

# Economic Combinations of Aggregates for Various Types of Concrete

CHARLES SARGENT, Chairman of Engineering, Department of Civil Engineering, University of Alaska

This paper demonstrates graphical methods for solving the usual trial-and-error problem of computing the percentages of several aggregates blended to make the combination conform to specifications; demonstrates how the same graphical solutions can be expanded to include calculations for the most economical blend; and explains how the diagrams can be combined with accounting records and routine test results to provide a method for selecting blends on the basis of the probability of rejection of the final product.

● THE GRAPHICAL METHOD for combining two aggregates is reviewed here to illustrate how the economic combination can be determined, and also to develop background information for the understanding of the methods for combining three aggregates.

## ECONOMIC BLENDING OF TWO AGGREGATES

Specifications for concrete aggregates (either portland cement or asphalt cement) are usually written in such a manner as to place limits for the ratio of any specified size to the whole aggregate. These limits are usually expressed as a percentage of the material retained on, or passing, certain sized sieves, the sieves being of a standard design either in the Tyler or U.S. Standard series. If a sample of the aggregate proposed for a certain concrete mix does not fall within the specified limits, it is necessary to blend the aggregate with another having the required sizes in other proportions. If the original aggregate is close to the place of use and the blending aggregate has to be imported from some other region, or manufactured by crushing, there is a definite advantage in knowing what minimum amount of the blending material need be added to make the blend meet the specifications, as the imported material will undoubtedly be more costly than the local material.

For the purpose of illustration, two aggregates are assumed where the only requirement is on one size. It is also assumed that it is necessary for size X to fall between 20 percent and 40 percent, that aggregate A has 60 percent of size X, and that aggregate B has 10 percent of size X. It is desired to find the combination of the two aggregates that will produce a blend having the specified requirements.

A graph such as shown in Figure 1 is constructed. The left centisimally divided vertical line represents aggregate A, and the right similarly constructed line represents aggregate B; the centisimally divided base line joins the two vertical lines as shown. The percentage of size X in aggregate A is plotted on the left vertical line, and the percentage of size X in aggregate B is plotted on the right vertical line. A straight line joining the percentages of size X in the two aggregates is drawn diagonally across the graph; the specification limits are plotted on the diagonal line, the upper limit at 40 percent and the lower limit at 20 percent. A projection of any point on the diagonal line between the specification limits to the base line represents a combination of the two aggregates which will meet the specifications. The percentage of A is measured from the left end and the percentage of B is measured from the right end of the base

line. If the most economic combination is desired, the selected point will be chosen as close to the line representing the least expensive aggregate as possible. In this case, if aggregate B is the least expensive, the point chosen will be very close to the point representing 80 percent of B and 20 percent of A.

An inspection of this process indicates that the aggregates blend as a straight-line function. A mathematical expression representing this operation would have provided just as accurate, and probably a quicker solution to the problem.

It is a rare situation when only one size is specified, however, and with several sizes to be considered and the relation between sizes being an exponential, it would not be so convenient to write an equation that would give the correct ratio of blended aggregates when more than one size need be considered. Hence, the analytical solution of the blending problem is usually by trial and error.

When several sizes are specified, the same type of diagram is constructed and each size is plotted in the same manner as the one size previously described. The specification limits for that size are similarly plotted on the diagonal lines.

Example 1 (see Appendix) shows a typical diagram for the combination of two aggregates with several size specifications. It should be noted that the upper specification limits on the several sizes are joined by lines, and the lower specification limits are similarly joined. These irregular lines form an envelope of specification limits; no combination may be selected on the base line if a vertical line constructed at that percentage figure cuts into either envelope. It should also be noted that if the envelopes overlap, or intersect, there can be no combination of the two aggregates which will meet the specifications. The range of combinations will be from the farthest right point of the left envelope to the farthest left point of the right envelope. The most economical combination will be the extreme point of the envelope adjacent to the most expensive aggregate.

The advantage of this system over the trial-and-error system (even though some highly practiced engineers can work out such a solution rapidly) is that all possible solutions are obtained by the operation and the selection of the actual combination can be made on any basis desired, such as economic solution, nearest the ideal gradation curve, or in the middle (to minimize the chance of specification infringement). Example 1 (Appendix) illustrates a typical problem involving the blending of two aggregates for economic solution.

