Mechanics of Continuously-Reinforced Concrete Pavements

M. M. MILLER, JR., Instructor in Civil Engineering, and
M. J. GUTZWILLER, Associate Professor of Civil Engineering, Purdue University

This investigation is primarily concerned with the deflections and the resulting stresses in a continuously-reinforced concrete pavement, loaded simultaneously with longitudinal and transverse loads. The theory developed for the computation of the deflections is based on the common assumptions used in the theory of continuous beams on elastic foundations, as well as three assumptions concerned with a cracked slab on an elastic foundation. The main assumptions in the first class are: (a) The deflection at a point some distance from the transverse load is zero, and (b) the subgrade modulus is constant throughout the full range of deflection. The assumptions peculiar to this problem are: (a) The cracks formed by volume changes in the pavement are equally spaced, (b) the segments between cracks are assumed to be straight, and (c) the moment at a crack is some function of the angle change (that is, $M/\phi = C$).

By considering the geometry of the deflected pavement and the equilibrium of the individual segments, a series of simultaneous equations may be written in terms of either the deflections or the angle changes at the cracks. Equations for shear and moment at the cracks are easily written.

The equations are in such a form as to consider any combination of pavement length, crack spacing, subgrade modulus, transverse load, and longitudinal load. In the solutions presented here, M/ϕ is considered equal to a constant, C. Experimentation is in progress to evaluate C for a series of pavement types.

THE THEORETICAL analysis of a pavement slab, either reinforced or unreinforced, is a highly complex problem. If all the possible physical characteristics of the loads, slab, and the support, that is, the subgrade, could be idealized, the remaining problem of computing deflections and stresses is very complicated if at all possible.

In the development of any basic theory for deflections and stresses in a pavement slab it is necessary to make certain assumptions as to the idealized nature of the physical characteristics of the problem. Having made these assumptions and having developed a theory based on the assumptions, it is necessary to verify by experimentation and observation the results of the theory. Any major discrepancies between theoretical and actual results must be accounted for by adjustments of basic theory. If it is not possible to reconcile the theory with actual conditions, then the theory must be discarded.

This is essentially the history of the analysis of concrete pavements. Early theories were unsuccessful at predicting with reasonable accuracy the stresses in a pavement slab and are of only historical value now (1).

Most of Westergaard's work on pavement slabs followed the pattern described earlier Westergaard developed equations for stresses in loaded pavement slabs, found that the computed stresses did not agree with experimental results and consequently modified his theory to bring about an agreement (2, 3, 4). The fundamental nature of his analysis for stresses is apparent in that the equations are of the form used in practically every other analysis, theoretical or empirical (5).

The development of a theory for continuously-reinforced concrete pavements will follow the same pattern. An idealization of the assumptions is necessary to arrive at a fundamental theory. The assumptions and the theory are then modified or adjusted to account for the variations from the ideal.

The first statement of the principles of continuously-reinforced concrete pavement was made in 1947 by W. R. Woolley who set forth the basic ideals governing crack spacing and steel stress (6). A number of field tests have been established to test some of his theories (7).. Many of the special problems of continuously-reinforced concrete pavements have been treated by Zuk (8). It is hoped that the theory developed in the following discussion will add to the fund of theoretical and experimental knowledge of continuously-reinforced concrete pavements.

PURPOSE AND SCOPE

The purpose of this paper is to develop a theory for the computation of stresses and deflections in continuously-reinforced pavements. A series of four pavement lengths varying from 20 to 50 ft, taken from the "middle" portion of a continuously-reinforced concrete pavement, are studied with respect to deflections, shears, and moments. Various combinations of load and subgrade support are used in the calculations.

The theory is intended to be a basic tool by which any combination of physical conditions in a continuously-reinforced concrete pavement may be represented mathematically. The great speed of calculation afforded by a digital computer will make it possible to represent the results of this theory in curve or table form for easy use by a highway engineer.

THEORETICAL DEVELOPMENT

This analysis of a continuously-reinforced pavement with transverse cracks occurring at intervals involves five major assumptions, as follows:

1. The transverse cracks occur in equally-spaced intervals;
2. The pavement is relatively straight between cracks;
3. The moment at a crack is some function of the angle change, that is, $M/\phi = C$;
4. The subgrade modulus is constant; and
5. The deflection at a point some distance from a transverse load is zero.

The first assumption is reasonably correct in that a pavement of more than a few hundred feet in length may be considered infinite in length and the middle portion, which is completely anchored against movement, will have more or less equally-spaced cracks. The second assumption is also quite correct. Observations on test specimens in the laboratory have shown that the slabs have very small curvatures between cracks. It would be expected that the cracks being relatively flexible compared to the pavement between cracks would account for most of the change in slope of the deflected slab. Assumption 3 is now the subject of laboratory research at Purdue University. Assumptions 4 and 5 are well-known assumptions used in most considerations of long beams on elastic foundations. In the following analysis of the slab an element 1 in. wide is considered.

