Computer Solution of Swedish Slip Circle Analysis for Embankment Foundation Stability

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Personnel of the Materials Laboratory and of the Computer Section of the Washington Department of Highways have developed a program for the IBM 650 magnetic drum data processing machine which will analyze a given foundation problem in a matter of minutes. At present the program will handle analyses of embankment stability where the foundation is composed of as many as 3 layers or strata of different material. Although the present program is restricted to homogeneous embankments, suitable modifications should enable it to handle any number of embankment or foundation materials if they are placed or occur in a known geometric pattern in the cross-section.

The following data are necessary for the machine analysis: cohesion, angle of internal friction, and unit weight of the soils involved; initial slope to be analyzed; thickness of the foundation soil strata; height of embankment; and design safety factor.

The program may be used in two ways: (1) To investigate a given range of slopes, automatically advancing to the next flatter slope if the safety factor against failure is found to be less than the predetermined value; and (2) To investigate a range of slopes, in individual analyses for each slope.

● THE SWEDISH SLIP CIRCLE method of analyzing slope and embankment foundation stability is based upon studies which have indicated that failures in slopes usually occur along a cylindrical surface within the soil rather than along a plane surface, and that rotation takes place about the center of the cylinder. Early work on this was done by Petterson (1) and Fellenius (2) and later, D. R. May (3) who proposed the use of a planimeter in connection with a graphical solution of the problem. Tables and charts by Fellenius, Taylor (4), and others were also developed to aid in solution of the problem, using the ϕ circle modification.

In the past, the most common method of analysis made use of the graphical solution of the problem. This approach, though simplified to a great extent, is a time-consuming trial and error procedure which often lacks thorough analysis. The complex mathematical capabilities of the electronic computers appeared to offer a method for further simplifying the analysis.

At the suggestion of the Materials Laboratory, the Computer Section of the Washington Department of Highways initiated the development of a computer program for the complete analysis of embankment foundation stability. It was intended that this program would analyze all potential failure circles, or arcs, so that confidence in the thoroughness of investigation could be realized. The potential saving in engineering time was obvious from the start, and this point alone would justify the development of the program.

Although time has not permitted as rigorous a check as might be desired, the completed program is thought to be thorough in its coverage. Computation of a single safety factor requires only 3 to 5 seconds. Complete analyses, involving many trials, may take anywhere from 1 to 5 minutes. However, as might be expected, any computer program which attempts to be all-inclusive, such as this one, has certain limitations, most of which appear to be potentially insignificant at this time. The limitations are covered in detail in a subsequent section.

METHOD OF ANALYSIS

It is not the intent of this paper to describe the theory of the Swedish Slip Circle analysis, but to discuss its implementation through use of a medium size electronic computer—in this particular case, the IBM 650.

The purpose of the program is to solve for the minimum safety factor for a designated slope, the safety factor being the ratio of the withholding force divided by the tendency to "slip." For equilibrium, or a safety factor of 1.0, the sum of the moments about point "0", the center of the revolving cylinder, must be equal to zero.

The determination of the minimum safety factor necessitates the investigation of a series of circles until the circle is found with the conditions that yield the minimum safety factor. This phase is ideally adapted to an electronic computer because of the decisions that can be built in to a computer program. Programming techniques have also made it possible to generate selected dimensions, by increments, which locate and describe the circles investigated.

There are two general ways in which the program may be used.

1. By using certain code numbers, in this case a code 8, the computer will determine the minimum safety factor beginning with the steepest slope designated for analysis. Computation of a safety factor less than 1.0 will cause the program to branch and investigate flatter slopes in succession until a minimum safety factor greater than 1.0 is found and punched out. Slopes of 1, 8:1, 2:1, 3:1, 4:1, 6:1, and 10:1 can be investigated in the present program form. Others may be included, or substituted, if desired.
2. If the code number 9 is used, the minimum safety factor, regardless of value, is punched out for each designated slope. Any slope flatter than 1:1 may be investigated in this case.

A typical cross-section of an embankment resting on questionable foundation soils is shown in Figure 1. The terminology of various dimensions used in the development of equations for the computer solutions is shown thereon. The coordinates of the center of the circle are represented by the distances A and B which are, respectively, the lateral distance from the crest of the slope and the vertical distance from the top of the embankment. All equations are in terms of A, B, R (the radius), the slope, and the values which describe the thickness and properties of embankment and foundation soil layers. The complete mathematical equations involved are shown in Appendix A, together with explanatory notes and sketches showing their derivation.

The program has been set up to handle 4 soil layers, one of which is the embankment. Input data and their significant figures consist of the following types of information:

W = Density, XXX lb/ft^3
C = Cohesion, XXX lb/ft^2
tan ϕ = Coef. of friction, .XXX
H = Height of fill or soil layers, XX.X ft

The following data are required for the problem:

1. Problem number;
2. Identification number to indicate method of analysis desired (automatic slope change to a safety factor greater than 1, or safety factor for beginning slope), Code 8 or 9;
3. Beginning slope;
4. Fill height (H_1);
5. Unit weight, cohesion and tangent of angle of internal friction of fill material (W_1, C_1 tan ϕ_1);