### ECONOMIC BLENDING OF THREE AGGREGATES

In the preceding, it was shown that an economic combination of two aggregates could be obtained on the basis of the straight-line relation of the sizes within each of the aggregates. With three aggregates, there can be no straight-line relation because three

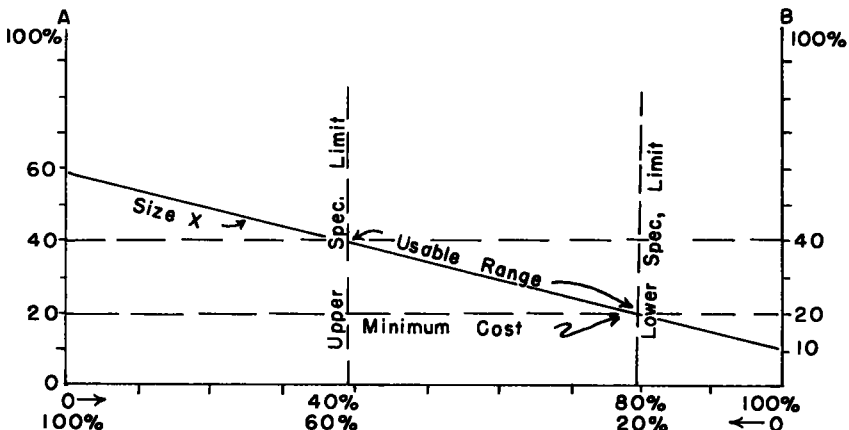


Figure 1.

random points are not joined by a straight line except by chance. It is possible to add a dimension to the previous idea, however, and join the three aggregates by planes rather than by lines, due to the well-known geometric principle that any three points in space lie on some mutual plane. To be helpful in the type of solution of interest, the plane used must have certain qualities, as follows:

1. The extremities of the plane must represent the size quantities under consideration.
2. The extreme points when projected to the horizontal must be equidistant.
3. It must be possible to represent the specification limits by straight lines on the plane.
4. When the planes are superimposed, it must be possible to construct a solid envelope representing the specification limits.

Reflection on this problem brings one to the conclusion that the solid formed by planes representing sizes, in a problem such as the one being considered, must be a right prism having an equilateral triangular base. The base must be centisimally divided to permit the plotting of sizes in percentage, as was done on the vertical lines in the two-aggregate example.

In Figure 2 it was assumed that three aggregates were to be combined, and there was only one size limited by specifications. Aggregate A was assumed to have 60 percent of size S, aggregate B was assumed to have 30 percent of size X, and aggregate C was assumed to have 10 percent of size X. Specifications on size X were assumed to be 40 percent to 50 percent. The plane joining the three points (A60, B30, and C10) represents how the aggregates are related. The 40 percent and 50 percent lines on the plane represent the specification limits. Any point selected within the specifications is related to the ratio of the three aggregates in the combination by the projected position of the point in the horizontal base plane. The percentage of any aggregate is represented by the perpendicular distance from the point to the side opposite the apex which represents the aggregate.

Before accepting the validity of these statements, one should be satisfied that the geometric relationships described are true. The Appendix offers one of the many possible analytical proofs of the geometric conditions used in the foregoing.

To use a solid diagram such as illustrated in Figure 2 involves considerable descriptive geometry, particularly when more than one size is to be considered. The planes representing the various sizes are not parallel, and in fact might intersect at various angles. It is much simpler to make an orthographic projection of the controlling prism

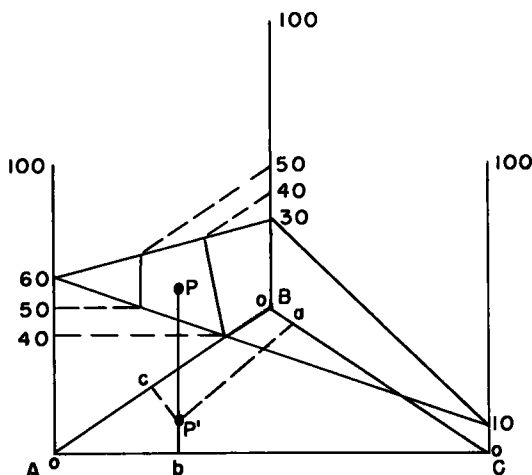


Figure 2.

so that all solutions may be made in one plane. The orthographic projection is made by folding the vertical planes of the prism outward and into the plane of the base, as shown in Figure 3.

