the cost obtained by the method outlined. Compute the value, using these trial figures, and see how they compare with actual prices paid. The difference in your price as calculated and the price paid in each sale sample is the v for that sale. If you have guessed right, or nearly right, your v's will be small and about evenly balanced as to over or under the actual price.

If, as you probably will, you develop v's that are too large, or too small, change your assumed values and try it again. By a little juggling, a set of unit values can be found that will be reasonably accurate. When a man can draw upon many years of real estate experience, it often is the case that very close approximations of values can be made without the bother of making the involved calculation required in the least square method.

The work of solving a dozen or more sales by this method is tedious, but the results more than justify the effort. Prices are shown in their true light--probable values only. The departure from the normal price of every sale is found. The use of variants will be touched upon later.

The important thing to remember is to express both land and buildings in terms of common value in respect to each other.

As to land--particularly building lots--there is little difficulty. Use equivalent front feet in residential and business property. Adjust the actual dimensions with any of the standard depth tables and corner influence factors. For industrial property, I use the square foot unit; but try to select samples that are reasonably comparable in size and location.

Getting buildings expressed in common terms of value is more difficult. It is generally recognized that, as the volume of a building goes up, its unit price goes down when the quality remains constant. From an extended study of building costs it ap-

Figure 3 - Appraisal Weight Sheet

Date:
Inspector

Owner ____________ Town. ________ Project: __________

ITEM	DESCRIPTION	WEIGHT
Excavation & Sewer Back fill - grading		
Piers	2.0	
Ring Wall	2.5	
Full Cellar, 2 floors or more	4.0	
Full Cellar, 1 floor	5.0	
Foundation		
Piers	1.0	
Ring Wall	2.5	
8" Wall	4.5	
10" Wall	5.5	
12" Wall	6.5	
Sheathing	2.0	
Side Wall Surface		
Novelty Siding	1.5	
Wood or Asbestos shingle	2.0	
Clapboard	2.5	
Stucco or Brick Veneer	3.5	
Stone Veneer	4.0	
Exterior Trim		
Simple	1.0	
Average	2.0	
Expensive	3.0	
Exterior Paint		
Poor	0.5	
Average	1.0	
Best	1.5	
Roof including boarding		
Rolled Paper	2.0	
Asphalt or wood shingle	3.0	
Asbestos shingle	4.0	
Slate or tile	4.5	
Sheet Metal		
Galvanized gutters, etc.	0.5	
Copper gutters, etc.	1.5	
Garage doors, fire-proofed	1.0	
Chimney		
Exterior	0.5	
Interior	1.0	
Exterior with fireplace	3.0	
Interior with fireplace	4.0	
Water Service		
City	1.0	
Country	4.0	
Concrete Floor		
1 Story	2.5	
2 Story or more	1.5	
1 Story with garage	3.0	
2 Story with garage	2.0	
Steel Work		
Lallys	0.5	
Steel Sash	0.5	
Framing & Bridging		
Sub-standard	6.0	
Standard	9.0	

Figure 3 - Continued

ITEM	DESCRIPTION	WEIGHT
Sub-floor	1.5	
Finish Floor Including Porch Floor & Ceiling		
Poor	2.0	
Good	3.0	
Excellent	4.0	
Asphalt Tile or Linoleum (Battleship)		
Kitchen	0.5	
Interior Walls		
Sheet rock or wall board	4.0	
Plaster except in basement	6.5	
Plaster including basement ceiling	8.0	
Tile Work		
Bath	2.0	
Bath & Lavatory	2.5	
Interior Trim		
Average	1.0	
Best	1.5	
(with wood paneling)	2.0	
Interior Doors		
Average	2.0	
Best	2.5	
Closets, Cabinets, etc.		
Closets only	0.5	
With kitchen cabinets	2.5	
with book cases, china closet, cedar closet, mantel etc.	4.0	
Stairs		
Fair	0.5	
Enclosed	2.0	
Open	2.5	
Interior Decoration		
Poor	2.0	
Average	4.0	
Best	6.0	
Windows		
Old type (thumb latch)	4.0	
Sash and weight	5.0	
Silentite type	7.0	
Hardware		
Average	1.0	
Best	2.0	
Insulation	2.0	
Electric Installation		
Poor	2.0	
Average	3.0	
Best	4.5	
Plumbing		
Poor	1.0	
Average	4.5	
Best (Bath & Lavatory)	6.5	
Heating		
Pipeless	1.0	
Hot Air	3.0	
Steam	4.5	
Domestic Air Conditioning	7.5	
Hot water, average	6.0	
Hot Water, best	7.5	
Extra, automatic heat	1.5	

Total Weight ______

Figure 3.

