Analysis of Viscoelastic Flexible Pavements

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To develop a better understanding of flexible pavement behavior it is believed that components of the pavement structure should be considered to be viscoelastic rather than elastic materials. In this paper, a step in this direction is taken by considering the asphalt concrete surface course as a viscoelastic plate on an elastic foundation.

Assumptions underlying existing stress and deformation analyses of pavements are examined. Representations of the flexible pavement structure, impressed wheel loads, and mechanical properties of asphalt concrete and base materials are discussed. Recent results pointing toward the viscoelastic behavior of asphaltic mixtures are presented, including effects of strain rate and temperature.

Using a simplified viscoelastic model for the asphalt concrete surface course, solutions for typical loading problems are given.

THE THEORY of viscoelasticity is concerned with the behavior of materials which exhibit time-dependent stress-strain characteristics. The principles of viscoelasticity have been successfully used to explain the mechanical behavior of high polymers and much basic work (1) has been developed in this area. In recent years the theory of viscoelasticity has been employed to explain the mechanical behavior of asphalts (2, 3, 4) and to a very limited degree the behavior of asphaltic mixtures (5, 6, 7) and soils (8). Because these materials have time-dependent stress-strain characteristics and because they comprise the flexible pavement section, it seems reasonable to analyze the flexible pavement structure using viscoelastic principles. This type of analysis should develop a better understanding of the behavior of flexible pavements, beyond that developed using elastic concepts.

Although this type of theoretical analysis will not result in a simple, universally applicable design formula, it can form the basis for a better understanding of flexible pavements subjected to loading conditions for which there is no precedent. For example, doubt exists among some designers that flexible pavements can accommodate heavy wheel loads and high tire pressures. If there existed a better understanding of the mechanics of behavior of the materials comprising the flexible pavement system, qualitative at least, then perhaps a definitive answer could be established.

It is hoped that this paper provides a step in this direction by illustrating some data pointing to the viscoelastic behavior of asphaltic mixtures and by illustrating the solution of typical loading problems.

BRIEF REVIEW OF FLEXIBLE PAVEMENT ANALYSIS

An analysis of flexible pavements with the intent of establishing a rational design procedure might include the following factors: (a) determination of mechanical properties of the components of the pavement system, (b) development of suitable methods of stress and deformation analysis, (c) identification of space and time characteristics of impressed loads and environmental conditions, and (d) establishment of appropriate failure criteria.

Although all the above factors are interrelated, factors (a) and (b) are perhaps most intimately connected. For example, the characteristic differences between the theories of elasticity, plasticity and viscoelasticity arise from differences in choice of a law relating stress and strain. To date most of the work related to the structural analysis of flexible pavements has been concerned with elastic theory of material behavior. This is not surprising because from a mathematical viewpoint this type of model of the mechanical properties of real materials is the simplest. As pointed out elsewhere in this paper, the inadequacy of the purely elastic model for asphalts and asphaltic mixtures has been demonstrated.

From the standpoint of structural analysis the concentrated normal force or pressure uniformly distributed over a circular area forms the simplest geometric idealization of an actual wheel load. Other space distributions (ellipse, tandem loads, etc.) can be included at the expense of computational effort. It appears, however, that little attention has been given to the effect of shear forces transmitted to the pavement surface, an effect which would seem to be particularly important during starting and stopping of vehicles. The effects of frequency and duration of load application as well as repetition of load have infrequently been considered. These factors are particularly significant for materials with frequency-dependent response mechanisms, such as asphaltic mixtures. Finally, it may be noted that the magnification effect on stress and deformation produced by moving loads does not seem to be significant at present, although substantial increases in landing speeds of aircraft coupled with increased wheel loads and pavement thicknesses could make this problem important. Accordingly, a quasi-static treatment of stress and deformation analysis is possible.

In formulating a specific boundary value problem for analytical solution, various levels of approach can be followed. The highest level (that is, fewest assumptions made) involves the treatment of the pavement system as a multi-layer continuous solid. This approach has been taken by Burmister (9), for example, in the case of an elastic material. The computational difficulties associated with this approach are quite severe, and it is difficult to obtain the effect of changes of various parameters of the system. A second level of approach, first applied to pavement analysis by Westergaard, treats the pavement as a plate on an elastic (set of independent springs) foundation. While simplifying the mathematics considerably, this method suffers from the disadvantage that stresses in the subgrade (elastic foundation) cannot be determined and further, that transverse normal and shearing stresses in the plate cannot be determined. Pickett (10) has extended Westergaard's work to permit determination of subgrade stresses. This is done by replacing the set of independent elastic springs (or equivalently, a dense liquid) by an elastic solid; however, the mathematics in this solution is also rather involved. A further approach (11) retains the simplification of the dense liquid subgrade but allows for the effect of transverse normal and shearing stresses. Results show that this effect is singificant in the range for which the ratio of radius of loaded area to pavement thickness unity.

It should be noted that each of the previously mentioned levels of approach has been applied only in the case of elastic material properties. (A recent paper by Hoskins and Lee (15) is an exception.) Extension of these results in part for application to viscoelastic materials is possible through the use of a correspondence principle due to Lee (12). Finally, it may be noted that the concept of the dense liquid subgrade has been generalized by Reissner (13) to allow for differential shear stiffness in the subgrade, a more realistic model than the independent spring idea. Some advantages of this approach for viscoelastic materials have been discussed by Pister and Williams (14).

While maximum stress or limiting strain (or deflection) theories of failure have frequently been applied as design criteria for flexible pavements, it is believed that these theories are inadequate for application to viscoelastic flexible pavements. A theory incorporating environmental effects on material properties as well as rate and repetition of loading and accumulation of deformation is needed. A preliminary step in this direction, applied to the related problem of failure of high polymer solid propellants, has been taken (16).