In the projected form the diagram is similar to that used for the combination of two aggregates, except that there are three such diagrams representing all the two-aggregate combinations and these are linked by the centisimally-divided triangular base.

To visualize how this diagram works, one must imagine the solid with the several planes superimposed, and each plane having the specification limits drawn on it. If the area on each plane outside the specification limits were made opaque, leaving the portion between transparent, the several superimposed planes would each have a window which represented the useful portion of the size represented. In order for there to be a possible solution, the windows in all planes must be lined up so that it is possible to see a portion of the base plane through all the planes when looking directly down on the prism from above. The visible portion of the base plane then covers all points which represent possible combinations of the three aggregates. Any point within the visible area represents a solution, but the most economical combination is determined by selecting a point closest to the apex of the base plane representing the least costly of the three aggregates, and farthest from the apex of the base representing the most costly of the three aggregates.

In the projected form of the prism, the visible area previously described is determined graphically by projecting the various points from the outside graphs to the edges of the base triangle. Connecting these projections across the base by lines, there will be one line for each specification limit for each size. Any area of the base which is above the greatest minimum specification line and below the least maximum specification is

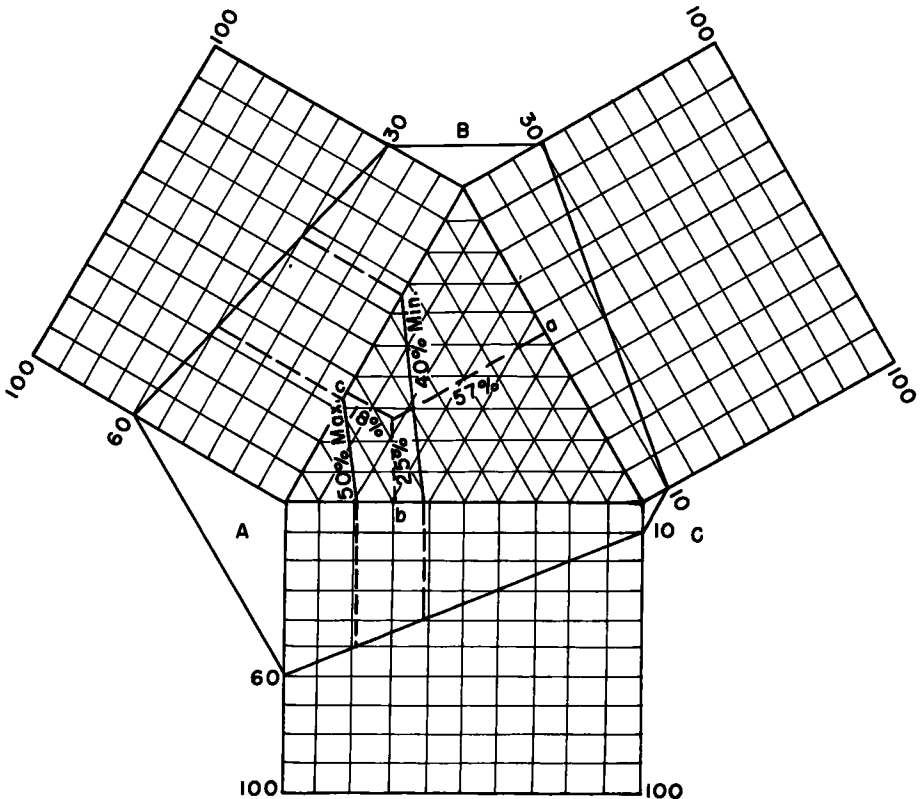


Figure 3.

the area representing possible solutions of the problem. Example 3 (Appendix) shows how a typical problem of this sort is solved.

### AGGREGATE BLENDING ON THE BASIS OF PROBABILITY

In the previous discussion it was assumed that each of the several aggregates to be blended to meet certain requirements for the combination had known values for the quantities on each sieve size. In actual practice, the only method whereby the exact amount in each size category can be assured is to separate the aggregates into as many sizes as there are specified sieve sizes and reblend according to the desired proportions. Such a separation into a great number of sizes would be expensive and time consuming, particularly for the fine material. In fact, it would be next to impossible to separate fine material on the No. 50 and smaller sieves in any quantity. Sampling, then, is the only practical method whereby the gradation of the aggregates can be determined.