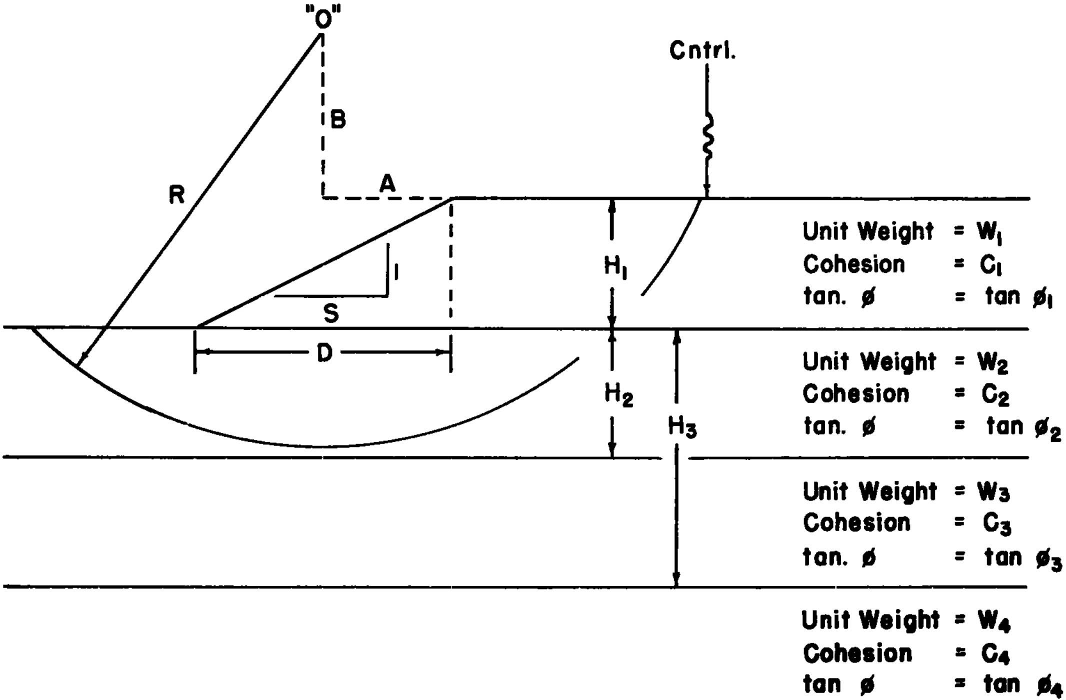

Figure 1. Terminology used in computer solution of Swedish Slip Circle.

6. Depth, unit weight, cohesion and tangent of angle of internal friction of first foundation soil layer (H_2, W_2, C_2 $\tan\phi_2$);
7. Depth from ground surface to bottom of second foundation soil layer (H_3); unit weight, cohesion and tangent of angle of internal friction of second foundation soil layer (W_3, C_3, $\tan\phi_3$);
8. Unit weight, cohesion, and tangent of angle of internal friction of third foundation soil layer (W_4, C_4, $\tan\phi_4$); and
9. Width of embankment to which analysis should be confined (CNTRL)

A sample input data form is shown in Appendix B, in conjunction with the sample problem. It will be noted that as many as 3 foundation soil layers may be considered. If the particular problem should not encompass that many foundation strata, only the data for the layers involved need be entered, subject to the condition that the last layer described automatically will be considered infinite in depth. For example, if the foundation profile for any problem should consist of 30 ft of organic clay underlain by relatively firm sand, which persists in depth beyond a point of concern, data for both the clay and the sand should be entered, even if those for the latter have to be assumed.

The flow chart for the IBM 650 program is shown in Figures 2 through 5. The first step of the calculation involves the computation of the initial values of A, B, and R which define the first trial circle. The following empirical equations are used to obtain these values:

$$A = \tfrac{1}{2}(H_1 + D) \qquad (1)$$

$$B = \frac{H_1 s^2 + 2D - 3H_1}{8} \qquad (2)$$

$$R = \sqrt{(H_1 + B)^2 + (D - A)^2} + 1 \qquad (3)$$

in which

s = slope
H_1 = ht of embankment

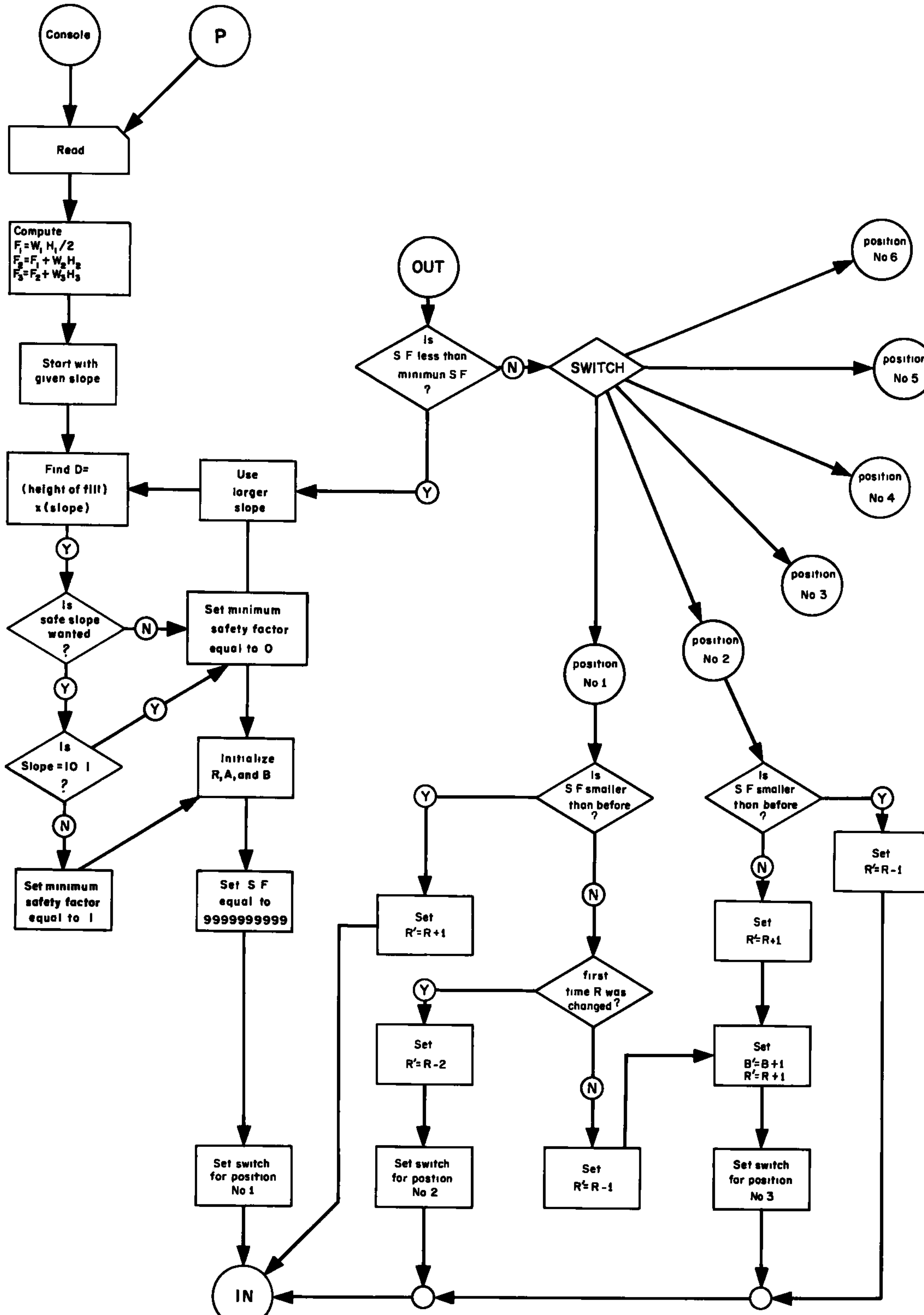

Figure 2. Flow chart (1), Swedish Slip Circle analysis on IBM 650.

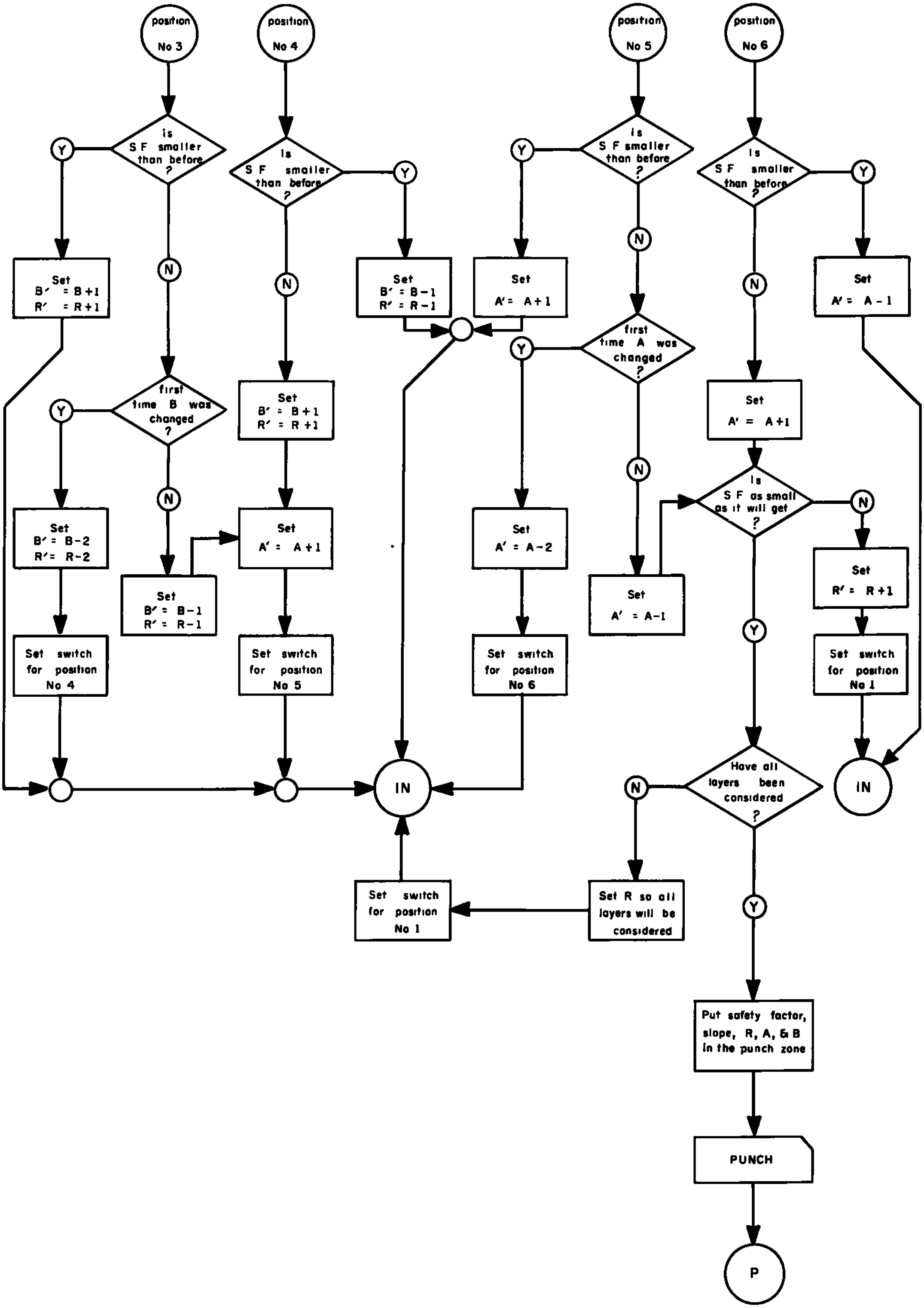

Figure 3. Flow chart (2), Swedish Slip Circle analysis on IBM 650.

