Mathematical Expressions for the Circular Arc Method of Stability Analysis

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● THE DETERMINATION of the conditions under which earth slopes will be stable represents one of the most important applications of soil mechanics. A number of useful methods for investigating this problem involve the classical theories of Coulomb, Rankine, and Boussinesq. Other methods involve trial and error analysis for determining the most critical location of flat and curvilinear failure planes. This paper deals specifically with the cylindrical failure plane as applied to the general solution of the slope stability problem.

Analyses involving use of the cylindrical plane are based upon the two-dimensional case (1) in which the failure surface is represented by an arc of a circle (sometimes referred to as the Swedish Circle). The ratio of resisting moment to driving moment (taken about the center of the failure circle) or resisting force to driving force (taken along the failure arc) is used as a basis for describing the relative stability of the soil loading system for each specific failure surface. By successive trials, the location of the weakest plane can be established, and its corresponding moment (or force) ratio serves as an indication of its factor of safety. (This ratio is often used in various forms to represent the factor of safety of the earth slope against sliding. Although there is a great deal to be said about the definition of factor of safety (2), it is not the purpose of this paper to rationalize the point. It is noted, however, that the equations presented herein may be used in expressing any desired definition of factor of safety applied to the circular arc type of stability analysis.) Inasmuch as there are an infinite number of trial planes available for any problem (Fig. 1), the complete solution is often

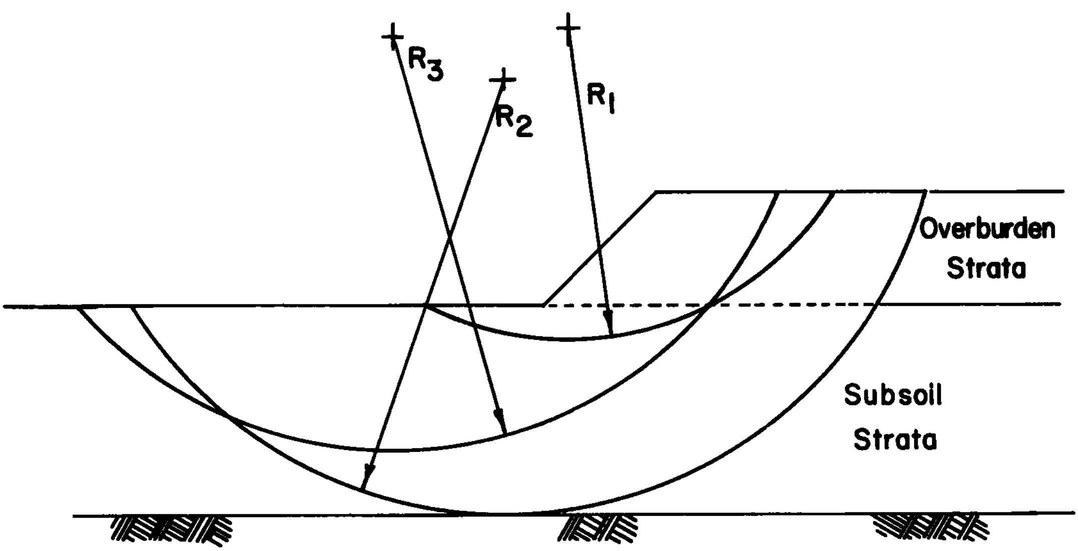


Figure 1. Typical circular arc failure surfaces.

tedious and time consuming. Where time limitations are imposed by design schedules, it is often impossible to properly locate the most critical failure plane. A number of attempts have been made to reduce the number of trials required (2) and, in the case of homogeneous soils, trial was eliminated (3). Unfortunately, the case of soil homogeneity is rarely if ever encountered, and to date the general solution for heterogeneous soil conditions defies simplification. The trial and error process is necessary for the present; however, the electronic computer now makes a rigorous approach economically feasible.

One of the first papers on the subject of the use of electronic computers for the solution of embankment stability was presented at the 37th Annual Meeting of the Highway Research Board (4). Other papers have appeared on the subject (5), particularly in England where the use of the Deuce electronic digital computer is widespread. These papers are technically excellent but are oftentimes limited in scope to suit the immediate needs of the users.

It is the purpose of this paper to present rigorous mathematical expressions which will permit direct application to computer programming as well as to organized manual computation for the determination of the weakest failure plane. Simplifying assumptions have been kept at a minimum to fully utilize the accuracy potential of the high-speed computer. Basic equations are presented for solution of the simple stability problem involving a constant earth slope of homogeneous material founded on a stratified subsoil. Special cases are also investigated, involving irregular or stratified slopes, the condition of toe failure, dam analysis, and related refinements demonstrating the flexibility of the derived expressions. In some instances, simplifying assumptions are made; however, it is left to the soils engineer to determine the suitability of the assumptions before attempting to use the equations for any specific problem. Specific examples of the use of these equations in the solution of a typical roadway embankment and an earth dam problem are presented in the Appendix. The Appendix also contains the basic forms from which the equations presented in the text are derived.

GENERAL DERIVATION

The method of investigating the stability of earth slopes, involving a multi-layer soil system, has generally been based upon the use of the method of slices. This involves the determination of forces developed by vertical segments of the soil system on corresponding incremental lengths of arc along the assumed failure plane. The method developed herein does not use the concept of slices but, instead, investigates the forces developed along the assumed failure plane due to the effect of each soil stratum. This approach to the problem results in completely rigorous mathematical expressions applicable to the general stability problem.

In deriving many of the equations presented, a coordinate system was used to relate the geometry of the soil system with that of the failure plane. The origin of this coordinate system is located at the toe of embankment slope (Fig. 2). To simplify application to electronic computer programming, the final form of the equations was often altered so that dimensions may be used as positive numbers without regard to coordinates. The term L (see "Glossary of Symbols"), which represents the horizontal distance from the toe of slope to the vertical axis through the center of the critical circle, is one of the few terms in the final equations that

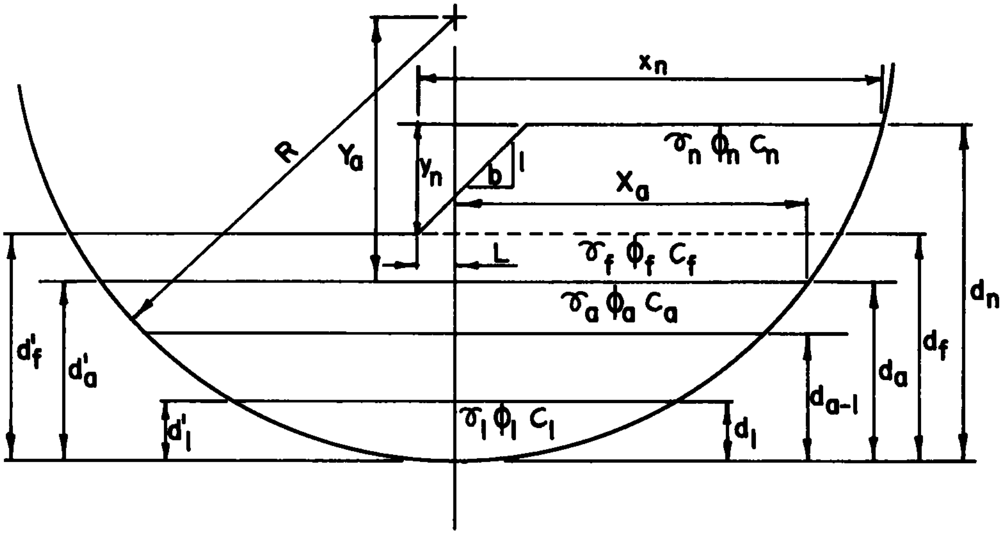


Figure 2. General earth slope problem.

are measured in accordance with the original coordinate system, and consideration must be given to its algebraic sign. When the term L is measured toward the embankment, its value is negative, whereas when it is measured away from the embankment, it is positive. When other exceptions are made, they are indicated in the text.

The basic solution of the slope stability problem, as presented herein, applies to the case of a uniformly sloping embankment of homogeneous material that is infinite in extent, situated on an unlimited number of layers of subsoil materials, as shown in Figure 2, for which the following simplifying assumptions are made:

1. The embankment is infinite in extent, has a uniform slope, and is homogeneous in nature.
2. The live loading, including its dynamic effects, may be represented by a surcharge applied to the embankment.
3. The subsoil can be represented as homogeneous horizontal strata, uniform in thickness.
4. The limits of the embankment slope, when projected downward onto the arc of the trial circle, fall within strata having the same angle of internal friction.
5. The stress distribution due to embankment loading is transmitted only vertically.

Where it is desired to avoid using these assumptions, supplementary mathematical expressions are presented in the section entitled "Special Cases." (Expressions to correct for the first and third assumptions as applied to embankments are found under the headings "Pore Pressure Distribution" and "Dam Analysis"; the second assumption may be eliminated in accordance with the explanation given under the heading "Concentrated Live and Dead Load." Assumption 4 can be eliminated by the method described under the heading "Shear Moments." A system for reducing the error incurred by assumption 5 is discussed under the heading "Shear Strength with Earth Slope Stress Distribution.")

Dead Load Moments

The dead load driving moment for a vertical embankment slope with the slope corresponding to the vertical axis (axis through center of trial circle) of the failure circle is expressed by Eq. 1, and the dead load resisting moment is expressed by Eq. 2.