VISCOELASTIC BEHAVIOR OF ASPHALTIC MIXTURES

As stated previously, the theory of viscoelasticity is concerned with the relation between stress as a function of time and strain as a function of time. To illustrate this behavior conveniently, several mathematical models have been proposed. Depending on the imposed conditions of stress or strain and time, these models can be related to the actual behavior of a particular material. Various models are discussed in the following paragraphs. To illustrate their suitability to describe the behavior of asphaltic mixtures, typical results of tests on a particular asphaltic mixture are presented. Data for the components of this mixture have been described elsewhere (20) and are therefore not included in this paper.

Figure 1a shows a Maxwell element, incorporating in series a spring (representing elastic behavior with a modulus E_1) and a dashpot (representing viscous behavior with a viscosity η_1).

This model represents a material which when subjected to stress, undergoes an instantaneous elastic deformation together with deformation increasing with time. The model can also be used to represent a material exhibiting relaxation of stress with time when the material is held at constant deformation. This type of behavior is shown in Figure 1b. When the model is quickly deformed to a strain ϵ_1 and then constrained so that ϵ_1 remains constant, the stress will gradually relax with time. The differential equation relating stress and strain for the Maxwell element is:

$$\frac{d\epsilon}{dt} = \frac{1}{\eta_1} \sigma + \frac{1}{E_1} \frac{d\sigma}{dt} \tag{1}$$

in which σ is the applied axial stress. For the condition described above, $\frac{d\epsilon}{dt} = 0$ and the solution of the equation is

$$\sigma = \sigma_0 \exp(-\frac{E_1}{\eta_1} t) \tag{2}$$

in which σ_0 = initial stress for the material deformed to strain ϵ_1; and

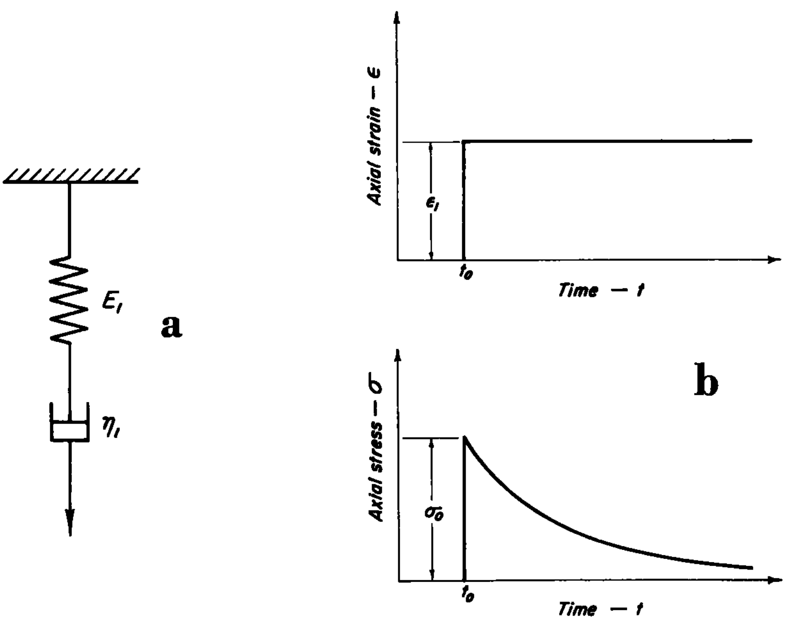

Figure 1. Representing (a) Maxwell model, and (b) relaxation of stress with time for simple Maxwell model.

σ = stress at any time, t, during relaxation of stress at constant strain. This equation indicates that the stress relaxes exponentially with time. The ratio $\frac{\eta_1}{E_1}$ has the dimensions of time and is called the relaxation time for the material, and designated by the symbol, τ. Thus the equation can be written:

$$\sigma = \sigma_0 \ \exp\left(-\frac{t}{\tau}\right) \quad \text{or} \quad \sigma = E_1 \ \epsilon_1 \exp\left(-\frac{t}{\tau}\right) \tag{2a}$$

It may be noted that τ represents the time required for the initial stress to decrease to the value $\frac{\sigma_0}{e}$. For asphaltic mixtures, there is evidence to show that a simple model is not sufficient to describe the behavior in stress relaxation. It is necessary to couple either a finite or an infinite number of Maxwell models in parallel. This type of model (shown schematically in Figure 2) is called a generalized Maxwell model. If the number of elements is allowed to approach infinity, it may be assumed that the material has a continuous distribution of relaxation time, that is, E is a continuous function of τ and the equation representing the relaxation of stress at constant strain becomes

$$\sigma = \epsilon \int_0^\infty E(\tau) \exp\left(-\frac{t}{\tau}\right) d\tau \tag{3}$$

Data from a stress relaxation test on an asphaltic mixture are shown in Figure 3. The general pattern of stress relaxation is an exponential decay with time. Thus, a Maxwell type model would probably be suitable for conditions pertaining to stress relaxation with time.

To illustrate behavior of an asphaltic mixture in creep and creep recovery, use can be made of a Burgers' model (Fig. 4a). The Burgers' model has the advantage that it displays under load, instantaneous elastic deformation, retarded elastic deformation and plastic or viscous deformation (Fig. 4b). Brown and Sparks (3) have found that this type of model defines the behavior of certain paving asphalts in creep and creep recovery experiments. In their work it was necessary to couple in series four Kelvin elements (spring and dashpot in parallel) rather than the one shown in Figure 4a. A similar type of model could be fitted to the data shown in Figure 5 for asphaltic mixtures in creep and creep recovery experiments.