A slab under the action of a transverse load will deflect until the sum of the subgrade reaction forces and the shearing forces on the ends of the slab equal the value of the load. Such a deflected configuration is assumed in Figure 1. The angle changes, ϕ_n, are shown in accordance with assumption 2. Deflections at each of the cracks are written in terms of the angle changes as follows:

$$\begin{aligned}
\Delta_1 &= \ell\phi_0 \\
\Delta_2 &= \Delta_1 + \ell(\phi_0 + \phi_1) = \ell(2\phi_0 + \phi_1) \\
\Delta_3 &= \Delta_2 + \ell(\phi_0 + \phi_1 + \phi_2) = \ell(3\phi_0 + 2\phi_1 + \phi_2) \\
\Delta_n &= \ell(n\phi_0 + (n-1)\phi_1 + (n-2)\phi_2 + \dots + \phi_{n-1})
\end{aligned} \tag{1}$$

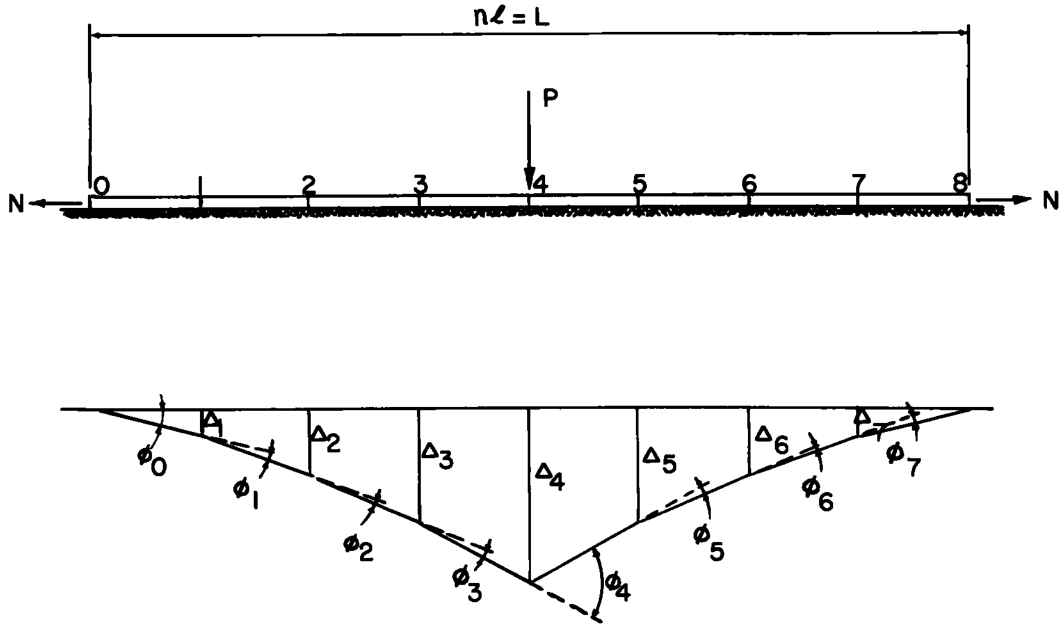

Figure 1. Typical deflected pavement.

The equations

$$\Delta_0 = 0;\ \Delta_n = 0 \qquad (2)$$

satisfy the requirements of assumption 5, and provide a necessary relationship in the final solution.

The total force exerted on each segment by the subgrade is equal to k times the volume of the subgrade displaced by that segment. Such a typical segment is shown in Figure 2 (a). The volume of the subgrade displaced is equal to the average of the end deflections of the segment times the length, ℓ. The resultant force is:

$$F_j = \ell/2\ (\Delta_j + \Delta_{j+1})k \qquad (3)$$

Figure 2 (b) shows the complete force system acting on a typical segment. The equilibrium of each segment, the vertical equilibrium of the slab as a whole, and the conditions of Eqs. 2 provide sufficient equations to solve for either the unknown deflections or the unknown angle changes. The unknown forces will be expressed in terms of the deflections, and then Eqs. 1 will be used to obtain the final form of the equations in terms of angle changes.

For simplicity the slab will be assumed to be loaded symmetrically, P being at the center line and a horizontal load N at each end. A certain number of segments, say eight, will be used for the purposes of definiteness. Because of the symmetry it is necessary only to consider half the length, the right half being a mirror image of the left half.