When sampling techniques are used as a means of quality control, there will be a range of results. The distribution of results within the range will follow some pattern which by statistical analysis may be reduced to probability figures that can be used to determine the mean values for the various sizes, and also the dispersion of values with relation to the mean value. The probability that one aggregate will have a gradation exactly equal to the mean value of the samples taken is remote. Why, then, should a composite aggregate have a gradation equal to the gradation of the composite of the means of the samples? Such a probability is even more remote than the probability of having the gradation of the component aggregates equal the gradation of their means.

By expanding the theory and technique used in the previous parts of this paper, it is possible to determine a method for combining aggregates on the basis of the probability of gradation rather than assuming that the mean of the samples taken is in fact valid.

As sampling is done for control purposes on the constituent aggregates, the test results are accumulated. After several tests have been made (the more tests, the more reliable will be the results) it is possible to determine the dispersion of results with respect to the mean of the samples. Knowing the dispersion, it is possible to determine the standard deviation of the samples from the mean. When the standard deviation is known, it is possible to determine the probability of occurrence of any particular variation from the mean. This assumes a normal distribution of results; however, if the distribution is not normal it is possible to determine the probability by additional statistical methods.

To use this system, one should be familiar with statistical quality control methods, discussion of which is beyond the scope of this paper.

If the probability of occurrence of a certain gradation is used, rather than a mean value for gradation, the relationship of sizes between aggregates will no longer be a straight line, but will be variable between the limits of probability selected for control. On the chart, this relationship will be shown by a region rather than a line. The region will be bounded by the probability limits, and it is these limits which will have a straight-line relationship. The actual relationship between aggregates will remain unknown, but this fact is unimportant.

In the prism representing the combination of aggregates, there will now be a separate prism for each sieve size, rather than a plane as was used before. This prism will be contained between two planes which represent the relationship of the probability limits rather than some definite mean value. The specifications are indicated on each face of the prism, and if the same analogy is used as before, the region in the prism beyond the specification limits is made opaque, and the region within the specification limits should be made translucent in proportion to the probability of occurrence of a certain frequency. If these prisms are superimposed, the top view of the containing prism will show a region of the base with various shades—the lighter the shade, the greater the probability of the combination being available; the darker the shade, the lesser the probability of a particular combination being within the specifications.

As stated in the previous discussion, use of a solid prism is not a practical method

for setting up the problem and again the orthographic projection is used. With the probability method, the upper and lower ranges of probability are plotted and connected, as were the mean values in the previous work. It is also advisable to plot the mean values, which will make three points for each sieve size rather than one as was done previously. Specification limits are plotted on each line and projections to the base are made as before.

In the probability solution, the base triangle will be crossed by regions, or bands. Each sieve size will have two bands, one for the upper specification limit, and one for the lower specification limit. The width of each of the bands will be determined by the probability limits chosen for the control of the aggregates.

If there is a region on the base triangle above the highest lower specification band and below the lowest upper specification band, then, as before, this region represents all the possible solutions for the problem where the probability of a chosen point representing a combination within the specifications will be equal to, or greater than, the probability limit used. Any region across the bands from the solution region will represent areas where the probability of a chosen point being within the specifications is one minus the probability limit or less. This last area may be immediately removed from consideration for any solution.

The regions of the base triangle covered by the bands themselves, and these may be overlain by one or more of the bands, are areas of ranging probability. If a region is covered by one band, the probability that a point chosen therein will meet specifications will be the range between the chosen probability figure squared to the probability figure. If two bands overlay the area, the range of probability will be from the probability figure cubed to the probability squared. Successive overlaps will decrease the probability on an exponential equal to the number of overlaps. This change in probability is due to the theory of compound probability, which states that if a compound event is made up of a number of separate and independent sub-events, and the occurrence of the compound event be the result of each of the sub-events happening, the probability of the occurrence of the compound event is the product of the probabilities that each of the sub-events will happen. Here the probabilities are all the same; therefore, the compound probability is an exponential of the probability figure. There may be some question as to whether or not these are unrelated events, and in some cases, depending on how the data are used, the events are more or less related. For instance, it is not possible for more material to pass the No. 200 sieve than passes the No. 100 sieve, etc. If the frequency charts for the separate sizes do not overlap, the relationship of the events is not so important. At any rate, the exponential method will give conservative results; that is, the probability will never be less than the value stated.