Due to the existence of a sloping embankment, and the fact that the vertical axis of the failure circle may be located to either side of the toe of slope, a correction must be applied to both the driving and resisting moments. When the vertical axis falls outside of the embankment (L is positive or zero), the correction expressed by Eq. 3 must be subtracted from the driving moment. When the vertical axis passes through the embankment slope (L is negative), the value obtained from Eq. 4 is to be subtracted from the driving moment, and that obtained from Eq. 5 added to the resisting moment.

Shear Moments

In non-cohesive materials, there is general agreement that shear strength is adequately represented by the expression:

$$S = p \tan \phi$$

where p is the intergranular pressure (total pressure minus pore pressure) normal to the plane of failure, and ϕ is the angle of internal friction of the material. However, the manner of treatment of shear strength applied to cohesive materials is a point of controversy among engineers (7). For cohesive soils, the expression most often given is similar to that for cohesionless soils, except for the addition of the term C, representing the cohesion of the material. Thus, the shear strength of cohesive materials may be represented as follows:

$$S = C + p \tan \phi$$

This relationship indicates that the shear strength of any material can be thought of as being represented by the addition of a constant (C) and a variable ($p \tan \phi$). Separate equations are derived representing the resisting moment for each of these components and are referred to as moments due to cohesion and friction, respectively. The resisting moment due to cohesion is given in Eq. 6, and that due to friction is given in Eq. 7.

Inasmuch as the derivation of Eq. 7 includes a constant embankment load, the moment due to friction must be corrected for the effect of the sloping embankment by subtracting the value obtained in Eq. 8.

This correction is proper if assumption No. 4 is correct. Where the error involved in assumption No. 4 is not permissible, then a more exact solution can be made by using Eq. 8 incrementally, using values of X_L and L corresponding to the intersections of the various strata with the assumed failure plane projected onto the embankment slope line. The application of this refinement, it is believed, may be necessary in a limited number of cases.

Where the pore pressure effects are already included in the values of C and ϕ used in Eq. 6 and Eq. 7, or where no pore pressure exists, no further correction is necessary. However, where pore pressure has not otherwise been taken into account, a further correction in shear strength is necessary. Because of the method used in describing pore pressure, the subject of moments ascribed to those forces is handled under a separate heading.

DERIVATIONS—SPECIAL CASES

The foregoing presentation permits the solution of most stability problems involving simple slopes and loading configurations. However, in order to analyze conditions which cannot be handled by the basic equations the following special cases have been investigated: (a) concentrated live and dead load, (b) non-uniform slope, (c) stratified slope, (d) finite berm, (e) finite embankment, (f) dam analysis, (g) pore pressure distribution, (h) shear strength with stress distribution, and (i) toe failure investigation. Methods of handling these cases are developed and summarized in the following paragraphs. This list of special cases is not to be construed as being the only special cases possible but are presented to demonstrate that the derived expressions and the approach used in their derivations may be extended to include many special cases.

Concentrated Live and Dead Load

The incorporation of loads concentrated on an earth embankment may be desirable where heavy live loads are encountered and dynamic effects become important. This type of loading is assumed to effect the driving and resisting moments only, as their effects on shear strength and friction are assumed to be taken into account by distribution factors described in the text. When the value of moment V is positive, the result is used as a driving moment, and when negative, it is used as a resisting moment.

$$\text{Concentrated load moment} = V = P_L(1 + I)\ (L - E_L)$$

where P_L = Concentrated load;
E_L = Horizontal distance from toe of slope to load P_L, with its algebraic sign; and
I = Load increase factor due to dynamic effects.

$$(E_L + L)^2 \leq 2R(d_f+H) - (d_f+H)^2$$

Although the above limiting equation is specifically for loads at the top of the slope, a similar expression can be used for loads existing on the original ground surface, by using the term H as the height of the point of load application above the toe of slope.

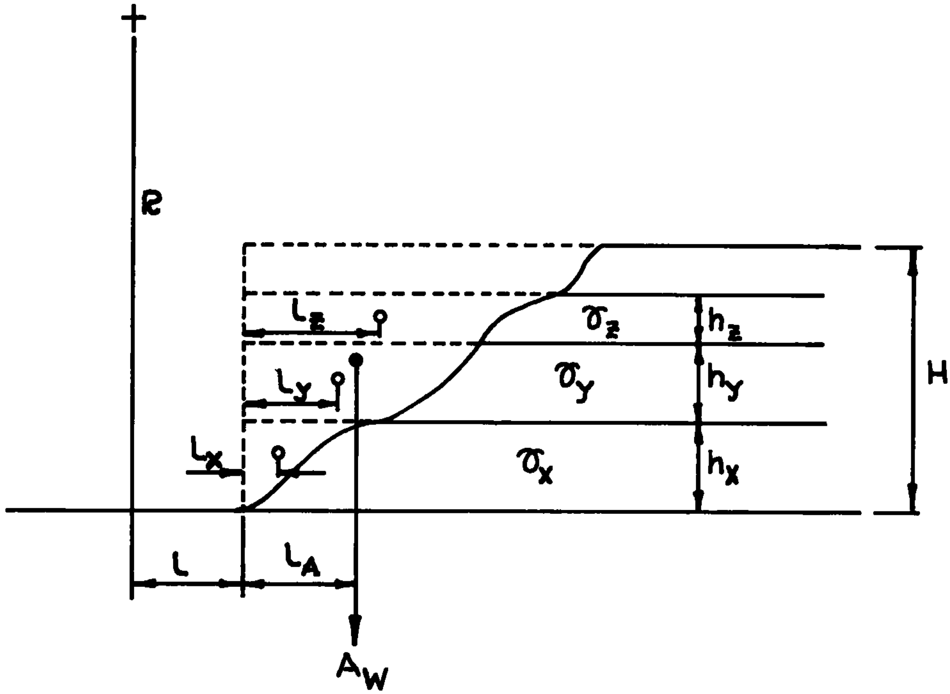
Non-Uniform Slope

The condition of non-uniformity in the embankment slope may be represented as a condition of a stratified embankment. Thus, Eq. 1 can be applied to determine the driving moments. The correction for the driving moments may be obtained by extending Eq. 3, using the proper values of X_L and L for each stratum, as shown in Figure 3A:

$$\text{Drive Correction} = \sum_{e=f+1}^{e=n} \frac{\gamma_n}{6b_e}\,(X_e{}^3 - L_e{}^3)$$

where the subscript e is used to denote embankment stratification. Where the vertical axis passes through the embankment, then the lower limit (f + 1) in the expression will be changed accordingly, and a correction to the resisting moment will be required. The identical expression to that given above will apply to the resisting moment correction except

3B



3A

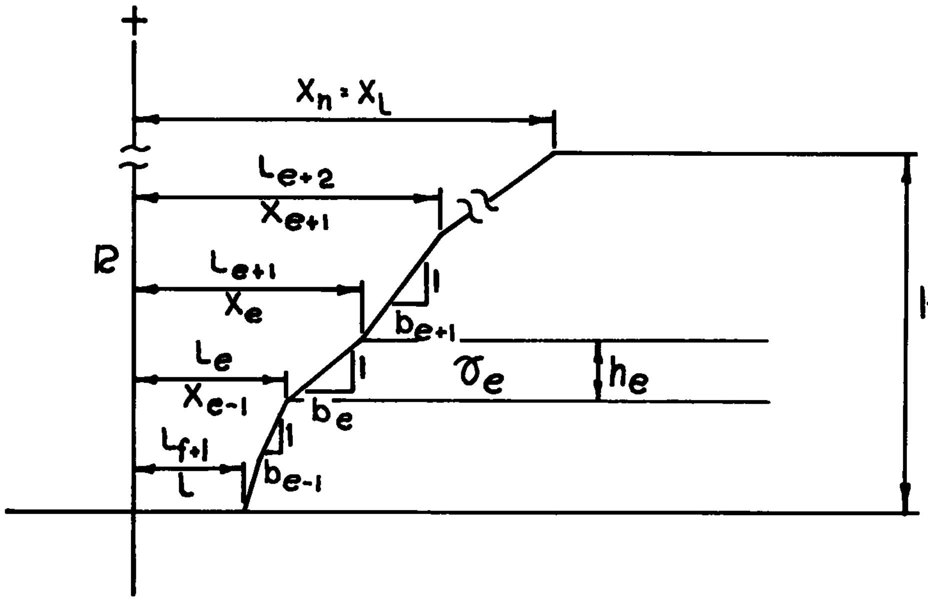


Figure 3. (A) Irregular earth slope, exact method; (B) Irregular earth slope, alternate method.

that the upper limit n will be reduced to correspond with the limiting stratum encountered. Further refinements may be required which may involve the use of Eq. 4 and Eq. 5; however, such detail will very rarely be required. (X_e and L_e take no sign.)

An alternate possibility for the correction of soil moments for a non-uniform slope may be obtained by Eq. 9, which equation is derived in accordance with Figure 3B. The derivation of this expression is based upon the determination of the weighted average soil characteristics of the area bounded by the embankment slope and the vertical axis. These characteristics are defined by Equations 9A, 9B and 9C, representing γ_A, A_W and L_A, respectively. The result obtained by means of these expressions represents the net driving moment correction due to the slope irregularity, and so is exact only for the condition where the vertical axis passes outside of the embankment. Where the vertical axis passes through the embankment, the correction to the driving moment obtained by Eq. 9 will be less than the actual, but this is offset by the fact that no correction is made to the resisting moment. When Eq. 9 is negative, its numerical value is added as a correction to the resisting moment. Although the use of Eq. 9 may slightly alter the position of the critical failure plane, its principal effect is in not presenting a true value of total soil driving and resisting moments when the vertical axis of the failure plane passes through the embankment.