The equations for deflections are:

$$\begin{aligned}
\Delta_0 &= 0\\
\Delta_1 &= \ell\phi_0\\
\Delta_2 &= \ell(2\phi_0 + \phi_1)\\
\Delta_3 &= \ell(3\phi_0 + 2\phi_1 + \phi_2) \qquad (4)\\
\Delta_4 &= \ell(4\phi_0 + 3\phi_1 + 2\phi_2 + \phi_3)\\
\Delta_8 &= \ell(8\phi_0 + 7\phi_1 + 6\phi_2 + 5\phi_3\\
&\quad + 4\phi_4 + 3\phi_5 + 2\phi_6 + \phi_7) \text{ or}
\end{aligned}$$

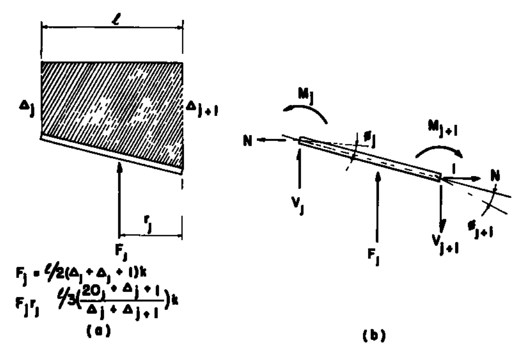

Figure 2. Forces on a typical segment.

$$\Delta_8 = \ell(8\phi_0 + 8\phi_1 + 8\phi_2 + 8\phi_3 + 4\phi_4)$$

The equations for the shears at each crack are:

$$V_0 = V_0$$
$$V_1 = V_0 + \ell k \Delta_1/2$$
$$V_2 = V_0 + \ell k (\Delta_1 + \Delta_2/2) \qquad (5)$$
$$V_3 = V_0 + \ell k (\Delta_1 + \Delta_2 + \Delta_3/2)$$
$$V_4 = V_0 + \ell k (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4/2)$$

The equilibrium of each segment taken individually yields the following equations:

$$M_1 + V_0\ell + F_1r_1 - M_0 - N\Delta_1 = 0$$
$$M_2 + V_1\ell + F_2r_2 - M_1 - N(\Delta_2 - \Delta_1) = 0$$
$$M_3 + V_2\ell + F_3r_3 - M_2 - N(\Delta_3 - \Delta_2) = 0 \qquad (6')$$
$$M_4 + V_3\ell + F_4r_4 - M_3 - N(\Delta_4 - \Delta_3) = 0$$

The form of the final equations is somewhat simplified if the following approximation is made. By substituting $V_n = V'_n - \frac{N\Delta n}{\ell}$ into Eqs. 6, they become:

$$M_1 + V'_0\ell + F_1r_1 - M_0 - N\Delta_1 = 0$$
$$M_2 + V'_1\ell + F_2r_2 - M_1 - N\Delta_2 = 0$$
$$M_3 + V'_2\ell + F_3r_3 - M_2 - N\Delta_3 = 0 \qquad (6)$$
$$M_4 + V'_3\ell + F_4r_4 - M_3 - N\Delta_4 = 0$$

The values for V_n may be calculated when the solution for the deflections is obtained.

In Eqs. 5, setting $V_4 = P/2$ and solving for V_0 yields:

$$V_0 = P/2 - \ell k (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4/2) \qquad (7)$$

Substitution of Eqs. 7 and 4 into the remaining Eqs. 5 and 6 yields the following expressions in terms of the ϕ's alone:

$$(47/6\ell^2k + N + C/\ell)\phi_0 + (4.5\ell^2k - C/\ell)\phi_1 + \ell^2k(2\phi_2 + 0.5\phi_3) = P/2$$
$$(41/6\ell^2k + 2N)\phi_0 + (13/3\ell^2k + N + C/\ell)\phi_1 + (2\ell^2k - C/\ell)\phi_2 + 0.5\ell^2k\phi_3 = P/2$$
$$(29/6\ell^2k + 3N)\phi_0 + (10/3\ell^2k + 2N)\phi_1 + (11/6\ell^2k + N + C/\ell)\phi_2 + (0.5\ell^2k - C/\ell)\phi_3 = P/2 \qquad (8)$$
$$(11/6\ell^2k + 4N)\phi_0 + (4/3\ell^2k + 3N)\phi_1 + (5/6\ell^2k + 2N)\phi_2 + (\ell^2k/3 + N + C/\ell)\phi_3 = P/2$$

In Eqs. 4, setting $\Delta_8 = 0$ yields:

$$\phi_0 + \phi_1 + \phi_2 + \phi_3 + 0.5\phi_4 = 0 \qquad (9)$$

This equation along with Eqs. 8 gives five equations with five unknowns which can be solved simultaneously.

In these equations the coefficients may be calculated after a choice is made of the values for the parameters ℓ, k, and N. The parameter C need not be a constant but might be some function of ϕ itself. In this instance the equations might be non-linear in terms of ϕ but would still yield a solution under normal physical conditions. The solutions discussed here are based on C equal to a constant for lack of better information.