The difficulty with application of the Burgers' model in this case, however, is that the instantaneous elastic recovery in the specimens is different from the instantaneous elastic deformation under load. Moreover, data for specimens subjected to various amounts of creep (not included in this paper) exhibited different amounts of elastic recovery. For the Burgers' model, the elastic recovery is the same regardless of

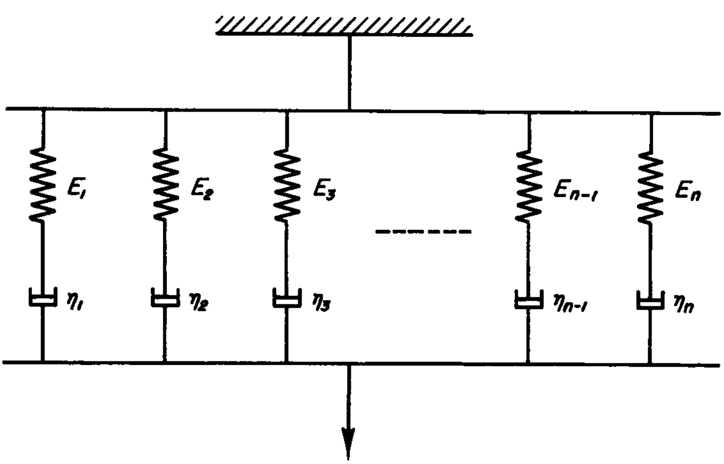

Figure 2. Generalized Maxwell model.

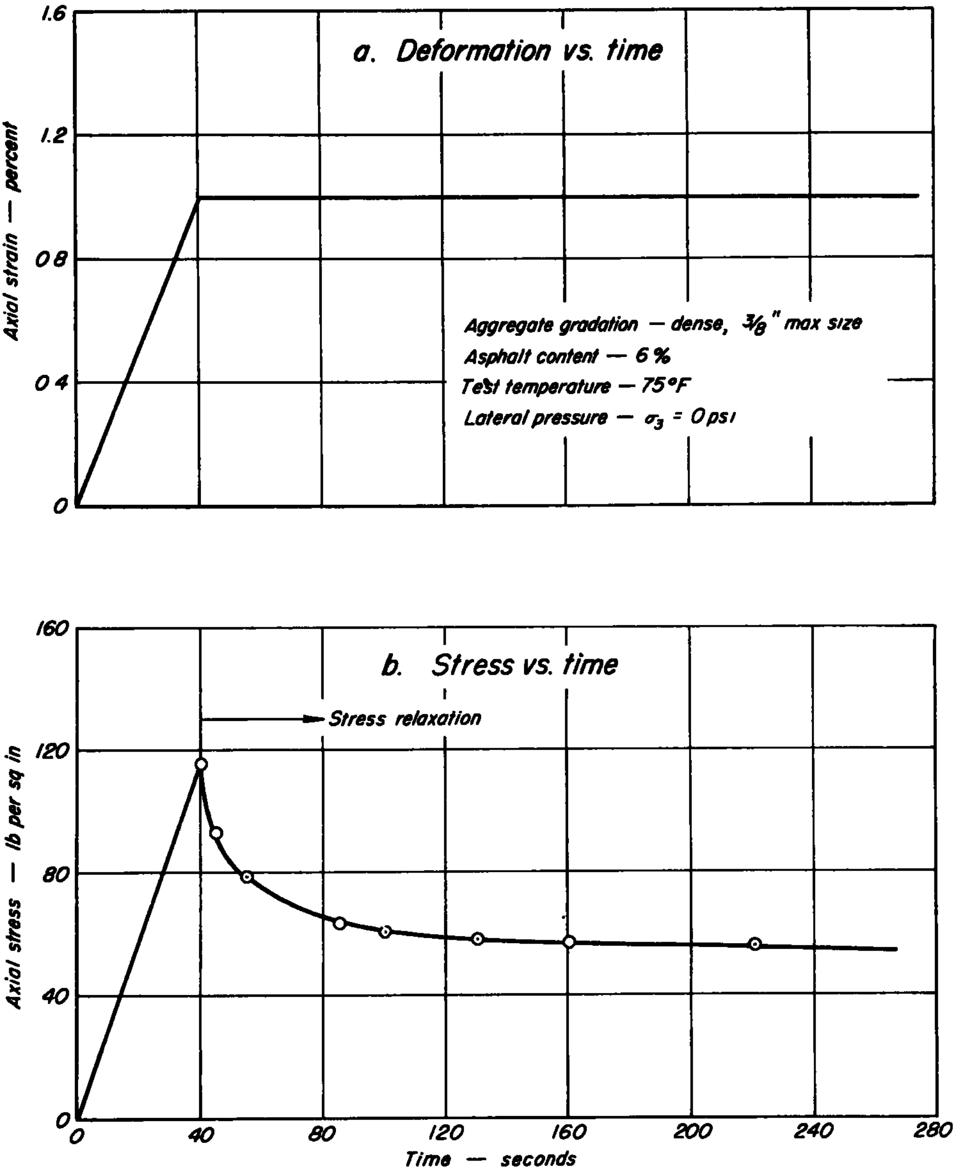

Figure 3. Stress relaxation with time—asphaltic mixture.

the time over which the specimen is allowed to deform under load. Hence it is believed that a model similar to the Burgers' model but incorporating this variability of elastic recovery with time would be more appropriate.

A model depicting such behavior is shown in Figure 6. It may be noted that this model was also suggested by Kühn and Rigden (4) for asphalts. The behavior of this model will be discussed in the section containing examples of stress analysis.

To illustrate that a generalized model can be used to represent the behavior of an asphaltic mixture under load, data from a series of triaxial compression tests at different rates of loading and temperatures of test on the same mixtures (Figs. 3 and 5) are shown in Figures 7 and 8. The technique for analysis and presentation is based on the treatment developed by Smith (17).

For a generalized Maxwell model (Fig. 2) subjected to a constant rate of strain, R, the relationship between stress, strain and time is

$$\sigma = R \int_0^\infty \tau\, E(\tau) \left[1 - \exp\left(-\frac{\epsilon}{R\tau}\right) \right] d\tau \tag{4}$$

Smith (17) has made use of the generalized Maxwell model in his studies of the visco-elastic behavior of a high polymer, polyisobutylene, under a constant rate of elongation in simple tension. In his presentation Smith has rewritten the foregoing equation into an equivalent form for more convenient analysis of the test data. The equivalent equation is

$$\frac{\sigma}{R} = \int_{-\infty}^{\infty} M(\tau)\, \tau \left[1 - \exp\left(-\frac{\epsilon}{R\tau}\right)\right] d\,\ln\tau \tag{5}$$

in which

$M(\tau)$ is a relaxation distribution function defined such that
$M(\tau)\, d\,\ln\tau$ is the contribution to the instantaneous modulus of those elastic mechanisms whose relaxation times lie between $\ln\tau$ and $\ln\tau + d\,\ln\tau$.