The method of obtaining the shear moment under the general case applies, except that the number of strata is increased. The shear moment correction as given in Eq. 8 must be altered by providing a summation using appropriate values of X_e and L_e for the embankment stratification. It is noted that in most cases, the embankment slope can be approximated by an average value of b without causing an undue error in the shear moment, which will then permit using Eq. 8 directly for correction purposes without summation.

<u>Stratified Slope</u>

The discussion presented under the heading "Non-Uniform Slope" is directly applicable to the stratified slope condition, except that density is a variable with respect to the strata involved. Thus, the following equation is subtracted from the driving moments in accordance with Figure 3A:

$$\text{Drive Correction} = \sum_{e=f+1}^{e=n} \frac{\gamma_e}{6b_e}\left(X_e^3 - L_e^3\right)$$

The above equation may be applied to the determination of the resisting moment in the same manner as described for the case of non-uniform slope.

The alternate possibility for moment correction, using Eq. 9 as described for the non-uniform slope, holds here as well. However, in applying the alternate method to shear moment correction, the use of a weighted average density factor leads to an error which will vary with the range of density values involved; however, the error should generally be small. The error in shear correction is due to the fact that the weighted density average will vary with the value of L. Although refinements can be made to reduce the error, such as relating L and γ_A, it is not believed necessary that this be done except where extreme accuracy is needed. Where such accuracy is required, then the alternate method should not be used.

Finite Berm

Where a finite berm is used, Eq. 3 is directly applicable as a resisting moment, and Eq. 8 is applicable as a shear moment. These equations represent the effect of trapezoidal earth forms and are therefore applicable to a single or multiple finite berm configuration, where the failure arc does not cut through the berm. The alternate possibility described for determination of moments in the case of the non-uniform slope applies. In the case of the finite berm, a restriction is placed upon the radius of the failure plane, such that:

$$R \geq \frac{(d_f + h_B)^2 + (B - L)^2}{2(d_f + h_B)}$$

where B is the horizontal distance from the toe of slope to the top of berm, as shown in Figure 4.

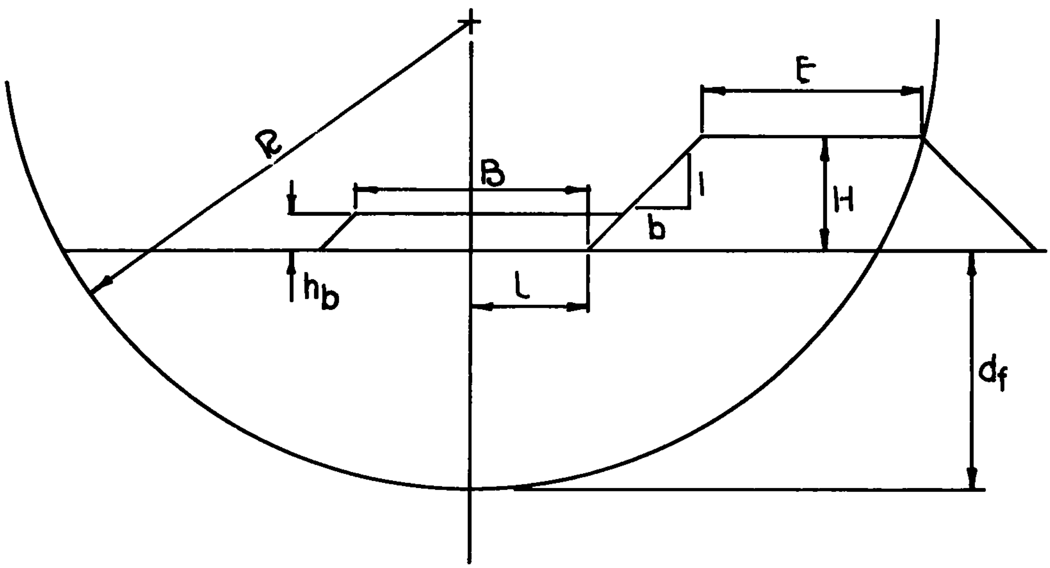


Figure 4. Finite embankment and berm.

Finite Embankment

The instance of the finite embankment is such that the following relationship holds, as shown in Figure 4:

$$R \leq \frac{(d_f + H)^2 + (L + bH + E)^2}{2(d_f + H)}$$

where E is the embankment width. This application is important in the investigation of the stability of roadway embankments along a plane transverse to its centerline.

The above equation is based upon the assumption that the failure circle does not intersect the far slope of the embankment. This assumption is valid for the vast majority of slope stability problems. Where the engineer prefers to provide a more detailed analysis by investigating circles which exceed the above limits for R, then the method of correction described in the Appendix under the heading "Investigation of Zoned Dam" may be used.

Dam Analysis

In order to apply these equations to the analysis of a dam, additional equations must be derived to express the geometry where the failure circle passes through the upstream face and also to account for the existence of a zoned (non-horizontally stratified) embankment.

Referring to Figure 5 for the variables, the distance above the toe of slope that the trial circle will pass through any interior slope is given by the following expressions (for a toe or deep circle):

$$h_a = \frac{-G_a + \sqrt{G_a^2 + (b_a^2 + 1)\,(2Rd_f - D_a^2 - d_f^2)}}{b_a^2 + 1}$$

where $G_a = b_a D_a - (R - d_f)$

and $D_a = L + m_a$

(The values of h_a can be determined in accordance with the coordinate system and therefore may be applied to subsurface stratification as well as earth slope stratification.)

The above equation is applied to determine the limiting points along the failure arc corresponding to the boundary lines for each zone.

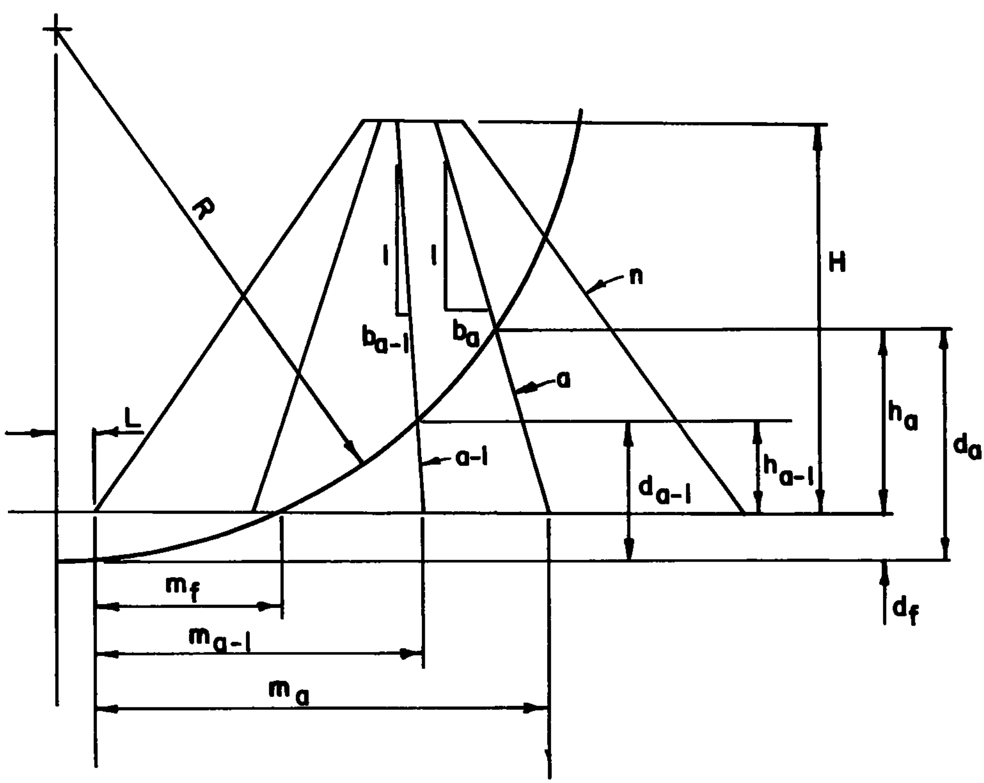


Figure 5. General zoned embankment problem.

In order to provide for the moment of an interior zone as a correction with respect to the original mass of the dam as a homogeneous material, the following equation is used to correct the driving moment obtained in using Eq. 1:

$$M_{Dc} = \frac{\gamma d - \gamma a}{6} \left[\frac{(D_a + b_a H)^3 - (D_a + b_a H_a)^3}{b_a} - \frac{(D_{a-1} + b_{a-1} H)^3 - (D_{a-1} + b_{a-1} h_{a-1})^3}{b_{a-1}} + (R - d_a)^3 - (R - d_{a-1})^3 + 3R^2 (d_a - d_{a-1}) \right]$$

where $d_a = b_a + d_f$.

A description of the application of the above equations to the solution of a specific problem is given in the Appendix.

Pore Pressure Distribution

It is not possible to express the distribution of pore pressure by mathematical symbols for application to all conditions, even where the case of soil homogeneity is assumed. However, an expression can be given based upon the assumption that pore pressure at any point can be described for a specific problem, if the position of that point is known with respect to the geometry of the system. Inasmuch as the failure plane is easily coordinated with respect to the toe of slope of the embankment, it follows that if tabular values of pore pressure are available and also coordinated with respect to the toe, that no rigorous mathematical expression is needed. Thus, if the pore pressure along the failure plane at a height d_a above the low point of the arc is known, and if this pressure is constant for a distance Δd equal to $(d_a - d_{a-1})$, then the smaller the value of Δd, the greater the accuracy in the analysis of the effect of pore pressure.