The equations for the cases of 20, 16, and 12 segments of equal length are given in Appendix A.

Due to the inherent symmetry of an infinitely long pavement it is unnecessary to consider a case where P is not at the center crack. However, for finite slabs with N = 0, this possibility might be critical. In this case a similar set of equations may be written involving all of the angle changes and deflections as well as two different end shears, V_0 and V_n. A more formal statement of the equilibrium and boundary conditions makes this problem clearer.

Consider a slab with n segments, and a load P at the jth point. Consider also as unknowns the deflections, (n + 1) in number; and the two end shears, V_0 and V_n. This is a total of (n + 3)unknowns. The equilibrium condition of each segment yields n equations. The two boundary conditions $\Delta_0 = 0$, $\Delta_n = 0$ yield two more equations. The last equation which is necessary is obtained by considering the shear condition at the point of load. The condition is that the numerical sum of the shears to the left and to the right must equal the applied loads or:

$$\left| V_j \text{ (right)} \right| + \left| V_j \text{ (left)} \right| = P \qquad (10)$$

By substituting appropriately, the results obtained previously for the symmetrical case may be verified and similar equations may be obtained for the unsymmetrical case.

Although the equations are quite simple when the crack spacing, ℓ, is considered a constant, they are not much more complicated if some other arrangement of cracks is assumed. With a given crack distribution, either symmetrical or unsymmetrical, it would be necessary only to express each interval as some multiple of a unit length and carry these multipliers along in the equations.

RESULTS

The equations which are listed in Appendix A were solved for the combinations of parameters shown in Table 1. In each case P equals 250 lb per in. and ℓ equals 30 in.

The six combinations are used for each of the four slab lengths: namely, 8ℓ, 12ℓ, 16ℓ, and 20ℓ, giving a total of 24 different solutions.

The curves in Figures 3 through 8 show the results of a slab of length 8ℓ. Figures 9 and 10 show partial results for a slab of length 20ℓ. In each of the figures are shown the deflection, shear, and moment diagrams.

A number of interesting observations can be made concerning the results of the computations. The most obvious is the reduction in maximum deflection and maximum moment with an increase in subgrade modulus k. However, the relationship among the maximum deflection, maximum moment, and the subgrade modulus cannot be deduced without more computations.

The presence of the horizontal load N, has a slight effect on the deflections but practically no effect on the moments. The effect of higher values of N on the deflections and moments might be more pronounced, but again to determine this effect will require more computations.

The general shape of the diagrams presented agrees very well with the exact solution for a continuous beam on an elastic foundation (9). The characteristic vanishing of the deflection, shear, and moment at points of increasing distance from the point of application of load is evident. The fact that the curves, in addition to being deflection, shear, and moment diagrams for the fixed position of load are also influence lines for deflection, shear, and moment at a point is useful for the consideration of more than one load.

Figures 9 and 10 are shown for comparison with Figures 3 and 5, respectively. The purpose for the comparison of these two cases is to show specifically a fact that is true generally: namely, that only a small number of crack intervals need be considered in the solution. The use of a number of crack intervals larger than, for example, 10 to 20, depending possibly on the length of the crack interval, will yield no additional information, although it will in-

TABLE 1

Combination No.	C-inch-lbs/ inch/rad.	N-lbs/ inch	k-lbs/ cubic inch
1	2.5×10^6	1,000	150
2	2.5×10^6	1,000	440
3	2.5×10^6	0	150
4	2.5×10^6	0	440
5	0	1,000	150
6	0	1,000	440

DEFLECTION, SHEAR, AND MOMENT DIAGRAMS—SLAB OF LENGTH 8ℓ

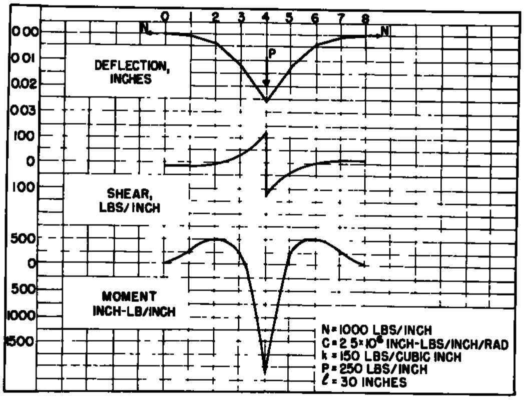

Figure 3.