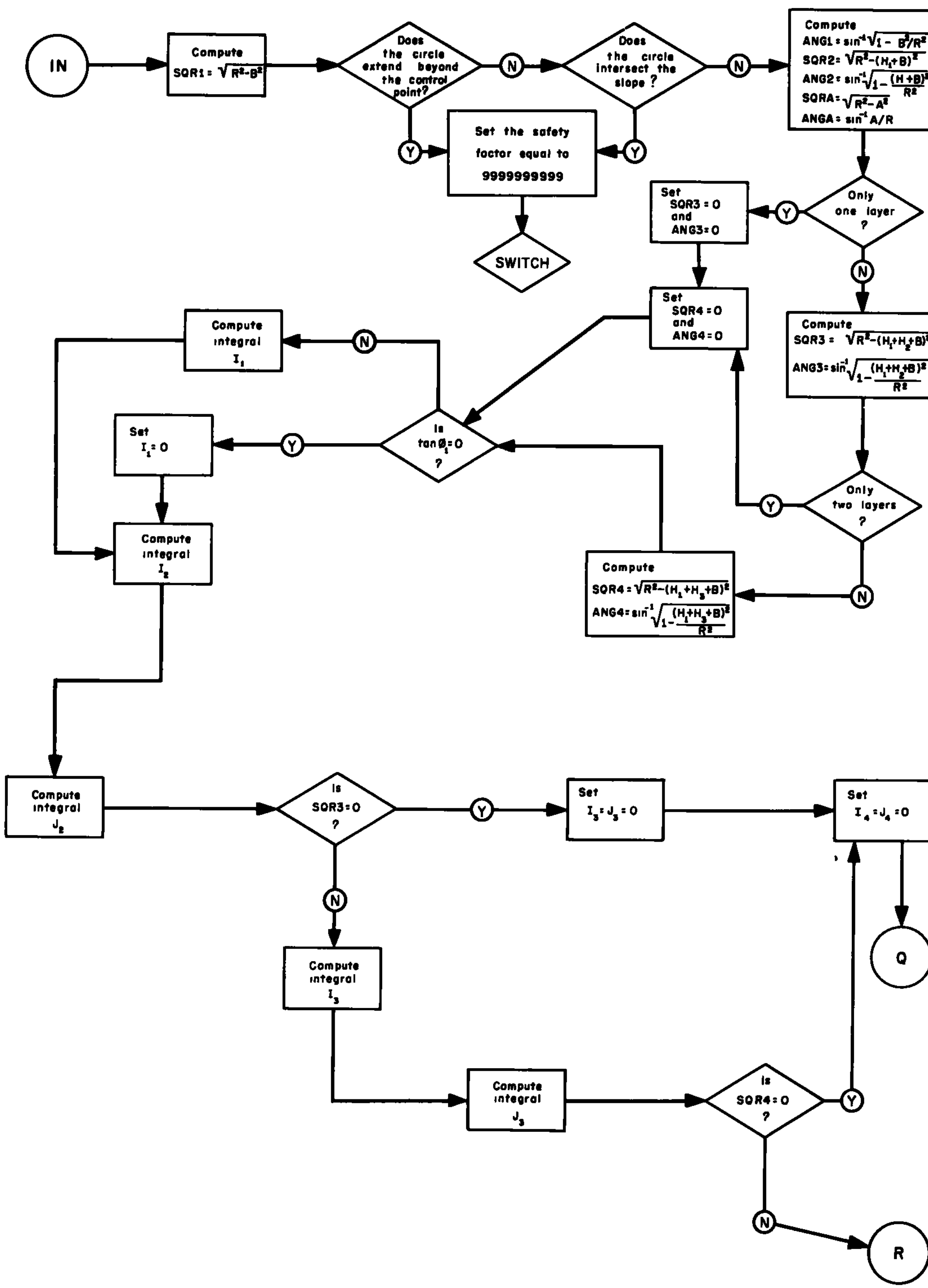

Figure 4. Flow chart (3), Swedish Slip Circle analysis on IBM 650.