The advantage of this equation from an experimental standpoint is that σ/R is a function only of ϵ/R and that data obtained at different strain rates should superpose to give a single curve on a plot of log σ/R vs log ϵ/R. That this occurs for an asphaltic mix in triaxial compression is shown in Figure 7. Three curves representing three

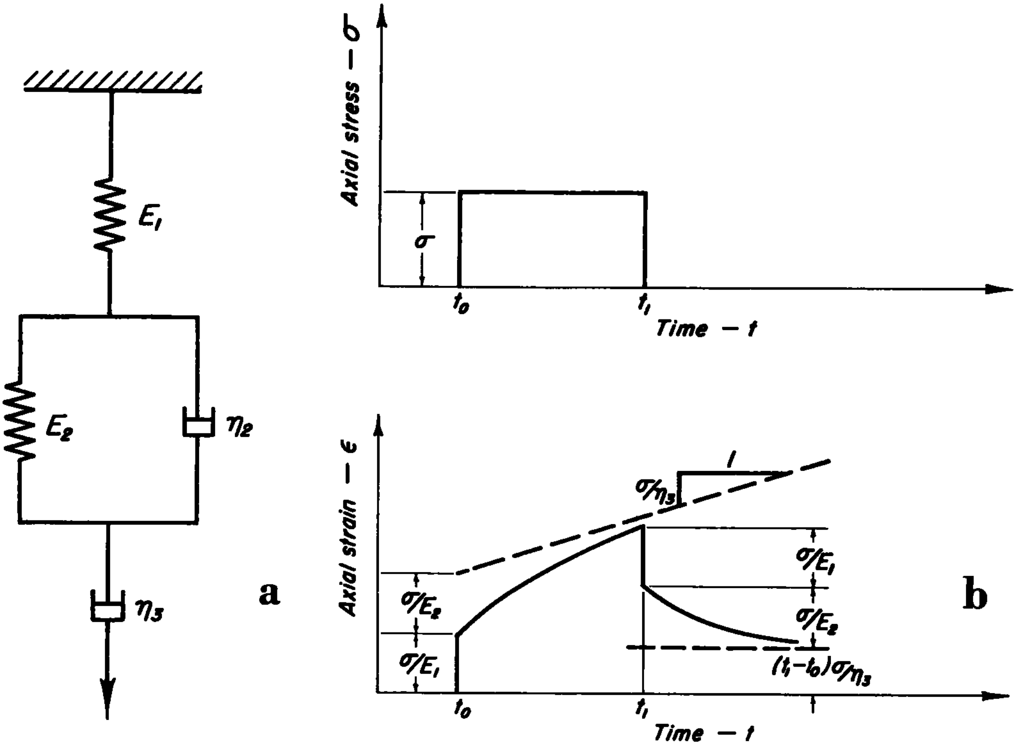

Figure 4. Schematic diagram of (a) Burgers' (four-element) model, and (b) axial stress and strain vs time relationships for Burgers' model subjected to stress in time interval t_0 to t_1.

different temperatures are illustrated. The stress-strain data shown in the figure are based on small deformations, up to approximately two percent strain.

To combine data obtained at various temperatures a reduced variable scheme proposed by Ferry (18) can be used. This analysis (17) is based on the assumption that all relaxation times have the same temperature dependence and that the modulus of each spring in the model is proportional to the absolute temperature. Introducing these assumptions Eq. 5 becomes

$$\frac{\sigma T_0}{RTa_T} = \int_{-\infty}^{\infty} M(\tau)\,\tau \left[1 - \exp\left(- \frac{\epsilon}{RTa_T}\right) \right] d\ \ln\tau \qquad (6)$$