The problem thus resolves itself to the description of pore pressure at point d_a. From Figure 2 it is evident that the coordinate of any point can be expressed with respect to the toe of slope as $(d_a - d_f)$, $(x_a + L)$. The latter designation is consistent with the definition that x_a and L are negative when measured toward the embankment. As stated previously, the availability of a master tabulation of pore pressure related to the coordinate system employed would be the most accurate approach to the problem feasible at this time. However, for application to electronic computers where limited storage capacity is available, it is desirable to approximate the value of pore pressure by relating it to two independent factors, as given in the following relation:

$$U_a = Q_a + F_a f(h)$$

where $f(h)$ represents a function of the overburden construction load. The values of Q_a and F_a are dependent only upon the geometry of the system, where the term F_a is used to express the effect of the earth slope and Q_a is a term encompassing all other factors affecting pore pressure. Assuming a uniform embankment height, the reduction in shear moment due to pore pressure is expressed by Eq. 10. The existence of a sloping embankment requires a reduction in the pore pressure. The latter correction requires that Eq. 11 be added to the shear moment. Inasmuch as Eq. 11 is related only to the earth slope, the Q factor is not involved in the expression.

In order to properly utilize Equations 10 and 11, it is necessary to relate the value of pore pressure with the geometry of the soil system under investigation. The values of Q_a and F_a are investigated for application to soils draining vertically, horizontally and to a case involving pressures in earth dams.

In the investigation of vertical draining subsoil, it is assumed that the value of Q is zero within the limits of the embankment. From the theory of consolidation, the pore pressure variation with depth is parabolic; and to demonstrate the case, it is assumed that the ground surface always permits free drainage. The value of F_a can then be expressed as follows:

$$F_a = M\left[1 - \left(\frac{md_a - (m-1)d_f}{d_f}\right)^2\right]$$

for $d_a \leq d_f$

where m is a constant varying from 1 to 2, depending upon whether the condition is one of single drainage, double drainage, or an intermediate drainage condition. The term M is a factor which represents the maximum pore pressure effect of a unit embankment load. (This equation may be altered to suit the condition where the upper stratum is not free draining, by substituting $d_f - d_a$ for d_a.) Pore pressure effects beyond the limits of the earth slope can be corrected by use of the Q-term.

Where only horizontal drainage is effective, the value of F_a would be constant with depth, and F_a would equal M. Although Q is considered as zero within the limits of the slope it may, nevertheless, be added when there is superimposed pressure through other means, such as ground water movement. Where both horizontal and vertical drainage occur, as well as water flow, the various terms may be combined to express the desired pore pressure distribution.

The conditions described above pertain to pore pressure below ground level; however, a necessary consideration is that of analyzing a combination of pore pressure distribudion in subsoils as well as slopes such as earth dams. To demonstrate this application, it is assumed that the equipotential lines, as well as the phreatic line, can be described algebraically, as in Figure 6.

Taking the toe of the slope to be the origin, the general equation for the equipotential line can be expressed as:

$$y_e = f(x_e) + c_e$$

Using the approximation that the pore pressure between any vertical increment at a specific horizontal distance from the toe of slope is constant, a sufficiently large number of increments is arbitrarily established to assure desired accuracy. For any increment or hypothetical stratum located h_a above or below the toe of slope the horizontal distance from the toe to the intersection of this limit with the failure plane is:

$$x_a = L - \sqrt{R^2 - (R-d_f - h_a)^2}$$

(The sign in front of the radical is $\pm$ for $h_a \gtrless 0$).

Since y_e must equal h_a on the equipotential line and x_e must equal x_a, then c_e must be:

$$c_e = h_a - f(x_a)$$

Thus, the equation for the equipotential line passing through the failure plane at the height h_a must be:

$$y_e = f(x_e) + h_a - f(x_a)$$

If the equation of the phreatic line can be expressed as:

$$y_p = f(x_p) + c_p$$

then the point on the phreatic line h_p corresponding to h_a on the failure plane can be determined, and the pore pressure at height h_a on the failure plane will be:

$$Q_a = (h_p - h_a)\, \gamma_w$$

with Q_a assumed constant for that portion of the failure plane passing through the stratum. When Q_a is negative, then the phreatic line has been exceeded and the pore pressure is zero.

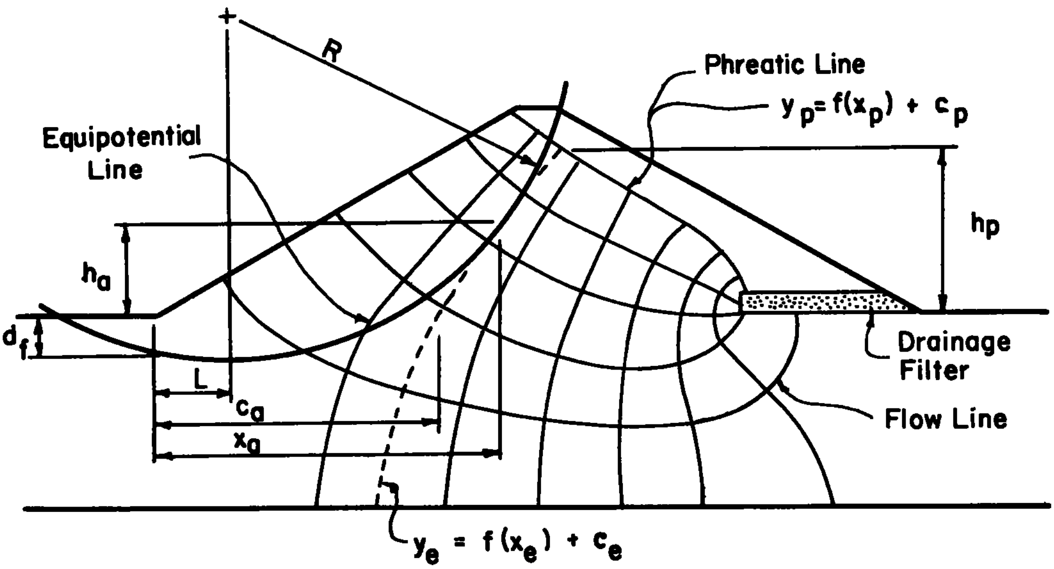


Figure 6. Typical flow net for earth dam.

The foregoing method may be applied to any system of equipotential and phreatic lines that can be expressed algebraically. Although most theoretical cases involve equipotential lines of the type shown in Figure 6, for demonstration purposes, an application of this method to the draw-down condition where the equipotential lines are vertical and the phreatic line follows the downstream face of the dam, will produce the following general expressions for toe failure investigations:

$$Q_a = \gamma_w \left(\frac{L}{b} - h_a - \frac{\sqrt{R^2 - (R-d_a)^2}}{b} \right)$$

where $L < -x_a < bH$

and

$$Q_a = \gamma_w (H - h_a)$$

where $bH + E > -x_a \geq bH$

For the case where the failure circle intercepts the downstream slope,

$$Q_a = \gamma_w \left[(H-h_a) + \frac{x_a + bH + E}{b_b} \right]$$

where $-x_a > bH + E$

b_b = downstream slope of dam
E = width of crest of dam

This method may be applied to many types of flow net conditions; however, it is emphasized that each case must be investigated separately to determine that the use of algebraic representation of the equipotential and phreatic lines will not result in undesirable error.

It is pointed out that the analysis of pore pressure may, in certain locations along the failure plane, produce a negative net frictional moment. However, this error is inherent to the procedure used in the method of slices and is compensated in part by an excess shear resistance effected along other portions of the arc by the neglect of stress distribution. In specific instances, it may be necessary to utilize stress distribution factors in the analysis. In such cases, where neutral stresses are large, the effect of such stresses acting on the vertical sides of hypothetical slices may be investigated analytically by using the pore pressure designation described herein.

Shear Strength with Earth Slope Stress Distribution

There may be instances where assumption 5 is undesirable. In such cases, it is necessary to take into account the effect of stress distribution due to the earth slope. The approximations recommended in this solution are similar to those discussed under the heading "Pore Pressure Distribution."

Although the total stress may be described as the summation of factors as in the case of pore pressure, unlike pore pressure the total stress at any point is not perpendicular to the plane of failure. It is therefore necessary to treat each stress value as two components:

$$P_v = Q_v + F_v(h)$$

$$P_h = Q_h + F_h(h)$$

where P_v and P_h represent the vertical and horizontal stress, respectively, at any point in the subsoil due to the earth slope. The procedure that may be followed is to determine the shear strength effects due to the subsoil loading by means of Eq. 7, taking the summation to the ground line (d_f) rather than to the top of slope (d_n).

The derivation of the equations for the effect of earth slope stress distribution on the shear strength along the failure plane are not given herein; however, the equation for the shear strength effect of the vertical component P_v is identical with that given in Eq. 7 applied so that P_v replaces the term $W_a - \gamma_a(R-d_a)$. The equation for the shear moment effect of the horizontal component P_h is:

$$M_{fh} = \sum_{a=1}^{a = f' \atop a \pm n} P_h \frac{\tan \phi_a}{2} (x_a^2 - x_{a-1}^2)$$

The qualifications applied to the use of this method are similar to those stated for pore pressure. Where the values of Q_v, F_v, Q_h, and F_h can be related algebraically with respect to the geometry of the soil system, then simplifications in the use of electronic computers may be possible. However, it is left to the engineer to determine whether or not the assumption of stress distribution is desirable as compared to the usual assumption that such distribution is entirely vertical.