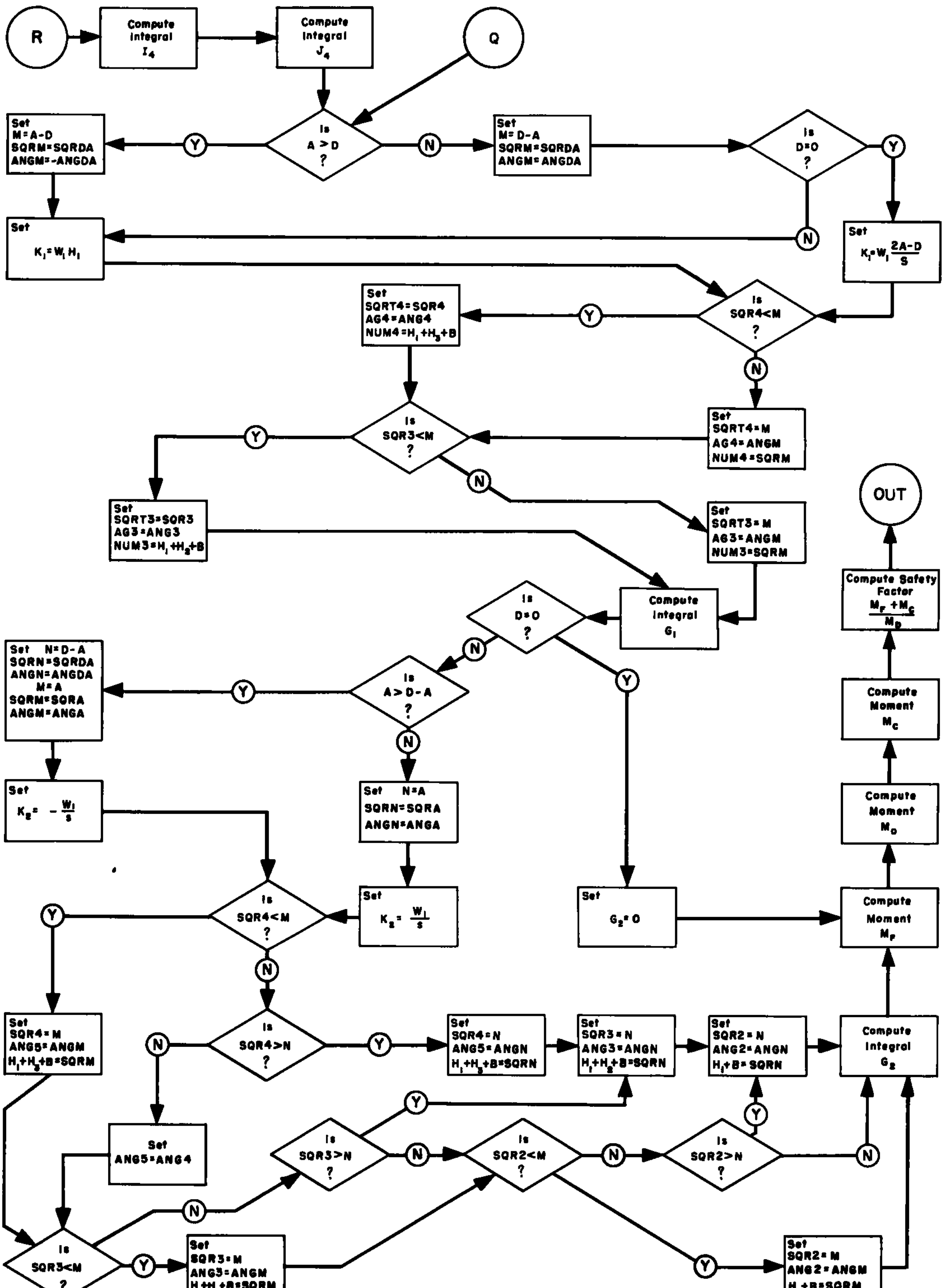

Figure 5. Flow chart (4), Swedish Slip Circle analysis on IBM 650.

These empirical equations give results such that the circle defined thereby will not intersect the slope, will not pass beyond the control point (cntrl), and will penetrate a minimum depth into the first foundation soil layer.

Using the initial values of R, A, and B together with the pertinent input data, a safety factor for the first circle is calculated. The value of R is then increased and another safety factor computed. The second safety factor is compared to the first, and if less, R is increased again until a minimum safety factor is reached. If the second safety factor is greater than the first, the value of R is decreased until a minimum safety factor is reached. After R is varied until a minimum safety factor is reached, A is varied, then B and R are varied together in the same manner. With the resultant values of R, A, and B, the entire procedure is repeated until the safety factor found equals the previous safety factor, thus establishing a minimum. This minimum safety factor, together with the related values of R, A, and B, is stored in a special comparison area.

The program then checks to see if all foundation layers have been investigated. Since the starting conditions are such that the upper foundation layer is investigated initially, this check operation consists mainly of resetting the initial values of R, A, B so that the circle passes into the second and/or third layer. The entire procedure is then repeated with these initial conditions, and the minimum safety factor resulting from all the above calculations is punched out of the special comparison storage area along with the radius and center coordinates of the circle providing this safety factor.

The above sequence of operations is followed in the calculation of the minimum safety factor, regardless of value, for any one designated slope (code 9). In the automatic operation (code 8) the same procedure is followed except that calculation of a safety factor less than 1.0 will cause the program to branch and investigate the next flatter slope. This will continue until a minimum safety factor greater than 1.0 is found or until a slope of 10:1 is investigated, whichever occurs first.

Output data consist of the problem number, the code number specifying the type of analysis, the initial slope investigated, the slope for which the minimum safety factor is calculated, the value of this safety factor, and the values of R, A, and B which locate and describe the circle that provides this safety factor. The form used in tabulating these data is shown in connection with the sample problem, Appendix B.

LIMITATIONS

It has been necessary to include certain limitations to the scope of the analysis. First, the stratification of foundation soil layers must be essentially horizontal, or capable of being represented so, for correct operation of the program.

A second limitation is that the circle should not intersect the slope. This does not appear to be a serious limitation inasmuch as the weakest circle, or the circle with the lowest safety factor rarely passes through the slope on a simple embankment cross-section.

Another control point is the width of embankment to which the analysis should be confined. Many limit the investigation to half the top width of the embankment. Since this distance is a variable, provision is made for this point to be included in the input data.

At present the program is compatible with homogeneous fill material only. It is not, as now constituted, capable of analyzing a heterogeneous embankment such as an earth fill dam, nor will it handle the analysis of an embankment incorporating berms in the cross-section. A modification to accomodate this latter feature is currently underway.