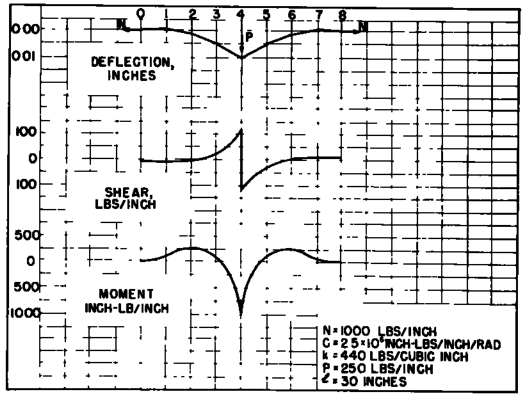

Figure 4.

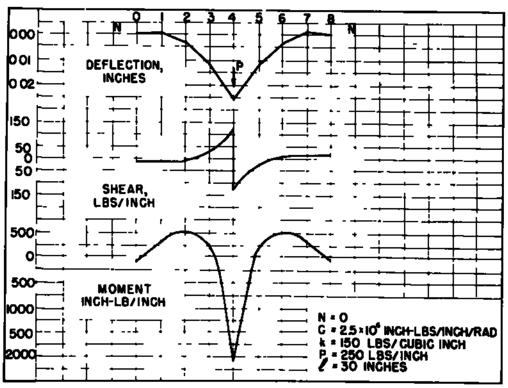

Figure 5.

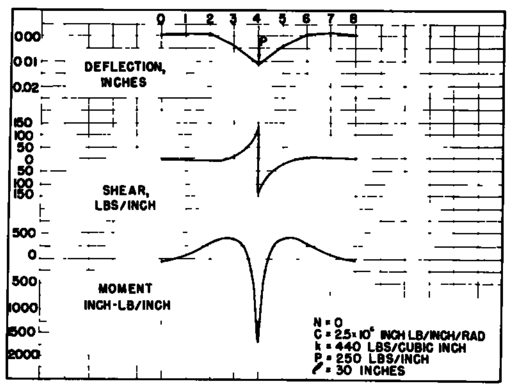

Figure 6.

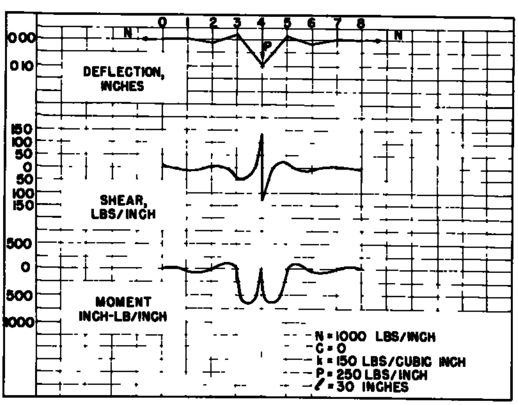

Figure 7.

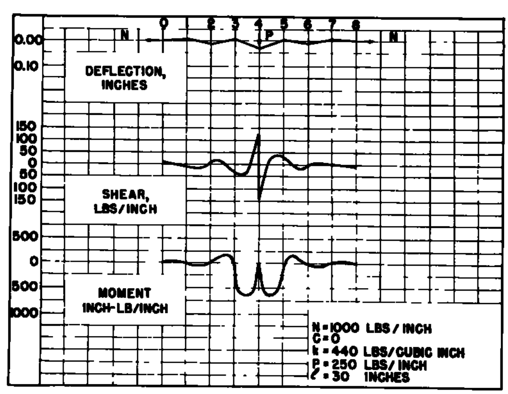

Figure 8.

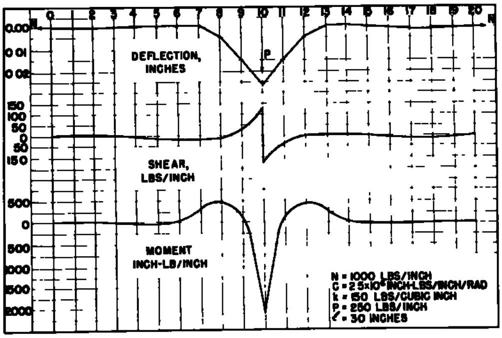

Figure 9.

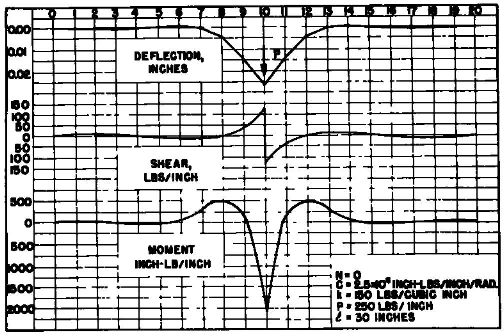

Figure 10.

crease the precision slightly. The optimum number of crack intervals in relationship to the crack spacing is still to be worked out. It seems unreasonable that the consideration of more than 20 intervals would ever be necessary to obtain an accurate solution in the region of the load.