Toe Failure Investigation

Where it is desired to investigate specifically for toe failure, as shown in Figure 7, the depth d_f to which the critical circle will pene-

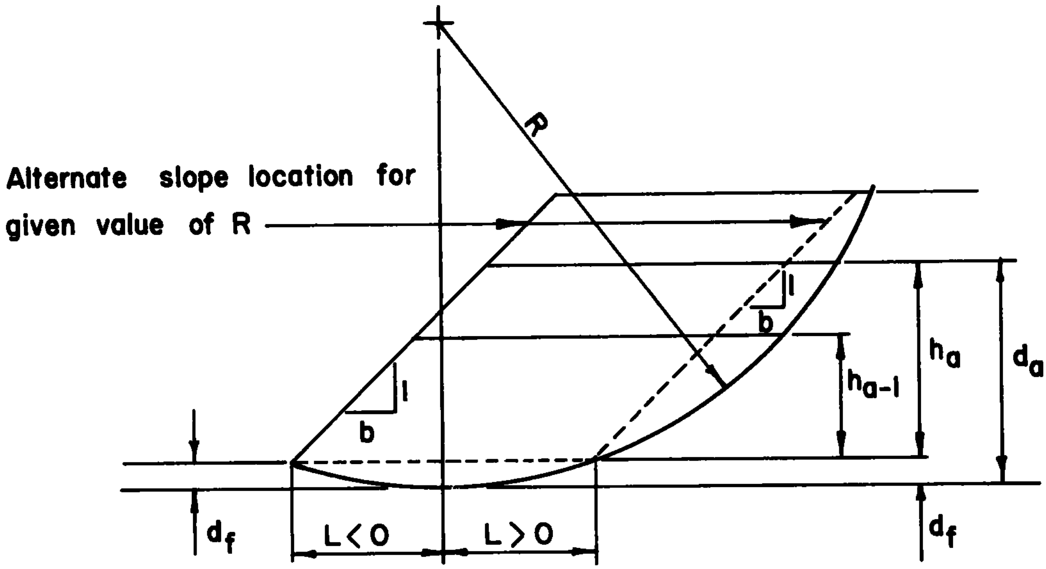


Figure 7. General toe failure problem.

trate the subsoil is not constant; therefore, the value of d for each stratum will vary with the location of the center of rotation. However, the height h of the top of each stratum above the toe of the slope is constant, as the stratification is assumed to be horizontal. To find d for any stratum, d_f must be added to h. The value of d_f for each circle is obtained as follows:

$$d_f = R - \sqrt{R^2 - L^2}$$

Thus, the value of d for any stratum becomes:

$$d_a = h_a + d_f$$

Using the appropriate values of d, Equations 1 through 11 may be used as previously described, applying the proper driving moment corrections.

LIMITATIONS

The mathematical expressions presented are limited by the assumptions made in their derivation. Where such assumptions are not permissible, they may be eliminated, or the associated inaccuracies reduced, by procedures outlined in the text. The assumption that is the most cumbersome to eliminate is that dealing with the horizontal stratification of the subsoil. However, suitable expressions may be obtained in a manner similar to that used for non-horizontal stratification in embankments.

The most difficult assumption to deal with is that concerning the representation of distributed pressure (excess hydrostatic as well as intergranular) where the theory can at best be presented as an approximation of the truth. Where such distribution can be expressed algebraically, the assumption that such pressures are constant within each stratum is a desirable postulate. If greater accuracy is desired, the number of strata may be increased to any practical limit. Due to the many complexities involved in theoretically analyzing pressure distribution, there is no certainty that the approach described herein is actually a limitation.

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APPLICATIONS

The expressions presented herein are believed to be the most general possible for application to the solution of the problem of embankment stability by the circular arc method of analysis. These equations are now being programmed to permit the use of Bendix G-15D electronic computer in this work.

It is estimated that for a ten-layer system, under the most adverse conditions as concerns the number of variables, it would require no more than five minutes to investigate any one circle location for a program set up on the interpretation system. With a reduction in variables and number of layers, the machine time would be reduced proportionately. It is expected that those who are familiar with computer operations will be able to find many simplifying methods in applying the equations presented.

Charts and curves, based upon these equations, are now being prepared, using parametric values of the variables assuming a circular failure plane having a unit radius. These will permit rapid solution of the stability problem where electronic computers are not available.

These equations cannot be used to replace the judgment of the soils engineer, as such judgment is required in properly interpreting the nature of the problem, the physical characteristics of the subsoil and embankment materials involved, as well as to decide upon the applicability of the circle arc failure plane. Although these equations are presented for a specific type of failure plane, the method of analysis by investigating the effect of each stratum is applicable to any type of assumed failure surface.

GLOSSARY OF SYMBOLS

a = Stratum and zone designation subscript;
A_w = Weighted area for correction of driving moments;
b = Horizontal distance for unit rise of earth slope (positive when sloping upward and away from the toe of slope); as subscript, represents downstream face of dam;
B = Horizontal distance from toe of slope to top of berm; as subscript, refers to berm;
c = Constant; as subscript, denotes correction;
C = Cohesion;
d = Height of top of stratum above low point of failure circle on driving side of arc; as subscript, refers to dam;
D_a = $(L+M_a)$;
e = Embankment stratification subscript;
E = Embankment width (finite case), or crest width (dams);
E_L = Horizontal distance from toe of slope to concentrated load (with sign);
F = Load distribution factor due to earth slope;
f = Subscript denotes stratum at toe of slope, or friction;
$f(\ )$ = Denotes algebraic function;
G_a = $b_a D_a-(R-d_f)$;
H = Height of earth slope;
h = Stratum thickness, in earth slope; as subscript, refers to horizontal component;
I = Load increase factor due to dynamic effects;
L = Horizontal distance from toe of slope to vertical axis of trial circle;
L_e = Same as L, except measured from intersection of bottom of stratum e and earth slopeline;
$L_{A,x,y,z}$ = Moment arm, measured horizontally to toe of slope;
m = Constant denoting drainage condition in subsoil;
M = Maximum pore pressure effect of unit load;
M_D = Driving moment, dead and live load;
M_{Dc} = Driving moment correction;
M_f = Frictional moment;
M_p = Moment reduction due to pore pressure;
M_{pc} = Correction in moment reduction due to pore pressure;
M_R = Resisting moment, dead load;
M_{Rc} = Resisting moment correction;
M_S = Moment due to shear;
M_{Sc} = Shear moment correction;
n = Subscript designation for uppermost stratum;
p = Intergranular pressure; subscript, designating phreatic line;
P = Distributed pressure;
P_L = Concentrated load;
Q = Load distribution factor;
R = Radius of trial circle;
S = Shear strength in subsoil stratum;
U = Pore pressure;
v = As subscript, vertical component;
V = Concentrated load moment;
W_a = Summation of unit loads above a given stratum;
x = Horizontal distance from toe of slope in coordinate system;
X = Horizontal distance from vertical axis to intersection of top

of stratum with the failure arc or earth slope line;

X_L = Horizontal distance from vertical axis to top of slope;

y = Ordinate location of point within coordinate system;

Y = (R-d); density of stratum;

γA = Average density of total stratified slope;

ϕ = Angle of internal friction;

Δd The incremental vertical distance between strata; and

' = As superscript, represents dimensions taken with respect to strata to the side of the vertical axis away from the toe of slope.

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APPENDIX

Mathematical Expressions for the Circular Arc Method of Stability Analysis

INVESTIGATION OF FIVE-LAYER SOIL SYSTEM WITH BERM

To demonstrate the application of the equations presented it is assumed that it is desired to check the stability of the proposed embankment construction shown in Figure 8. An infinite embankment is assumed where

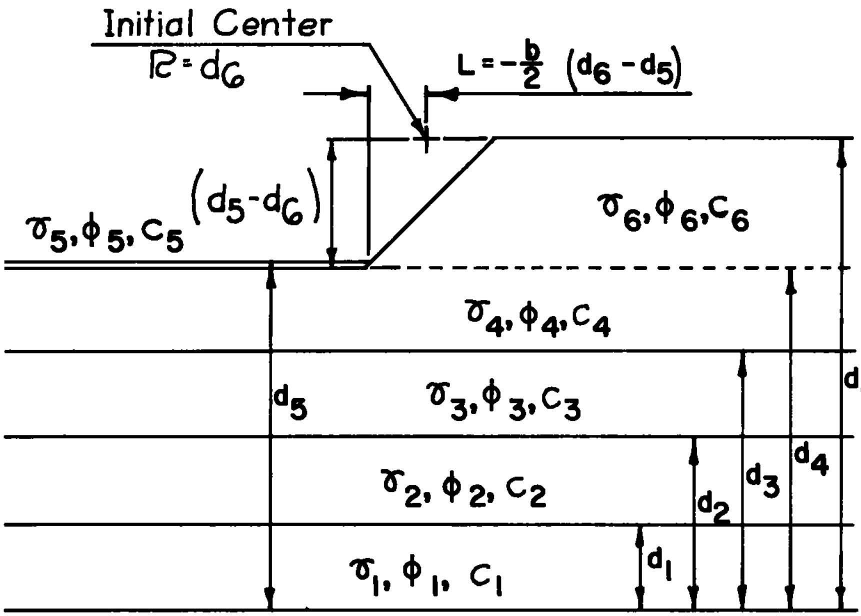


Figure 8. Typical 5-layer soil system stability problem.

the slope is 1/b, with four subsoil strata and one embankment stratum. Assuming that a berm may be required for stability purposes, a fictitious stratum is added, such that $d_4 = d_5$ initially. The expression for the driving moment M_D is obtained by means of Eq. 1. The driving moment needs to be corrected by subtracting M_{Dc}, using Eq. 3 or Eq. 4 whichever is applicable.