TABLE 1
REPRODUCTION COST TABLE
prepared by
CONNECTICUT STATE HIGHWAY DEPARTMENT
MAY 1, 1948

Cubic Feet	Price per Cubic Foot		
	Modest 60	Average 80	Excellent 100
1,000	68.9	82.3	94.3
2,000	64.7	79.8	91.7
3,000	61.2	77.4	89.5
4,000	58.2	75.5	87.6
5,000	55.9	73.7	85.8
6,000	53.6	72.1	84.0
7,000	51.7	70.5	82.5
8,000	50.2	69.2	81.2
9,000	48.7	67.9	79.9
10,000	47.4	66.7	78.5
11,000	46.3	65.6	77.4
12,000	45.3	64.6	76.5
13,000	44.3	63.6	75.4
14,000	43.5	62.8	74.5
15,000	42.7	61.9	73.7
16,000	42.0	61.3	72.8
17,000	41.4	60.5	72.1
18,000	40.7	59.8	71.3
19,000	40.2	59.2	70.6
20,000	39.6	58.5	70.0
21,000	39.1	58.0	69.3
22,000	38.5	57.4	68.8
23,000	38.2	56.9	68.2
24,000	37.7	56.3	67.8
25,000	37.4	56.0	67.2
26,000	37.1	55.6	66.7
27,000	36.7	55.0	66.2
28,000	36.4	54.7	65.8
29,000	36.1	54.2	65.3
30,000	35.8	54.0	64.9
31,000	35.5	53.6	64.6
32,000	35.2	53.2	64.1
33,000	35.0	52.9	63.9
34,000	34.8	52.6	63.5
35,000	34.5	52.3	63.1
36,000	34.3	52.0	62.8
37,000	34.2	51.7	62.6
38,000	33.9	51.5	62.2
39,000	33.8	51.3	61.9
40,000	33.6	50.9	61.6

TABLE 1--Concluded
REPRODUCTION COST TABLE
prepared by
CONNECTICUT STATE HIGHWAY DEPARTMENT
MAY 1, 1948

Cubic Feet	Price per Cubic Foot		
	Modest 60	Average 80	Excellent 100
41,000	33.4	50.7	61.4
42,000	33.2	50.6	61.2
43,000	33.1	50.3	60.8
44,000	32.9	50.1	60.7
45,000	32.8	49.9	60.5
46,000	32.6	49.6	60.2
47,000	32.5	49.5	60.0
48,000	32.3	49.3	59.8
49,000	32.2	49.1	59.5
50,000	32.1	48.8	59.4
51,000	31.9	48.7	59.2
52,000	31.9	48.6	59.0
53,000	31.8	48.4	58.7
54,000	31.7	48.2	58.6
55,000	31.6	48.1	58.5
56,000	31.5	48.0	58.2
57,000	31.5	47.7	58.1
58,000	31.2	47.6	58.0
59,000	31.1	47.5	57.9
60,000	31.0	47.4	57.6
61,000	31.0	47.3	57.5
62,000	30.9	47.1	57.3
63,000	30.9	47.0	57.2
64,000	30.8	46.9	57.0
65,000	30.6	46.8	56.9
66,000	30.6	46.6	56.8
67,000	30.5	46.6	56.7
68,000	30.5	46.4	56.5
69,000	30.4	46.3	56.3
70,000	30.4	46.2	56.3
71,000	30.3	46.1	56.2
72,000	30.3	46.1	56.1
73,000	30.1	46.0	56.0
74,000	30.1	45.8	55.9
75,000	29.9	45.7	55.7
76,000	29.9	45.7	55.7
77,000	29.8	45.5	55.6
78,000	29.8	45.4	55.4
79,000	29.8	45.4	55.3
80,000	29.7	45.3	55.3

pears that this relationship is on a hyperbolic curve. Equally, it is true that cost and quality vary directly in respect to each other.

If we can devise a scheme to express the quality of a building numerically and consistently, even though the number plan is arbitrary, we are on safe ground.

Figure 3 is a plan for expressing the quality of a residence on a numerical scale. Various items of the building are listed, and a weight assigned to each. The sum of the weight numbers is the total quality weight of the building. Not all items are covered, but there are enough to sample fairly the entire building.

Table 1 is a calculated table of relationship for size. This gives the cubic foot cost in Hartford, Connecticut, about May 1, 1948 for buildings with quality weights of 60, 80 and 100. To use the plan, simply interpolate. Do not be confused about adopting, as value, these cubic foot costs. They are simply used to express buildings in common terms. Calculate the building cost on this plan, and use it for your building's symbol coefficient. When you have solved the equation, the building symbol will work out to have a value of one plus or minus a fraction. The plus or minus fraction represents the depreciation or appreciation above or below this cost figure in the actual normal market your sample sales represent.

In respect to the variances that are developed for each sale, they arouse some interesting speculations. On a number of cases worked out, they suggest that "The magnitude of a variance is inversely proportional to the square of the frequency of its occurrence." If this were provable, it would offer a possibility of expanding a small sample infinitely. Having in mind the results of the presidential poll samplers in this year's election, we had better forego comment along this line.

More seriously--it seems to be a part of the American way of life to give the individual the benefit of every reasonable doubt when he is dealing with the public. Some of the text writers have gone so far as to say in connection with public acquisition, "Fair Market Price is the highest price a person could have obtained for his property in reasonable probability had he been willing to sell it."

As far as I know, a thing is reasonably probable when the chances of its occurring are the same as the chance of its not occurring.

Assuming that the sales sample is representative, we must admit that there are the same probabilities of the next sale being below the average as above. There is much less probability of the sale falling exactly on the line than there is that it will fall above or below the normal line.

If, however, we divide our variances into quarters two above the median line and two below, we can say, truthfully, that there is the same probability of the next sale falling within the two quarters next to the median line as there is of falling in the outer quarters. Anything within the two center quarters is reasonably probable.

Now, if we want the highest of reasonable probability, take the top of the quarter above the line. Conversely, when it is necessary to give the individual the benefit of reasonable doubts the other way, say for tax assessment purposes, use the bottom of the lower center quarter. Both are reasonably probable. Both seem to satisfy our general plan of doing public business.

Actually, when we use the higher figure, we say 3/4 of the buyers of our sample got bargains--only 1/4 overpaid. When we use the lower figure, 3/4 overpaid and 1/4 got a bargain. In a fairly realistic market, the variances tend to be small. The top of the upper central quarter will be only 20 to 25 percent above the bottom of the lower central quarter.