The curves in Figures 7 and 8 are for the case where C = 0. In this case the only interaction between the segments is shear transfer across the crack. This case is likely to arise when, in a continuously-reinforced pavement, the crack width or opening is so large as to prevent the two concrete faces from coming into contact and thus provide resistance to moment. Such a condition in an actual pavement is very near if not at the state of failure. It is of interest here since it is the extreme possibility of the moment-angle change relationship.

The unusual moment diagrams in Figures 7 and 8 are the moment diagrams for the individual segments when the moments at the cracks equal zero (C = 0). Each segment is somewhat like a simply-supported beam loaded with a varying distributed load from the subgrade. The resulting moment diagrams are quite reasonable when considered in this manner.

CONCLUSIONS

Many questions must be answered before the method of analysis developed in this research can be applied to the design of continuously-reinforced concrete pavements.

1. What is the moment-angle change relationship for a wide range of pavement design variables (thickness of concrete, position of reinforcement, percentage of reinforcement)?
2. What is the necessary number of segments to be considered for the necessary accuracy with a given crack spacing?
3. What is the effect of high values of N on the moments?
4. What is the effect of the plate action of each segment?
5. What is the effect of repeated loads on the pavement design?
6. What is the optimum percentage and best position of steel for given loads and field conditions?

The first question is susceptible to laboratory investigation. Experimentation with relatively small specimens which contain only one crack will eliminate some variables which obscure the nature of the moment-angle change relationship. A specimen may be tested by the application at the crack of pure moment, pure shear, or a combination of moment and shear. Longitudinal loads to control crack openings may also be applied. Measurements of the relative angle changes between the two segments formed by the crack may be made. The relative vertical displacement of the segments may also be measured. By varying the percentage and position of the reinforcement as well as the loads on the crack, complete information about crack behavior can be obtained. Such an investigation is now in progress at Purdue University.

The answers to questions 2 and 3 can be readily obtained when the moment-angle change relationship has been established. In this investigation the optimum number of crack intervals is 8 using a crack spacing of 30 in., while it may be some other number for other values of ℓ. By the use of the Purdue University Digital Computer a large number of solutions may be easily worked out. The solutions may be programed in such a manner as to provide the final values of moments, shears, deflections, and stresses. With such a computer available it is possible to consider in a relatively short time the large range of parameters necessary to answer these questions.

In this investigation the pavement is treated as a beam 1 in. wide. In the actual pavement the cracks divide the slab into a series of transverse strips. The action of these strips under the load must be investigated. For rather closely-spaced cracks the strip acts somewhat as a beam, while the larger spacings of cracks yield segments which act more like plates. The effects of this plate action must be determined and appropriate modifications must be incorporated in the final design procedure.

The effect of repeated loads on the pavement design can be investigated in the laboratory. Repeated application of loads in the research outlined above would provide information on the fatigue characteristics of various designs. This information can be combined with traffic surveys for proposed highways in the final design of the pavement.

The answer to the last question is really the ultimate purpose of all these investigations. What combinations of slab thickness, percentage of reinforcement, and position of reinforcement will support the given load in a given field condition? The existence of an answer to this question implies the existence of a design procedure which can account for all of the variables in the problem. It is obviously very difficult experimentally to account for each variable separately and then combine their effects to arrive at a design for each given set of conditions. Perhaps the best design procedure then, is one which utilizes the results of verified mathematical solutions. The results may be presented in table and chart form so that engineers may design safe and economical continuously-reinforced concrete pavements. It is sincerely hoped that this research will help make this possibility a reality.

ACKNOWLEDGMENTS

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Appendix A

The general equations for the solution for the values of the angle changes in the cases of L equal to 12ℓ, 16ℓ, and 20ℓ, respectively, are as follows:

$L = 12\ell$

$$(107/6\ell^2k + N + C/\ell)\phi_o + (12.5\ell^2k - C/\ell)\phi_1 +$$
$$\ell^2k(8\phi_2 + 4.5\phi_3 + 2\phi_4 + 0.5\phi_5) = P/2$$

$$(101/6\ell^2k + 2N)\phi_o + (37/3\ell^2k + N + C/\ell)\phi_1 + (8\ell^2k - C/\ell)\phi_2 +$$
$$\ell^2k(4.5\phi_3 + 2\phi_4 + 0.5\phi_5) = P/2$$

$$(89/6\ell^7k + 3N)\phi_o + (34/3\ell^2k + 2N)\phi_1 + (47/6\ell^2k + N + C/\ell)\phi_2 +$$
$$(4.5\ell^2k - C/\ell)\phi_3 + \ell^2k(2\phi_4 + 0.5\phi_5) = P/2$$