Some difficulty is being experienced in efforts to apply the program to analyses of embankments placed over a relatively thin layer of soft foundation soil, particularly where this layer is underlain by a fairly firm material. Although the extent of this limitation has not been pinpointed, it does appear that the program will be satisfactory for all cases where the ratio of the effective embankment width to the thickness of the soft foundation layer is less than 4.0. When this ratio exceeds 4.0, it is probable that the failure surface will be other than cylindrical. Such conditions should be analyzed by other methods. Answers given by this program analysis are subject to possible error in these cases, and should be checked in light of the above statements.

Another limitation to the present program is that the water table cannot be above ground surface. Modifications to handle this condition are also underway.

APPLICABILITY

In its present form the program is quite versatile. With very minor modifications, it can be used to calculate a safety factor from a given set of coordinates defining any desired circle—and can do so in less than 5 seconds. With the machine on automatic operation, slight changes in the program can be made so that each safety factor calculated, together with data on the slope and coordinates defining the circle, will be punched out. This gives a check on the thoroughness of scanning.

This program should be a valuable aid to those who desire to construct tables or charts for use in design or control of embankments. It will eliminate the long and tedious calculations usually associated with this procedure.

The program provides a means of evaluating situations involving 3 different foundation soil layers, which are often encountered in the work. Water table elevation is considered as a foundation soil boundary since the effective weight of material below this point is equal to the submerged weight. This, then, represents a change in soil properties necessitating the consideration of this portion of the foundation soil as a different material in the analysis.

CONCLUSION

It is felt that the program for use of the IBM 650 computer in analyzing embankment foundation stability by the Swedish Slip Circle method will be of definite value to those concerned with the design of embankment foundations. This program is available for use and is offered to those who might desire it. It is hoped that it will serve as a stepping stone to even more versatile programs for this analysis—programs brought about by the modifications others will discover and incorporate to meet their particular needs.

ACKNOWLEDGMENT

The major part of the program development was accomplished by Jon Petersen, presently an undergraduate at Stanford University, while employed with the Washington Department of Highways during the summer of 1957. The credit for the program is rightfully his. Recognition should also be given to personnel of the Computer Section, namely Paul Yeager, Charlene Travis, L. C. Reynolds and Eugene Lee for a great amount of assistance in the incorporation of minor modifications and to M. P. O'Neill and Harold Dunn of the Planning Division Drafting Section for the preparation of flow charts and drawings. Acknowledgment of the beneficial suggestions of Carl E. Minor and Lloyd Morgan of the Materials Laboratory relative to the development of the program and the prepartion of the paper is gratefully made. And last, but far from least, sincere appreciation of the able assistance and advice in checking the program and suggesting modifications is given to Henry Sandahl of the Materials Laboratory.

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Appendix A

<u>Equations</u>

$$I_1 = 6 \tan\phi_1 W_1 \int_{\sqrt{R^2 - (H_1 + B)^2}}^{\sqrt{R^2 - B^2}} \left(R^2 - X^2 - B\sqrt{R^2 - X^2}\right) dx$$

$$I_2 = 12 \tan\phi_2 W_2 \int_{\sqrt{R^2 - (H_1 + H_2 + B)^2}}^{\sqrt{R^2 - (H_1 + B)^2}} \left(R^2 - X^2 - [H_1 + B]\sqrt{R^2 - X^2}\right) dx$$

$$I_3 = 12 \tan\phi_3 W_3 \int_{\sqrt{R^2 - (H_1 + H_3 + B)^2}}^{\sqrt{R^2 - (H_1 + H_2 + B)^2}} \left(R^2 - X^2 - [H_1 + H_2 + B]\sqrt{R^2 - X^2}\right) dx$$

$$I_4 = 12 \tan\phi_4 W_4 \int_{0}^{\sqrt{R^2 - (H_1 + H_3 + B)^2}} \left(R^2 - X^2 - [H_1 + H_3 + B]\sqrt{R^2 - X^2}\right) dx$$

$$J_2 = 12 \tan\phi_2 F_1 \int_{\sqrt{R^2 - (H_1 + H_2 + B)^2}}^{\sqrt{R^2 - (H_1 + B)^2}} \sqrt{R^2 - X^2}\, dx$$

$$J_3 = 12 \tan\phi_3 F_2 \int_{\sqrt{R^2 - (H_1 + H_3 + B)^2}}^{\sqrt{R^2 - (H_1 + H_2 + B)^2}} \sqrt{R^2 - X^2}\, dx$$

$$J_4 = 12 \tan\phi_4 F_3 \int_{0}^{\sqrt{R^2 - (H_1 + H_3 + B)^2}} \sqrt{R^2 - X^2}\, dx$$

$$G_1 = 6 K_1 \Bigg[\tan\phi_4 \int_{0}^{\sqrt{R^2 - (H_1 + H_3 + B)^2}} \sqrt{R^2 - X\, dx} + \tan\phi_3 \int_{\sqrt{R^2 - (H_1 + H_3 + B)^2}}^{\sqrt{R^2 - (H_1 + H_2 + B)^2}} \sqrt{R^2 - X^2}\, dx + \tan\phi_2 \int_{\sqrt{R^2 - (H_1 + H_2 + B)^2}}^{M} \sqrt{R^2 - X^2}\, dx \Bigg]^{a}$$

$$G_2 = 6 K_2 \Bigg[\tan\phi_4 \int_{M}^{\sqrt{R^2 - (H_1 + H_3 + B)^2}} (A - X)\sqrt{R^2 - X^2}\, dx + \tan\phi_3 \int_{\sqrt{R^2 - (H_1 + H_3 + B)^2}}^{\sqrt{R^2 - (H_1 + H_2 + B)^2}} (A - X)\sqrt{R^2 - X^2}\, dx$$