Due to the constant thickness of each strata, the resisting moment is equal to the driving moment less the driving moment of the embankment, such that:

$$M_R = M_D - \frac{\gamma_6}{2}\left[R^2 \, (d_6 - d_5) - \frac{(R - d_5)^3 - (R - d_6)^3}{3}\right]$$

This is corrected by adding M_{Rc} as given in Eq. 5, when L is negative.

Discounting the effect of the embankment, the shear moments on either side of the vertical axis of the trial circle are equal. Thus, using Eq. 6 and Eq. 7 to take care of moments on both sides of the vertical axis, the desired shear moment is obtained.

The values of W_1 through W_4 used in Eq. 7 are as follows:

$$W_1 = \gamma_2 h_2 + \gamma_3 h_3 + \gamma_4 h_4$$
$$W_2 = \gamma_3 h_3 + \gamma_4 h_4$$
$$W_3 = \gamma_4 h_4$$
$$W_4 = 0 \text{ (initially)}$$

To the above determined shear moment must be added the shear moment effect of the embankment, given by the following expression:

$$\frac{\gamma_6 (d_6 - d_5)}{2} \sum_{a=1}^{a=5} \tan \phi_a \left[X_a Y_a - X_{a-1} Y_{a-1} + R^2 \left(\tan^{-1} \frac{X_a}{Y_a} - \tan^{-1} \frac{X_{a-1}}{Y_{a-1}} \right) \right]$$

plus the values obtained from Eq. 6 and Eq. 7 for a = 6.

Due to the sloping embankment, Eq. 8 is used to obtain the shear moment correction M_{Sc}, which is to be subtracted from the shear moment effect of the embankment.

Having obtained the necessary general equations and using a suitable definition for the factor of safety, a program can be set up for solution by electronic computer. Although a program is now being set up to investigate a more general case of soil stratification, its application to this would be approximately as described hereinafter.

Inasmuch as the problem is to investigate an infinite earth slope, the program will start at the minimum radius for the deepest stratum to be investigated:

$$R = d_6$$

having its center located at:

$$L = -\frac{b}{2}(d_6 - d_5)$$

(Equations are available that will permit a more desirable starting point; however, the use of such equations is left to the individual.) After this initial computation, the value of R will be varied in increments, and L maintained as a constant, and the factor of safety (ratio of resisting moments to driving moments) determined for each successive position. When a point is reached where the factor of safety at one location is greater than the preceding, the computer is returned to the previous location, R held constant and L is varied incrementally away from the slope, until the factor of safety increases, at which point the machine will revert back to the lower value. This cycle is repeated automatically, with the computer searching until the location of the minimum safety factor is determined.

Using the critical center so determined, the computer will automatically progress to the higher subsoil layers. The investigation will be continued until the factor of safety at a shallower stratum increases, at which point the machine reverts back to the lower depth.

Should the minimum factor of safety result in a value less than that desired, then d_5, which initially was set equal to d_4, is set to incrementally increase by any desired value Δh_B. The value of L will automatically be increased by the value b Δh_B, and the value of W_1 through W_4 is increased by the value $\gamma_5 h_B$.

Such trials can be investigated for the established critical failure plane and the berm height determined when the desired factor of safety is obtained. If necessary, the entire problem, including the berm, can be checked by returning the machine to any desired value of R and L.

Where desired, instead of investigating for a suitable berm height, the problem may be set up to determine the minimum slope for a specific factor of safety.

INVESTIGATION OF ZONED DAM

A typical zoned earth dam section on an impermeable base is shown in Figure 9. The required investigation is based on a toe circle analysis.

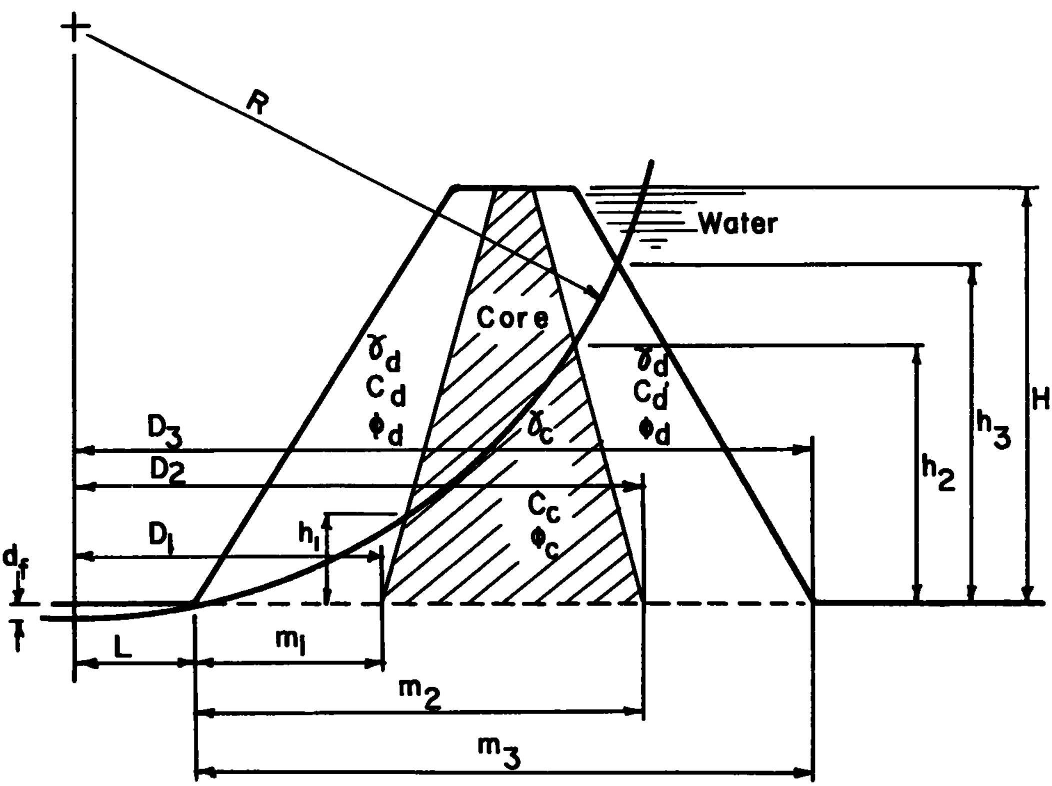


Figure 9. Typical zoned dam stability problem.

For simplicity, it is assumed that $\gamma_c = \gamma_d$ and that the trial circle intersects the upstream face of the dam, at a point h_3 located by the equation given under the heading "Derivations—Special Cases: Dam Analysis." The analysis proceeds as it would for a homogeneous embankment. To determine the driving moment, Eq. 1 and Eq. 3 are used from d_f to $(H + d_f)$. By repeating the use of these equations from h_3 to H a correction for the effect of water may be obtained, using $(\gamma_d - \gamma_w)$ in place of γ_a. No correction is needed for the interior zone as $\gamma_c = \gamma_d$. (When L is positive, no moment correction is made for subsoil moments.)

The frictional resistance is then determined by means of Eq. 7, assuming a constant friction angle ϕ_d for the entire length of arc and going in one step from d_f to $(H + d_f)$. The frictional correction for the downstream slope is made by means of Eq. 8, by going from L to $(L + bH)$. The frictional correction for the water on the upstream slope is made by going from $(h_3 + d_f)$ to $(H + d_f)$ in Eq. 7, and then from $(D_3 + b_3h_3)$ to $(D_3 + b_3H)$ in Eq. 8, with b_3 taking its algebraic sign.

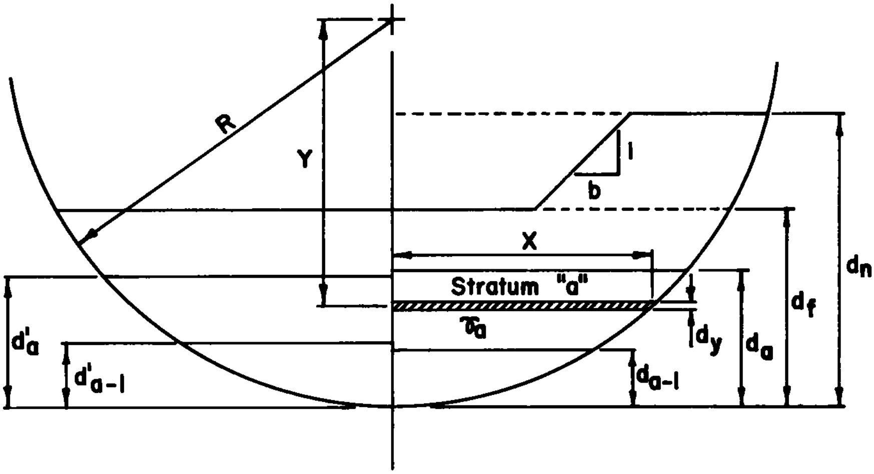
Knowing the values of h_1 and h_2 (equation for h_a, given under the heading "Derivations—Special Cases: Dam Analysis"), from Eq. 7 an approximate intergranular pressure correction is obtained by replacing $\gamma_a \tan \phi_a$ with $(\gamma_d \tan \phi_d - \gamma_c \tan \phi_c)$, where $W_a = (\gamma_d - \gamma_c)\ (H - h_2)$.