Once the areas of probability have been determined on the base chart for any particular set of specifications, the chart can be used in the same manner as previously described. Minimum points can be selected and tested for economy. Now, however, the probability enters in and pricing must consider the probability of rejection. Knowing the probability of rejection (one minus the probability of acceptance) the aggregate cost times the probability of rejection gives a figure to add to the price to insure that the cost of both accepted and rejected material will be covered. (See Appendix for example problem where this method is further explained.)

## SUMMARY

In this paper an attempt has been made to describe graphical methods for the solution of a problem which has been, for the most part, solved by trial-and-error methods. It also has been shown that the method, presented gives all possible solutions to the problem, thereby enabling the engineer to make economic selections which he can be confident will give the best results for the least amount of money.

In addition, it has been shown that it is not necessary to use mean or average values for the solution of this problem, but that it is possible and highly desirable to use probability ranges.

The three important features of these methods have been pointed out, as follows:

1. By using probability methods for combining aggregates, the supplier may choose combinations of aggregates over a wider range than would be possible under the method of the means. This wider selection should result in greater economy and a more realistic price of aggregate, as with this method it is possible to calculate the risk of rejection and to allow for the risk in determining price.

2. By using probability methods, the concrete mix designer can determine the probable range of variation of the aggregate he intends to use. Also, the variation of certain aggregates can be predicted. This will permit a tightening of specifications in some cases with a resulting increase in the quality of the product.

3. Statistical methods of quality control are important to the production of aggregates (as they are in many other fields of manufacturing), for to be able to adequately design concrete mixes, or to economically produce concrete mixes, the designer or producer should be familiar with these methods.

## Appendix

### MATHEMATICAL DERIVATION FOR THE TRILINEAR CHART

The trilinear chart used throughout this paper is an orthographic projection of a right triangular prism, the base of which is an equilateral triangle. In theory, any plane cutting the prism, but not cutting the base, has a certain relationship to the elevations above the base where the plane cuts the edges of the prism. The relationship assumed has been that the elevation of any point in the cutting plane, and not outside the vertical projection of the base, will have an elevation equal to the sum of the products of each edge elevation times the horizontal distance measured perpendicular from the opposite side to the point.

In Figure A1,  $A'B'C'$  represents the horizontal projection of a plane  $A B C$ , each corner of which is at a different elevation above the plane of projection.  $P'$  represents the projection in the plane  $A'B'C'$  of any point  $P$  in the plane  $A B C$ . Let the altitude of the triangle  $A'B'C'$  equal 100 percent.

Prove: The elevation  $E_p$  of the point  $P$

$$\text{is equal to } \frac{E_A(aP') + E_B(bP') + E_C(cP')}{100}$$

From Figure A1 (by geometric relationship):

$$x'n = P'a$$

$$B'm = P'c$$

$$P'b = P'b$$

$$\text{or } P'a + P'b + P'c = B'B'' = 100\%$$

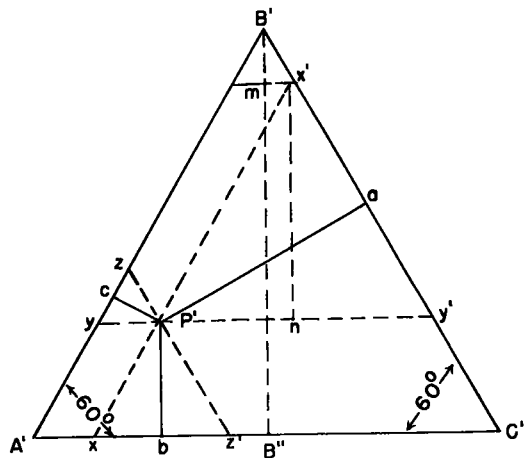


Figure A1.

From Figure A1:

$$E_p = E_A + A'y \text{ (slope A B)} + P'y \text{ (slope A C)} \quad \text{Slope AB} = \frac{E_B - E_A}{100 \sqrt{3}} \quad (1)$$

$$E_p = E_B + B'x' \text{ (slope B C)} + P'x' \text{ (slope B A)} \quad \text{Slope BC} = \frac{E_C - E_B}{100 \sqrt{3}} \quad (2)$$

$$E_p = E_C + C'z' \text{ (slope C A)} + P'z' \text{ (slope B C)} \quad \text{Slope CA} = \frac{E_A - E_C}{100 \sqrt{3}} \quad (3)$$