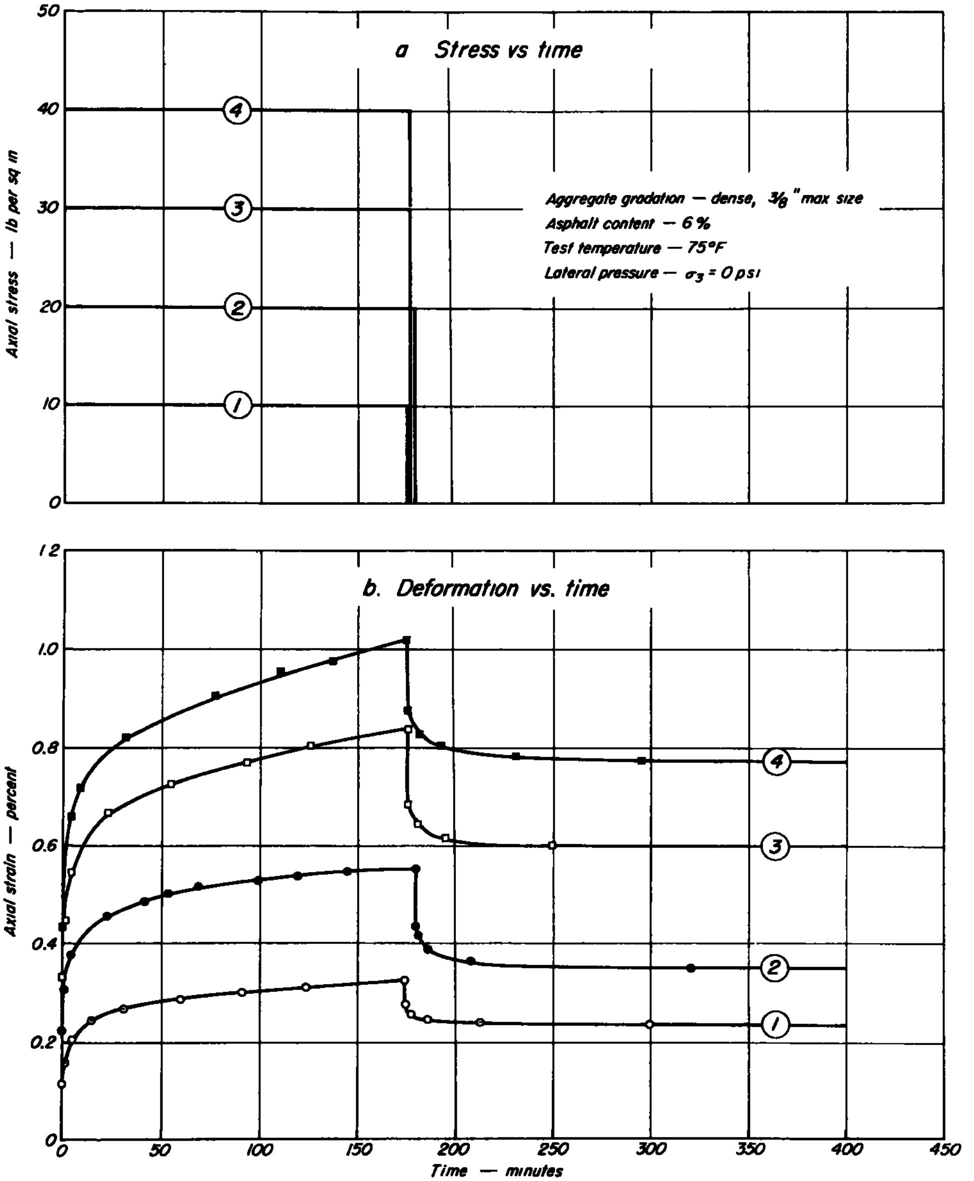

Figure 5. Creep and creep recovery for asphaltic mixtures subjected to different magnitudes of axial stress.

in which

T_0 = reference temperature (25°C or 298°K in this case);
T = temperature of test; and
a_T = ratio of relaxation time at temperature T to value at T_0.

This interpretation has been applied to the data in Figure 7 and the resultant plot of $\log \frac{\sigma T_0}{RTa_T}$ vs $\frac{\epsilon}{Ra_T}$ is shown in Figure 8.

Brodnyan (19) has used a similar technique to plot the dependence of shear modulus of asphalt as a function of frequency. The range in values of a_T vs temperature for a number of different asphalts is shown in Figure 9. Also in the figure are plotted the values of a_T required to obtain the curve shown in Figure 8 for the particular mixture under investigation. It can be noted that the values fall within the band, indicating that the temperature dependence of the viscoelastic characteristics of the mixture is related to that of the asphalt—a not unreasonable conclusion.

In general the data presented in this section indicate that asphaltic mixtures are viscoelastic (at least for small deformations) and that to depict the visco-

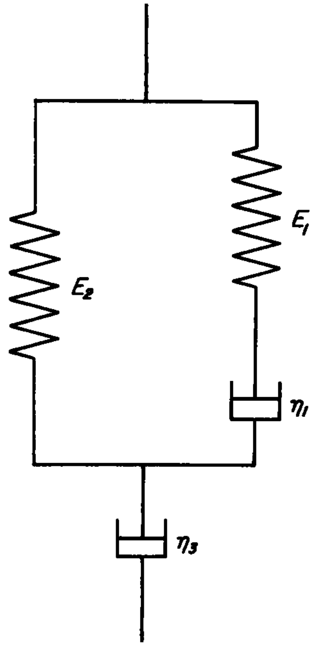

Figure 6. Model representing viscoelastic behavior of asphalt in extension.

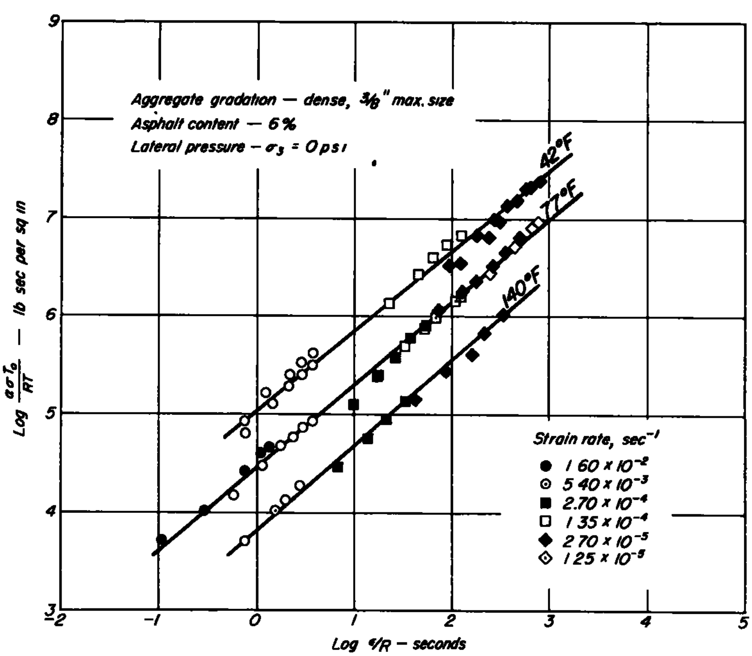

Figure 7. Stress vs strain data for asphaltic mixtures in triaxial compression reduced to unit strain rate at three temperatures.