The shear resistance due to cohesion is obtained by means of Eq. 6, using the value of S_d from d_f to $(H + d_f)$ and then correcting for the upstream water by going from $(h_3 + d_f)$ to $(H + d_f)$. The correction for the central core is obtained by using $(S_c - S_d)$ in place of S_a in Eq. 6 and going from $(h_1 + d_f)$ to $(h_2 + d_f)$.

If h_3 were found to be greater than H, the circle arc would not intersect the dam backslope and the correction for water would be omitted from the analysis.

The effect of pore pressure is the remaining factor to be established, and this is done as in the case of the embankment analysis, by assuming as many horizontal layers as is desirable and applying Eq. 10 and Eq. 11.

The equations may be used in a similar manner for other configurations of dams and for deep-seated failures in dams as well.



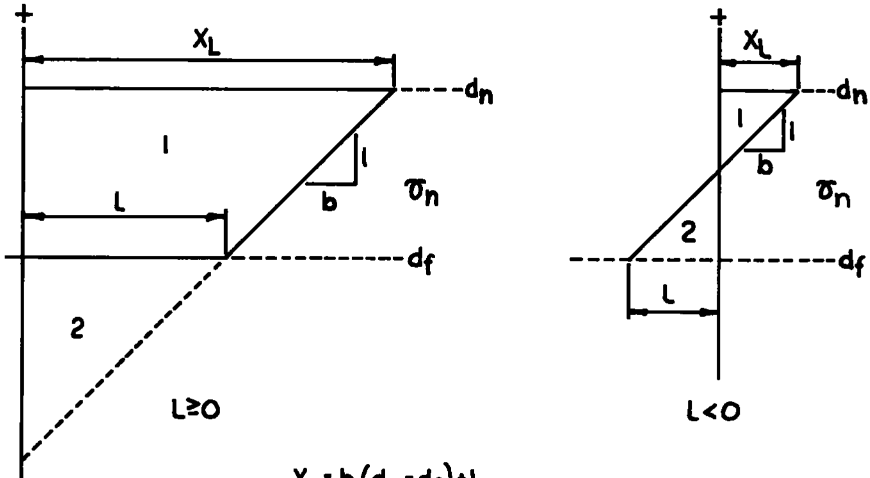
Equation of Arc $X^2 + Y^2 = R^2$

Moment of Stratum "a" about "O" = $\tau_a \int_{Y_a}^{Y_{a-1}} X \cdot \frac{X}{2} \cdot dy = M_a$

Eq. 1 $$M_D = \sum_{a=1}^{a=n} \frac{\tau_a}{2} \int_{R-d_a}^{R-d_{a-1}} \left(R^2 - Y^2\right) dy$$

Eq. 2 $$M_R = \sum_{a=1}^{a=f'} \frac{\tau_a}{2} \int_{R-d'_a}^{R-d'_{a-1}} \left(R^2 - Y^2\right) dy$$

Figure 10. Driving and resisting moments.



$$X_L = b(d_n - d_f) + L$$

$L \geq 0$

Subtract from Drive Moments

Eq. 3
$$\left.\begin{aligned} M_{1+2} &= \tau_n \frac{X_L}{b}\cdot\frac{X_L}{2}\cdot\frac{X_L}{3} = \frac{\tau_n X_L^3}{6b} \\ M_2 &= \tau_n \frac{L}{b}\cdot\frac{L}{2}\cdot\frac{L}{3} = \frac{\tau_n L^3}{6b} \end{aligned}\right\} \quad M_{Dc} = \frac{\tau_n}{6b}\left(X_L^3 - L^3\right) = M_1$$

$L < 0$

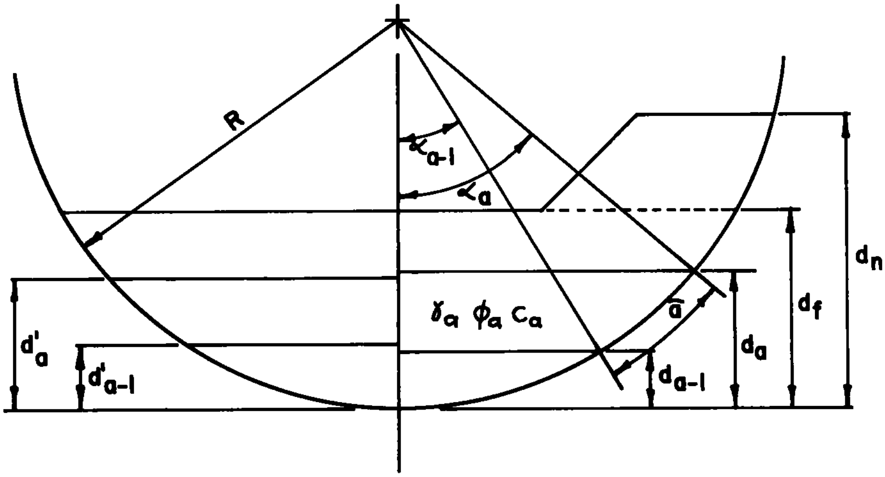
Subtract from Drive Moments

Eq 4
$$M_{Dc} = \tau_n \frac{X_L}{b}\cdot\frac{X_L}{2}\cdot\frac{X_L}{3} = \frac{\tau_n X_L^3}{6b}$$

Add to Resisting Moment

Eq 5
$$M_{Rc} = -\tau_n \frac{L}{b}\cdot\frac{L}{2}\cdot\frac{L}{3} = -\frac{\tau_n L^3}{6b}$$

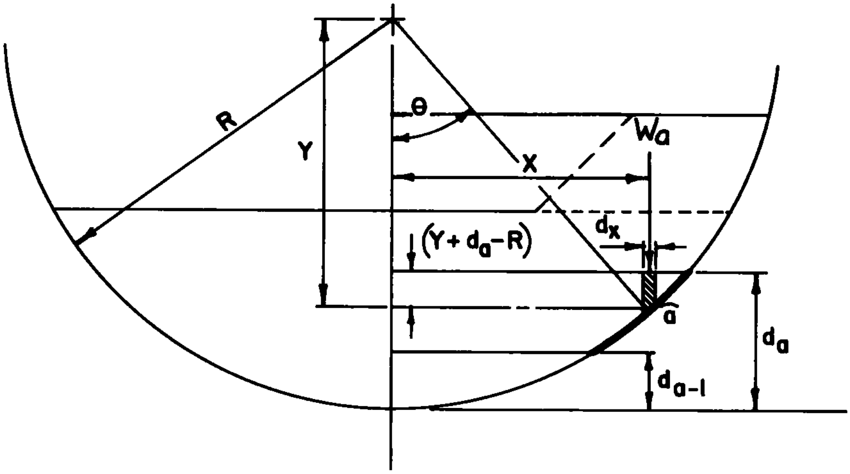
Figure 11. Correction in moment-drive and resistance.



Eq. 6
$$M_S = \sum_{a=1}^{a=n} R\,\widehat{a}_a\,c_a + \sum_{a=1}^{a=f'} R\,\widehat{a}_a\,c_a$$

$$\text{where } \widehat{a}_a = R\left[\tan^{-1}\frac{\sqrt{2Rd_a - d_a^2}}{R-d_a} - \tan^{-1}\frac{\sqrt{2Rd_{a-1} - d_{a-1}^2}}{R-d_{a-1}}\right]$$

Figure 12. Shear moment due to cohesion.



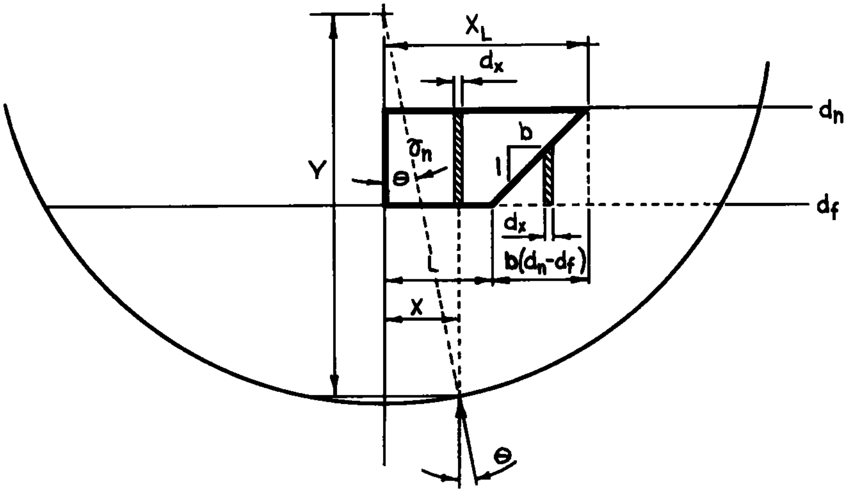
Eq. 7

$$M_F = \sum_{a=1}^{\substack{a=f' \\ + \\ a=n}} \left[W_a - \gamma_a \left(R - d_a \right) \right] \tan \phi_a \int_{X_{a-1}}^{X_a} Y \, d_x$$

$$+ \sum_{a=1}^{\substack{a=f' \\ + \\ a=n}} \gamma_a \tan \phi_a \int Y^2 \, dx$$

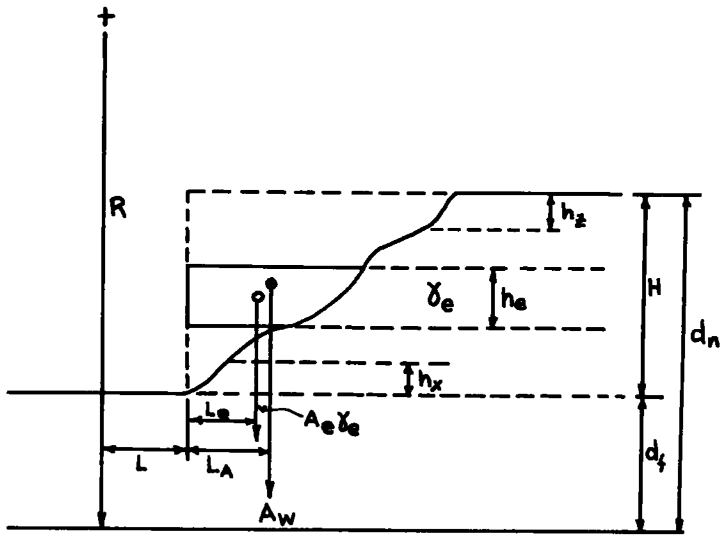
$$\text{where } W_a = \sum_{a=a}^{a=n} \gamma_{a+1} \left(d_{a+1} - d_a \right)$$

Figure 13. Shear moment due to friction.