$$A'y = P'b \sqrt{3}, \quad yP' = P'c \sqrt{3}, \quad B'x = P'c \sqrt{3} \quad \text{Slope AC} = \frac{E_C - E_A}{100 \sqrt{3}}$$

$$P'x' = P'a \sqrt{3}, \quad C'z' = P'a \sqrt{3}, \quad z'P' = P'b \sqrt{3} \quad \text{Slope CB} = \frac{E_B - E_C}{100 \sqrt{3}}$$

$$\text{Slope BA} = \frac{E_A - E_B}{100 \sqrt{3}}$$

Using the preceding equivalents, Eqs. 1, 2, and 3 may be rewritten:

$$E_p = E_A + \frac{P'b \sqrt{3} (E_B - E_A) + P'c \sqrt{3} (E_C - E_A)}{100 \sqrt{3}} \quad (4)$$

$$E_p = E_B + \frac{P'c \sqrt{3} (E_C - E_B) + P'a \sqrt{3} (E_A - E_B)}{100 \sqrt{3}} \quad (5)$$

$$E_p = E_C + \frac{P'a \sqrt{3} (E_A - E_C) + P'b \sqrt{3} (E_B - E_C)}{100 \sqrt{3}} \quad (6)$$

Adding Eqs. 4, 5, and 6 and collecting terms (note that  $\sqrt{3}$  drops out) gives

$$300 E_p = E_A (100 - P'b + 2P'a - P'c) + E_B (100 - P'a + 2P'b - P'c) + \dots + E_C (100 - P'a - P'b + 2P'c) \quad (7)$$

But

$$P'a + P'b + P'c = 100$$

therefore,

$$300 E_p = E_A (3P'a) + E_B (3P'b) + E_C (3P'c) \quad (8)$$

or

$$E_p = \frac{E_A P'a + E_B P'b + E_C P'c}{100} \quad (9)$$

### Example I

A contractor has two stockpiles of material from which he proposes to blend an aggregate to meet the Asphalt Institute specifications for Type IVd mix. The cost of material A is \$6.00 per ton, and material B is \$7.50 per ton. What percentage of each material should he use if he wishes to maintain a safe tolerance on any sieve size of 1 percent of the limiting specification, and still produce the most economical blend?



Sieve Analysis and Specifications (% passing, by wt.)			
Sieve Size	Spec.	Aggr. A	Aggr. B
1 1/2 "	100	100	100
1 "	80 - 100	95	100
3/4 "	70 - 90	79	100
1/2 "	—	70	100
3/8 "	55 - 75	50	95
No. 4	45 - 62	20	81
No. 8	35 - 50	10	70
No. 30	19 - 30	6	52
No. 50	13 - 23	4	35
No. 100	7 - 15	2	20
No. 200	0 - 8	1	12

From Figure A2 it is apparent that the possible combinations lie in the range of

41.5 to 52 percent of A  
48 to 58.5 percent of B

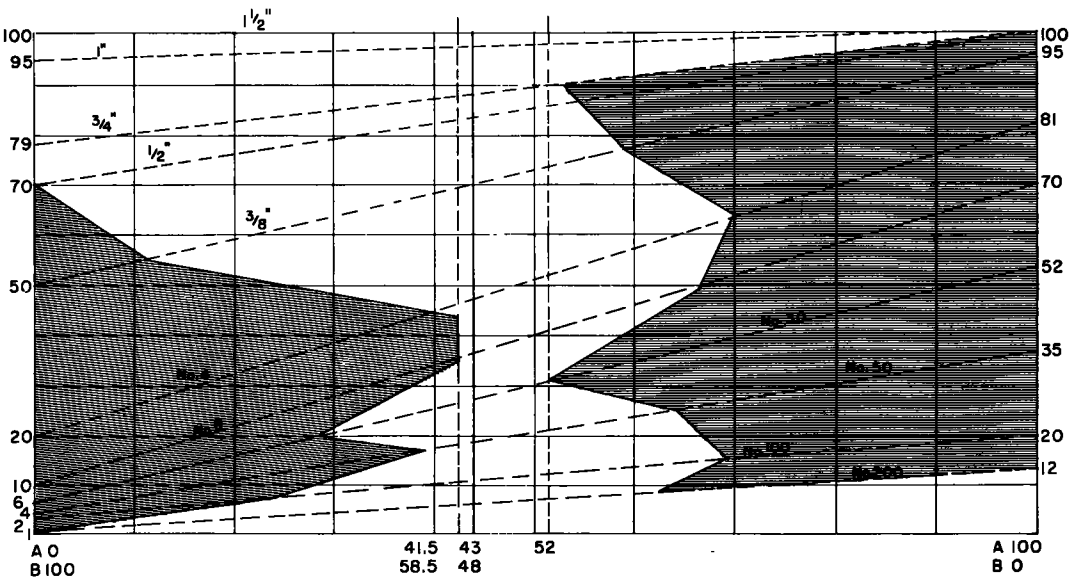


Figure A2.