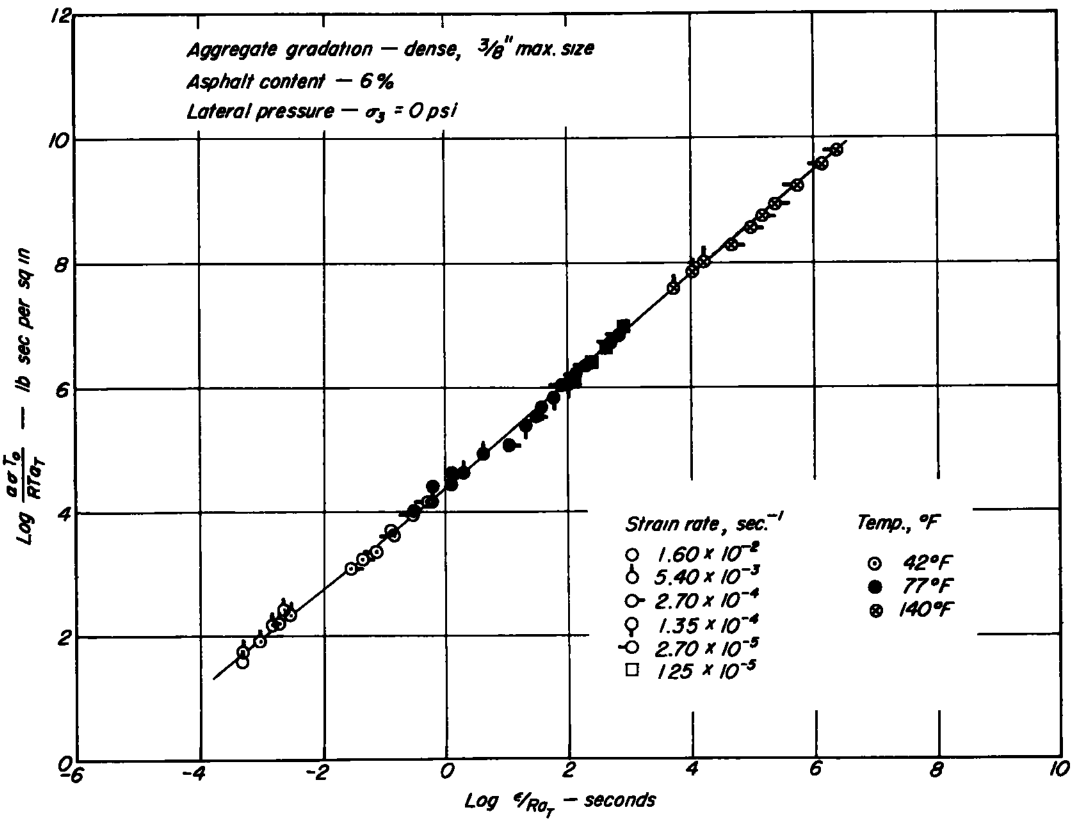

Figure 8. Stress vs strain data for asphaltic mixtures in triaxial compression reduced to unit strain rate and 77 F (298 K).

elastic behavior of an asphaltic mixture a complex type of model is probably required.

EXAMPLES OF STRUCTURAL ANALYSIS OF VISCOELASTIC PAVEMENTS

Two examples which provide qualitative insight into the behavior of flexible pavements are discussed. It must be recognized that the selection of the model for representing the viscoelastic mechanical properties of the surface course as well as the selection of the type of load has been dictated by the desire to present the results unencumbered by excessive mathematical argument. More realistic problems, for example, distributed pressure on a viscoelastic plate, including effects of transverse normal and shear stress, repeated loads, etc., and more realistic representations of material properties are currently under study.

Viscoelastic Beam on an Elastic Foundation

The problem of an infinite beam resting on an elastic foundation and loaded with a time-dependent load will serve to illustrate the effect of time and material properties on the deflection in a viscoelastic beam (Fig. 10). Considering a beam of unit width, subjected to a load q(x, t) per unit width, the Bernoulli-Euler equation for the elastic beam deflection is

$$\frac{E\,h^3}{12}\,\frac{\partial^4 w}{\partial x^4} + k\,w = q \tag{7}$$

in which E is the elastic modulus of the beam, h the beam depth, w the transverse deflection, q the load intensity on the beam and k the subgrade modulus. In the case of a viscoelastic beam, the modulus of elasticity must be replaced by the time-dependent relation between stress and strain. For the present example a 3-element model, Figure 10b (or Fig. 6 with $\eta_3 = 0$) exhibiting instantaneous elasticity, creep and recovery, has been selected. The stress-strain relation for this model can be written

$$\left[E_1 + \eta_1 \frac{\partial}{\partial t}\right] \sigma(t) = \left[E_1 E_2 + \eta_1 (E_1 + E_2) \frac{\partial}{\partial t}\right] \epsilon(t) \tag{8}$$