Eq. 8 $$M_{fc} = \int_0^{X_L} \sigma_n (d_n - d_f)\sqrt{R^2 - X^2}\ \tan\phi\, d_x - \int_L^{X_L} \sigma_n \frac{(X-L)}{b}\sqrt{R^2 - X^2}\ \tan\phi\, d_x$$

Figure 14. Correction in shear moment due to friction.



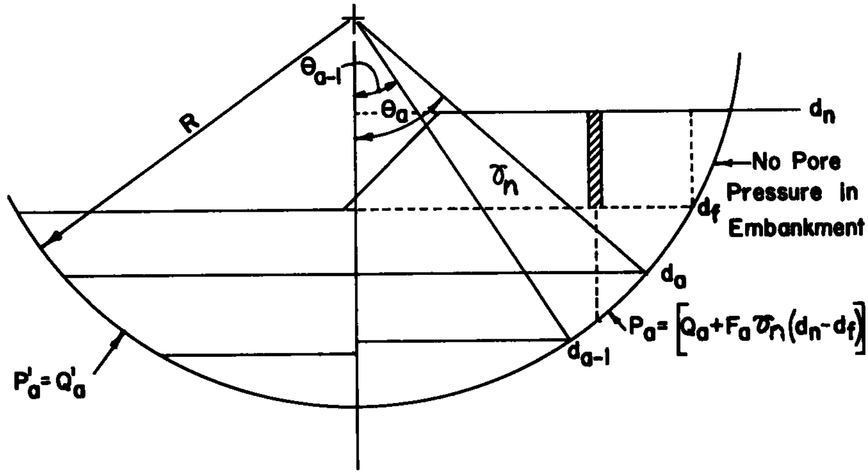
Eq. 9 $\quad M_{DC} = A_W(L+L_A) + \frac{1}{2}\gamma_A H L^2$

Eq. 9A $\quad \gamma_A = \dfrac{\sum_{e=x}^{e=z} \gamma_e h_e}{\sum_{e=x}^{e=z} h_e}$

Eq. 9B $\quad A_W = \sum_{e=x}^{e=z} \gamma_e A_e$

Eq. 9C $\quad L_A = \frac{1}{A_W} \sum_{e=x}^{e=z} \gamma_e A_e L_e$

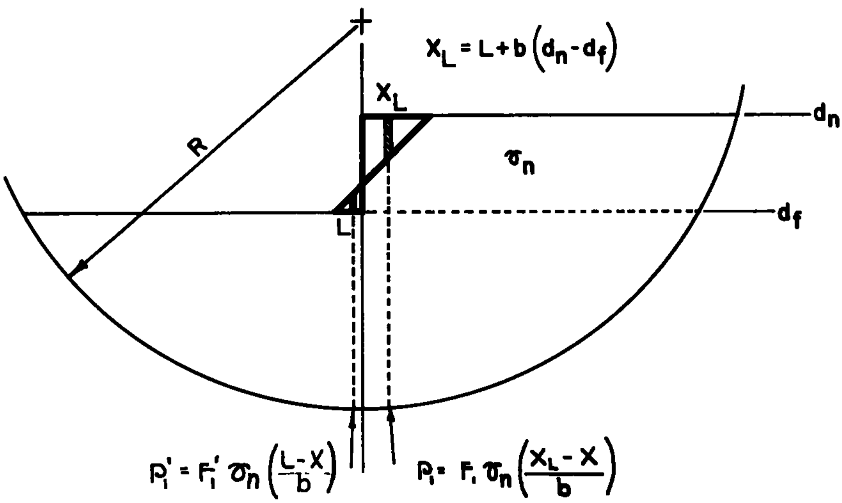
Figure 15. Driving moment correction—alternate method.



$$\text{Eq. 10} \quad M_p = \int_{\theta_{a-1}}^{\theta_a} P_a R^2 \tan \Phi_a \, d\theta + \int_{\theta'_{a-1}}^{\theta'_a} P'_a R^2 \tan \Phi_a \, d\theta$$

$$\text{where } \theta_a = \tan^{-1} \frac{X_a}{Y_a}$$

Figure 16. Shear moment reduction due to pore pressure.



$$\text{Eq. 11} \quad M_{pc} = \int_{0}^{\theta} P_1 R^2 \tan \Phi_1 \, d\theta - \int_{0}^{\theta'} P'_1 R^2 \tan \Phi \, d\theta$$

Figure 17. Correction for shear moment reduction due to pore pressure.

Tabulation of Equations

Eq. 1 $$M_D = \sum_{a=1}^{a=n} \frac{\gamma_a}{6}\left[3R^2\left(d_a - d_{a-1}\right) + \left(R - d_a\right)^3 - \left(R - d_{a-1}\right)^3\right]$$

Eq 2 $$M_R = \sum_{a=1}^{a=f'} \frac{\gamma_a}{6}\left[3R^2\left(d'_a - d'_{a-1}\right) + \left(R - d'_a\right)^3 - \left(R - d'_{a-1}\right)^3\right]$$

Eq 3 $$M_{DC} = \frac{\gamma_n}{6b}\left(X_L^3 - L^3\right)$$

Eq 4 $$M_{DC} = \frac{\gamma_n}{6b} X_L^3$$

Eq. 5 $$M_{RC} = -\frac{\gamma_n}{6b} L^3$$

Eq 6 $$M_s = R^2 \sum_{a=1}^{a=f'+a=n} C_a \left(\tan^{-1}\frac{X_a}{Y_a} - \tan^{-1}\frac{X_{a-1}}{Y_{a-1}}\right)$$

Eq 7 $$M_f = \sum_{a=1}^{a=f'+a=n} \frac{\tan\phi_a}{2}\left[W_a - \gamma_a\left(R - d_a\right)\right]\left[X_aY_a - X_{a-1}Y_{a-1} + R^2\left(\tan^{-1}\frac{X_a}{Y_a} - \tan^{-1}\frac{X_{a-1}}{Y_{a-1}}\right)\right]$$
$$+ \sum_{a=1}^{a=f'+a=n} \gamma_a \tan\phi_a \left[R^2\left(X_a - X_{a-1}\right) - \frac{1}{3}\left(X_a^3 - X_{a-1}^3\right)\right]$$

Tabulation of Equations (Continued)

Eq. 8
$$M_{sc} = \frac{\gamma_n}{2b}\tan\phi_1\left(X_L^2\sqrt{R^2-X_L^2} + X_L R^2 \tan^{-1}\sqrt{\frac{X_L^2}{R^2-X_L^2}}\right) + \frac{\gamma_n}{3b}\tan\phi_1\sqrt{(R^2-X_L^2)^3}$$
$$-\frac{\gamma_n}{2b}\tan\phi_1\left(1^2\sqrt{R^2-L^2} + R^2\sqrt{L^2}\tan^{-1}\sqrt{\frac{L^2}{R^2-L^2}}\right) - \frac{\gamma_n}{3b}\tan\phi_1\sqrt{(R^2-L^2)^3}$$

Eq. 9
$$M_{DC} = A_W\left(L + L_A\right) + \frac{1}{2}\gamma_A H L^2$$

Eq. 9A
$$\gamma_A = \frac{\gamma_x h_x + \gamma_y h_y + \gamma_z h_z + \text{etc.}}{h_x + h_y + h_z + \text{etc.}}$$

Eq. 9B
$$A_W = \gamma_x A_x + \gamma_y A_y + \gamma_z A_z + \text{etc.}$$

Eq. 9C
$$L_A = \frac{\gamma_x A_x L_x + \gamma_y A_y L_y + \gamma_z A_z L_z + \text{etc}}{\gamma_x A_x + \gamma_y A_y + \gamma_z A_z + \text{etc.}}$$

Eq. 10
$$M_P = R^2 \sum_{a=1}^{\substack{a=f'\\+\\a=n}} P_a \tan\phi_a\left(\tan^{-1}\frac{X_a}{Y_a} - \tan^{-1}\frac{X_{a-1}}{Y_{a-1}}\right)$$

Eq. 11
$$M_{PC} = \frac{R^2 F_1 \gamma_n \tan\phi_1}{b}\left(X_L \tan^{-1}\sqrt{\frac{X_L^2}{R^2-X_L^2}} + \sqrt{R^2-X_L^2} - \sqrt{L^2}\tan^{-1}\sqrt{\frac{L^2}{R^2-L^2}} - \sqrt{R^2-L^2}\right)$$