$$(71/6\ell^2k + 4N)\phi_o + (28/3\ell^2k + 3N)\phi_1 + (41/6\ell^2k + 2N)\phi_2 +$$
$$(13/3\ell^2k + N + C/\ell)\phi_3 + (2\ell^2k - C/\ell)\phi_4 + 0.5\ell^2k\phi_5 = P/2$$

$$(47/6\ell^2k + 5N)\phi_o + (19/3\ell^2k + 4N)\phi_1 + (29/6\ell^2k + 3N)\phi_2 +$$
$$(10/3\ell^2k + 2N)\phi_3 + (11/6\ell^2k + N + C/\ell)\phi_4 + (0.5\ell^2k - C/\ell)\phi_5 = P/2$$

$$(17/6\ell^2k + 6N)\phi_o + (7/3\ell^2k + 5N)\phi_1 + (11/6\ell^2k + 4N)\phi_2 +$$
$$(4/3\ell^2k + 3N)\phi_3 + (5/6\ell^2k + 2N)\phi_4 + (0.5\ell^2k + N + C/\ell)\phi_5 = P/2$$

$$\phi_o + \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + 0.5\phi_6 = 0$$

$L = 16\ell$

$$(191/6\ell^2k + N + C/\ell)\phi_o + (24.5\ell^2k - C/\ell)\phi_1 + \ell^2k(18\phi_2 + 12.5\phi_3 +$$
$$8\phi_4 + 4.5P_5 + 2\phi_6 + 0.5\phi_7) = P/2$$

$$(185/6\ell^2k + 2N)\phi_o + (73/3\ell^2k + N + C/\ell)\phi_1 + (18\ell^2k - C/\ell)\phi_2$$
$$+ \ell^2k(12.5\phi_3 + 8\phi_4 + 4.5\phi_5 + 2\phi_6 + 0.5_7) = P/2$$

$$(173/6\ell^2k + 3N)\phi_o + (70/3\ell^2k + 2N)\phi_1 + 107/6\ell^2k + N + C/\ell)\phi_2 +$$
$$(12.5\ell^2k - C/\ell)\phi_3 + \ell^2k(8\phi_4 + 4.5\phi_5 + 2\phi_6 + 0.5\phi_7) = P/2$$

$$(155/6\ell^2k + 4N)\phi_o + (64/3\ell^2k + 3N)\phi_1 + (101/6\ell^2k + 2N)\phi_2 + (37/3\ell^2k +$$
$$N + C/\ell_3) + (8\ell^2k - C/\ell)\phi_4 + \ell^2k(4.5\phi_5 + 2\phi_6 + 0.5\phi_7) = P/2$$

$$(141/6\ell^2k + 5N)\phi_o + (55/3\ell^2k + 4N)\phi_1 + (89/6\ell^2k + 3N)\phi_2 +$$
$$(34/3\ell^2k + 2N)\phi_3 + (47/6\ell^2k + N + C/\ell)\phi_4 + (4.5\ell^2k - C/\ell)\phi_5 +$$
$$\ell^2k(2\phi_6 + 0.5\phi_7) = P/2$$

$$(101\ell^2k + 6N)\phi_o + (43/3\ell^2k + 5N)\phi_1 + (71/6\ell^2k + 4N)\phi_2 +$$
$$(28/3\ell^2k + 3N)\phi_3 + (41/6\ell^2k + 2N)\phi_4 + (13/3\ell^2k + N + C/\ell)\phi_5 +$$
$$(2\ell^2k - C/\ell)\phi_6 + 0.5\ell^2k\phi_7 = P/2$$

$$(65/6\ell^2k + 7N)\phi_o + (28/3\ell^2k + 6N)\phi_1 + (47/6\ell^2k + 5N)\phi_2 +$$
$$(19/3\ell^2k + 4N)\phi_3 + (29/6\ell^2k + 3N)\phi_4 + (10/3\ell^2k + 2N)\phi_5 +$$
$$(11/6\ell^2k + N + C/\ell)\phi_6 + (0.5\ell^2k - C/\ell)\phi_7 = P/2$$

$$(23/6\ell^2k + 8N)\phi_o + (10/3\ell^2k + 7N)\phi_1 + (17/6\ell^2k + 6N)\phi_2 +$$
$$(7/3\ell^2k + 5N)\phi_3 + (11/6\ell^2k + 4N)\phi_4 + (4/3\ell^2k + 3N)\phi_5 +$$
$$(5/6\ell^2k + 2N)\phi_6 + (0.5\ell^2k + N + C/\ell)\phi_7 = P/2$$

$$\phi_o + \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6 + \phi_7 + 0.5\phi_8 = 0$$