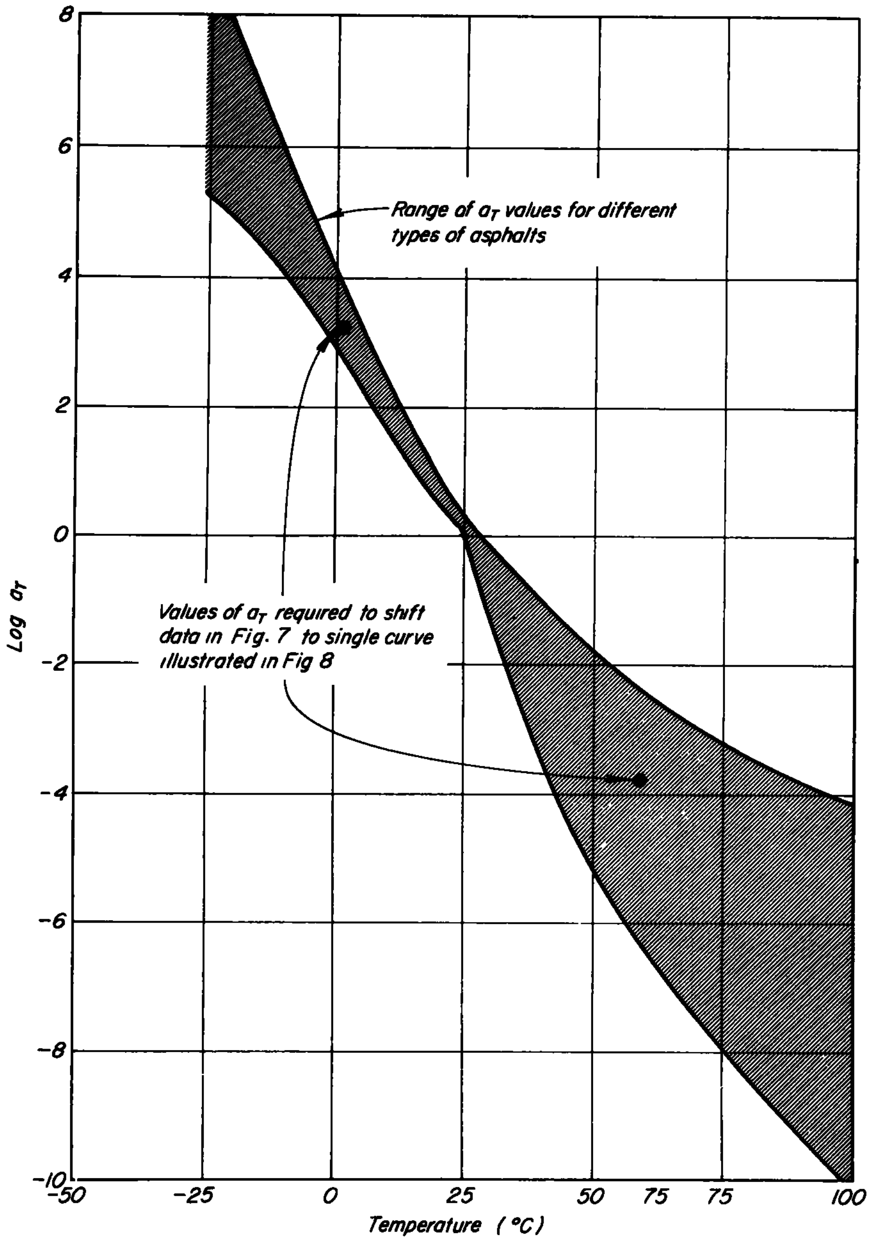

Figure 9. Temperature dependence of a_T values for various asphalts (after J.G. Brodnyan).

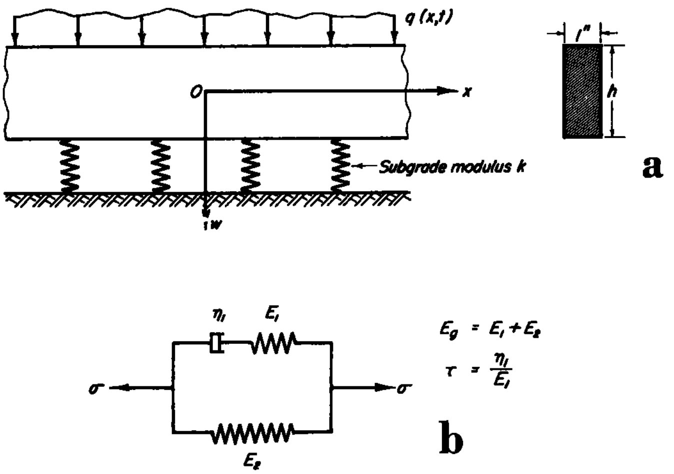

Figure 10. Schematic diagram of (a) viscoelastic beam on elastic foundation, and (b) mechanical model of material.

In terms of differential operators L_1 (t), L_2 (t) Eq. 8 can be written

$$L_1 \sigma = L_2 \epsilon \tag{8a}$$

By replacing the modulus of elasticity E in Eq. 7 by the ratio of the operators L_2/L_1, with time dependence of w and q implied, an equation for the viscoelastic behavior of the beam is obtained

$$\frac{h^3}{12}\,\frac{L_2}{L_1}\left\{\frac{\partial^4}{\partial x^4}\left[w(x,t)\right]\right\} + k\,w(x,t) = q(x,t) \tag{9}$$

To this equation are added the boundary conditions

$$w(\pm\infty,\ t) = \frac{\partial w}{\partial x}\bigg|_{\pm\infty,\ t} = 0$$

Further, the beam is assumed to be at rest at t = 0. The solution of the differential equation subject to the prescribed boundary and initial conditions can be brought about through the use of repeated Laplace and Fourier Cosine Transforms. The details will not be recorded inasmuch as the procedure is not novel. However, it may be noted that in performing the inverse transformations to recover the time and space dependence of w, it is more convenient to perform the Laplace inversion before the Fourier inversion. The final result for the deflection due to a concentrated load applied at t = 0 and held constant is

$$\frac{\pi k}{P\left(\frac{k}{E_g I}\right)^{\frac{1}{4}}}\, w(x,\tfrac{t}{\tau}) = \int_0^\infty \left\{\left[\frac{1}{1+z^4} - \frac{1}{1+nz^4}\right] \exp\left[-\frac{(1+nz^4)}{1+z^4}\right]\frac{t}{\tau} + \frac{1}{1+nz^4}\right\}\cos\left[\left(\frac{k}{E_g I}\right)^{\frac{1}{4}} xz\right]dz \tag{10}$$

in which

$$n = \frac{E_2}{E_1 + E_2} = \frac{E_2}{E_g}$$

is the ratio of the long-time and short-time elastic moduli of the material in bending and $\tau = \frac{\eta_1}{E_1}$ is the relaxation time of the material under static loading.

The center deflection as a function of time for two values of the ratio of long-time and short-time elastic moduli, $n = \frac{1}{10}, \frac{1}{100}$ was obtained by numerical integration. The results are shown in Figure 11. It will be noted that elastic deflection of a beam with elastic modulus E_g is reached instantaneously. The beam then deforms viscoelastically attaining asymptotically the deflection of a beam with elastic modulus E_2.