S = slope

$$F_1 = \frac{W_1 H_1}{2}$$

$$F_2 = W_2 H_2 + F_1$$

$$F_3 = W_3(H_3 - H_2) + F_2$$

$$K_1 = W_1 H_1 \quad (A > D)$$

$$K_1 = W_1 \frac{2A - D}{S} \quad (A \leq D)$$

$$K_2 = \frac{W_1}{S} \quad (A \geq D - A)$$

$$K_2 = -\frac{W_1}{S} \quad (D - A > A)$$

If $D - A > A$, $N = D - A$, $M = A$

If $A > D - A$, $N = A$, $M = D - A$

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[a] The limits of integration are as given if these values are $\leq M$. If a limit is $> M$, then it is set equal to M.

$$+\tan\phi_2 \int_{\sqrt{R^2-(H_1+H_2+B)^2}}^{\sqrt{R^2-(H_1+B)^2}} (A-X)\sqrt{R^2-X^2}\,dx \qquad + \tan\phi_1 \int_{\sqrt{R^2-(H_1+B)^2}}^{N} (A-X)\sqrt{R^2-X^2}\,dx \Bigg]^{b}$$

$$M_D = W_1H_1 \left[3\,R^2 + 3\,A(D-A) - 3\,B(H_1+B) - D^2 - H_1^2 \right]$$

$$M_C = 6R^2 \left[C_1 \ \mathrm{Sin}^{-1} \frac{\sqrt{R^2-B^2}}{R} + (2C_2 - C_1)\ \mathrm{Sin}^{-1} \frac{\sqrt{R^2-(H_1+B)^2}}{R} \right.$$

$$\left. + \, 2(C_3 - C_2)\ \mathrm{Sin}^{-1} \frac{\sqrt{R^2-(H_1+H_2+B)^2}}{R} + 2(C_4 - C_3)\ \mathrm{Sin}^{-1} \frac{\sqrt{R^2-(H_1+H_2+B)^2}}{R} \right]$$

$$M_F = I_1 + I_2 + I_3 + I_4 + J_2 + J_3 + J_4 - G_1 - G_2$$

DISCUSSION OF EQUATIONS

In the equations modified for computer use, all the forces are multiplied by 6R to eliminate fractions and a number of divisions by R. This multiplication gives results which are six times the moments due to these forces. The intergral giving M_d, which is six times the driving moment, can be easily evaluated to give the result shown above.

The frictional moment, however, cannot be explicitly presented as easily, so all equations for the I, J and G integrals are left in integral form. The integrals are all of the form: $6R\cos\theta\ dW$, or $6\sqrt{R^2 - X^2}\ dW$. The program initially considers the problem to have the geometrics shown in Figure 6(a). The regions with which each of the

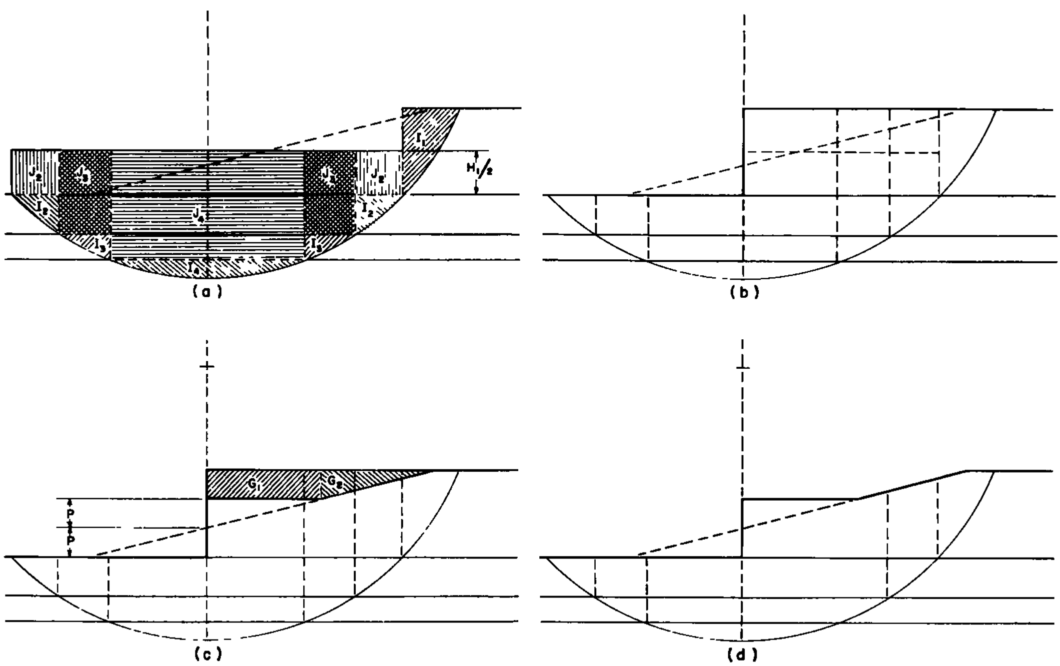

Figure 6.

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[b] The limits of integration are as given if these values are $\geq$ M and $\leq$ N. If a limit is <M, then it is set equal to M. If a limit is >N, then it is set equal to N.