$L = 20\ell$

$$(299/6\ell^2k + N + C/\ell)\phi_o + (40.5\ell^2k - C/\ell)\phi_1 + \ell^2k(32\phi_2 + 24.5\phi_3 +$$
$$18\phi_4 + 12.5\phi_5 + 8\phi_6 + 4.5\phi_7 + 2\phi_8 + 0.5\phi_9) = P/2$$

$$(293/6\ell^2k + 2N)\phi_o + (121/3\ell^2k + N + C/\ell)\phi_1 + (32\ell^2k - C/\ell)\phi_2 +$$
$$\ell^2k(24.5\phi_3 + 18\phi_4 + 12.5\phi_5 + 8\phi_6 + 4.5\phi_7 + 2\phi_8 + 0.5\phi_9) = P/2$$

$$(281/6\ell^2k + 3N)\phi_0 + (118/3\ell^2k + 2N)\phi_1 + (191/6\ell^2k + N + C/\ell)\phi_2 + (24.5\ell^2k - C/\ell)\phi_3 + \ell^2k\,(18\phi_4 + 12.5\phi_6 + 18\phi_6 + 4.5\phi_7 + 2\phi_8 + 0.5\phi_9) = P/2$$

$$(263/6\ell^2k + 4N)\phi_0 + (112/3\ell^2k + 3N)\phi_1 + (185/6\ell^2k + 2N)\phi_2 + (73/3\ell^2k + N + C/\ell)\phi_3 + (18\ell^2k - C/\ell)\phi_4 + \ell^2k(12.5\phi_5 + 8\phi_6 + 4/5\phi_7 + 2\phi_8 + 0.5\phi_9) = P/2$$

$$(239/6\ell^2k + 5N)\phi_0 + (103/3\ell^2k + 4N)\phi_1 + (173/6\ell^2k + 3N)\phi_2 + (70/3\ell^2k + 2N)\phi_3 + (107/6\ell^2k + N + C/\ell)\phi_4 + (12.5\ell^2k - C/\ell)\phi_5 + \ell^2k(8\phi_6 + 4.5\phi_7 + 2\phi_8 + 0.5\phi_9) = P/2$$

$$(209/6\ell^2k + 6N)\phi_0 + (91/3\ell^2k + 5N)\phi_1 + (155/6\ell^2k + 4N)\phi_2 + (64/3\ell^2k + 3N)\phi_3 + (101/6\ell^2k + 2N)\phi_4 + (38/3\ell^2k + N + C/\ell)\phi_5 + (8\ell^2k - C/\ell)\phi_6 + \ell^2k(45\phi_7 + 2\phi_8 + 0.5\phi_9) = P/2$$

$$(173/6\ell^2k + 7N)\phi_0 + (76/3\ell^2k + 6N)\phi_1 + (141/6\ell^2k + 5N)\phi_2 + (55/3\ell^2k + 4N)\phi_3 + (89/6\ell^2k + 3N)\phi_4 + (34/3\ell^2k + 2N)\phi_5 + (47/6\ell^2k + N + C/\ell)\phi_6 + (4.5\ell^2k - C/\ell)\phi_7 + (2\phi_8 + 0.5\phi_9) = P/2$$

$$(131/6\ell^2k + 8N)\phi_0 + (58/3\ell^2k + 7N)\phi_1 + (101/6\ell^2k + 6N)\phi_2 + (43/3\ell^2k + 5N)\phi_3 + (71/6\ell^2k + 4M)\phi_4 + (28/3\ell^2k + 3N)\phi_5 + (41/6\ell^2k + 2N)\phi_6 + (13/3\ell^2k + N + C/\ell)\phi_7 + (2\ell^2k - C/\ell)\phi_8 + 0.5\ell^2k\phi_9 = P/2$$

$$(83/6\ell^2k + 9N)\phi_0 + (37/3\ell^2k + 8N)\phi_1 + (65/6\ell^2k + 7N)\phi_2 + (28/3\ell^2k + 6N)\phi_3 + (47.6\ell^2k + 5N)\phi_4 + (19/3\ell^2k + 4N)\phi_5 + (29/6\ell^2k + 3N)\phi_6 + (10/3\ell^2k + 2N)\phi_7 + (11/6\ell^2k + N + C/\ell)\phi_8 + (0.5\ell^2k - C/\ell)\phi_9 = P/2$$

$$(29/6\ell^2k + 10N)\phi_0 + (13/3\ell^2k + 9N)\phi_1 + (23/6\ell^2k + 8N)\phi_2 + (10/3\ell^2k + 7N)\phi_3 + (17/6\ell^2k + 6N)\phi_4 + (7/3\ell^2k + 5N)\phi_5 + (11/6\ell^2k + 4N)\phi_6 + (4/3\ell^2k + 3N)\phi_7 + (5/6\ell^2k + 2N)\phi_8 + (0.5\ell^2k + N + C/\ell)\phi_9 = P/2$$

$$\phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6 + \phi_7 + \phi_8 + \phi_9 + 0.5\phi_{10} = 0$$