Repeated Load on an Unconfined Compression Cylinder

As a final example of deformation analysis of viscoelastic materials, the repeated compressive loading of an unconfined cylinder (Fig. 12b) is discussed. The nature of the loading (Fig. 12a) is that of a series of load pulses. The axial stress and strain, using the model discussed in the previous example, are related by Eq. 8. It develops in this application that it is convenient to define the relaxation time on the basis of the combined elastic modulus of the two springs rather than for E_1 only as before. Accordingly, with

$$\tau^* = \eta_1 \left[\frac{E_1 + E_2}{E_1 E_2}\right]$$

where the bracketed expression is the compliance of the two springs in parallel, the above equation is

$$\left[\frac{E_1}{\eta_1} + \frac{\partial}{\partial t}\right]\sigma(t) = (E_1 + E_2)\left[\frac{1}{\tau^*} + \frac{\partial}{\partial t}\right]\epsilon(t) \tag{11}$$

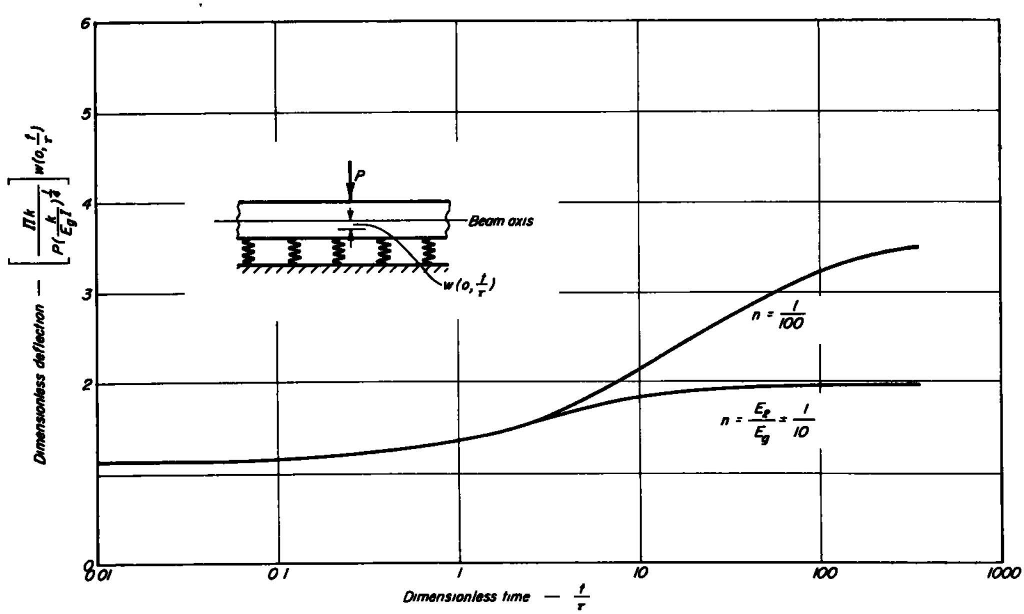

Figure 11. Center deflection vs time for viscoelastic beam on elastic foundation.

The solution of this differential equation for one cycle of loading and recovery (Figs. 12a and 12c) can be conveniently discussed in four steps as follows:

Referring to Figure 12c:

1. Segment a b, instantaneous elastic response at $t = 0$, with

$$\epsilon(0) = \frac{\sigma}{E_1 + E_2}$$

2. Segment b c, viscoelastic strain at constant stress, $0<t<t_0$, with

$$\epsilon(t) = \frac{\sigma}{E_1 + E_2}\left\{1 + \frac{E_1}{E_2}\left[1 - \exp\left(-\frac{t}{\tau^*}\right)\right]\right\}$$

3. Segment c d, instantaneous elastic response at $t = t_0$ on unloading, with change in strain $\Delta\epsilon(t_0)$ where

$$\Delta\epsilon(t_0) = \frac{\sigma}{E_1 + E_2} \quad \exp\left(-\frac{t_0}{\tau^*}\right)$$

4. Segment d e, strain recovery at zero stress, $t_0<t<t_1$, with

$$\epsilon(t) = \frac{\sigma}{E_2}\left[1 - \exp\left(-\frac{t_0}{\tau^*}\right)\right] \exp\left(-\frac{t - t_0}{\tau^*}\right)$$

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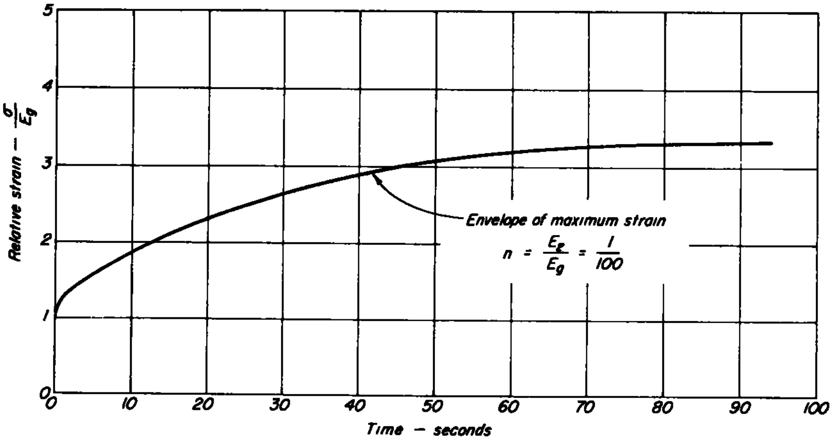

Figure 13. Axial strain vs time—repeated load on unconfined compression cylinder.

the possibility of accumulative buildup of strain occurs (Figs. 12c and 13). It may be noted that the addition of a dashpot in series with the present model (Fig. 6) would contribute a permanent deformation proportional to the total number of load repetitions. This model may possibly be suitable for determining the permanent deformation developed in flexible pavements subjected to repeated applications of load. Results of this type for triaxial compression repeated load tests are presented by Monismith and Secor (20).

Finally, it must be emphasized that more elaborate models must be used for analysis of the behavior of materials subjected to loads with differing frequency distributions. For example, to compare the effects of two different rates of load repetition, two sets of viscoelastic coefficients, each appropriate to one frequency must be employed.

SUMMARY

In this paper limitations imposed by purely elastic analysis of flexible pavements are reviewed. The importance of inclusion of time-dependent material properties and loading conditions in formulating a rational method of pavement design is emphasized. Some experimental data illustrating the viscoelastic behavior of asphaltic mixtures under various types of loading are presented. Representation of viscoelastic material properties by means of mechanical models is discussed and several typical models are shown. Two simple examples illustrating effects of time-dependent loading and material properties on the formulation and solution of viscoelastic boundary value problems are presented.

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