I and J integrals are associated, are labeled on this drawing. The shape of the embankment cross-section in Figure 6(a) is equivalent (as far as friction forces are concerned) to the shape in Figure 6(b). From this shape the G integrals (shown in Figure 6(c) are subtracted, leaving the configuration shown in Figure 6(d), which is frictionally equivalent to that shown in Figure 1—the usual embankment cross-section. Thus M_F, which is 6 times the frictional moment, equals the sum of the I and J integrals, minus the sum of the G integrals.

The cohesive force is calculated by multiplying the length of the arc by the cohesion of the material through which the arc is passing. The sum of these products for each materials considered is multiplied by 6R to obtain M_c. The formula for this is given explicitly in the section, "Equations."

The safety factor then equals:

$$(M_c + M_f)/M_d$$

Appendix B

Sample Problem

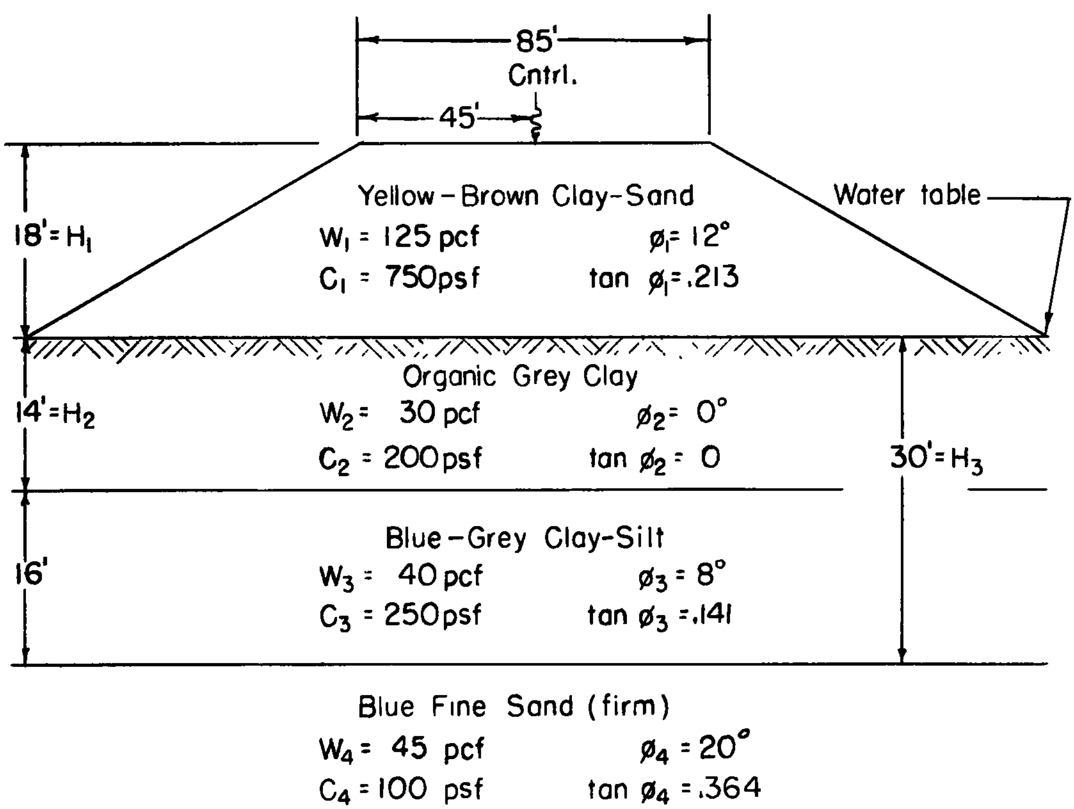

Figure 7. Sample problem.

INPUT DATA
SLIP CIRCLE PROGRAM

8/9	
8 - Punch values for safe slope only	
9 - Punch values for starting slope only	

IDENT.					8/9	STARTING SLOPE (To one)				FILL SOIL PROPERTIES W_1			C_1			Tan ϕ_1			H_1			FIRST LAYER SOIL PROP. W_2			C_2			Tan ϕ_2		
						x	x.	x	x	x	x	x	x	x	x	.x	x	x	x	x.	x	x	x	x	x	x	x	.x	x	x
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	0	1	1	1	9	0	1	8	0	1	2	5	7	5	0	2	1	3	1	8	0	0	3	0	2	0	0	0	0	0
0	0	1	1	1	8	0	1	8	0	1	2	5	7	5	0	2	1	3	1	8	0	0	3	0	2	0	0	0	0	0

Leave Blank if there is Only One Layer H_2			Second Layer Soil Prop. W_3			C_3			Tan ϕ_3			Leave Blank if There are Only Two or Less Layers H_3			Third Layer Soil Prop. W_4			C_4			Tan ϕ_4			Cntrl.		
x	x	.x	x	x	x	x	x	x	.x	x	x	x	x.	x	x	x	x	x	x	x	.x	x	x	x	x	x.
32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
1	4	0	0	4	0	2	5	0	1	4	1	3	0	0	0	4	5	1	0	0	3	6	4	0	4	5
1	4	0	0	4	0	2	5	0	1	4	1	3	0	0	0	4	5	1	0	0	3	6	4	0	4	5

OUTPUT DATA
SLIP CIRCLE PROGRAM

IDENT.					8/9	STARTING SLOPE					FINAL SLOPE					SAFETY FACTOR					R				A				B		
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
0	0	1	1	1	9	0	1.	8	0		0	1.	8	0		0.	8	9	7		0	4	9.		0	1	7.		0	1	7.
0	0	1	1	1	8	0	1.	8	0		0	4.	0	0		1.	1	1	6		0	8	0.		0	3	6.		0	4	8.

Figure 8.