Inasmuch as A is the most expensive material, the selection should be near 41.5 percent of A to 58.5 percent of B. But because a safe tolerance of 1 percent on the controlling size (in this case, the No. 8 size) was required, the minimum should be moved to 43 percent of A and 57 percent of B. The cost of this blend will be  $0.43 (6.00) + 0.57 (7.50) = \$6.85$  per ton. If it is desirable to compute precise values on this type of problem, it is a simple matter to write the equation for the line representing the limiting specification and from this single equation compute the intercepts desired.

### Example II

Find the most economic combination of aggregates A, B, and C which will meet the Asphalt Institute Specification Type IVa. Aggregate A costs \$3.00 per ton, aggregate B costs \$7.00 per ton, and aggregate C costs \$35.00 per ton.

This would be a typical situation where aggregate A is found close at hand, aggregate B needs some haul or processing, and aggregate C must be hauled a considerable distance or requires extensive manufacturing processes. It quite often happens that natural aggregate deposits have a gap in the gradation, making such a blend necessary.

Sieve Analysis and Specifications (% passing, by wt.)				
Sieve Size	Spec.	Aggr. A	Aggr. B	Aggr. C
$\frac{1}{2}$ "	100	100	100	100
$\frac{3}{8}$ "	80 - 100	90	80	100
No. 4	55 - 75	64	73	55
No. 8	35 - 50	40	40	50
No. 30	18 - 29	10	38	45
No. 50	13 - 23	0	30	45
No. 100	8 - 16	0	17	35
No. 200	4 - 10	0	0	35

Referring to Figure A3, it will be noted that there is a considerable area of possible solutions with minimums of about 2 percent of B, 12 percent of C, and 36 percent of A, and maximums of 68 percent of A, 30 percent of C, and 50 percent of B. Each of the maximums or minimums will have a set of coordinates in the other two directions, not necessarily the same as the maximums and minimums listed here. In this particular example there is a minimum value for C which can be combined over a considerable range of the other two aggregates. With a value of 12 percent of C, the range of values for A are from 36 percent to 63 percent and the values for B could range from 25 percent to 53 percent. If 12 percent is used for the value of C, the values used for A and B should add to 88 percent. Because B is more expensive than A, it would be more economical to choose a value of about 25 percent for B. The economic solution would then be A = 63 percent, B = 25 percent, and C = 12 percent.

This solution, although the most economical, might not be the most logical. Observing that the minimum cost point indicates that it is close to the border (if the samples tested were not truly representative), the combination could very well be out of specification on the minimums for the No. 50 and No. 200 sizes, and it would probably be better to allow a slight margin of safety and choose a value of 13 percent for C, 25 percent for B, and 62 percent for A.

The cost of the combined aggregate chosen would be  $35.00 (0.13) + 7.00 (0.25) + 3.00 (0.62)$ , or  $4.55 + 1.75 + 1.86$ , = \$8.16 per ton. The gradation of the final blend would be

$\frac{1}{2}$ "	100%
$\frac{3}{8}$ "	$90 (0.62) + 80 (0.25) + 100 (0.13) = 89.9\%$
No. 4	$64 (0.62) + 73 (0.25) + 55 (0.13) = 66.1\%$
No. 8	$40 (0.62) + 40 (0.25) + 50 (0.13) = 41.3\%$

No. 30	10 (0.62) + 38 (0.25) + 45 (0.13) = 21.5%
No. 50	0 (0.62) + 30 (0.25) + 45 (0.13) = 13.1%
No. 100	0 (0.62) + 17 (0.25) + 35 (0.13) = 8.7%
No. 200	0 (0.62) + 0 (0.25) + 35 (0.13) = 4.5%

This example shows other features of the system which might not be apparent in a casual study. In the case of the No. 8, No. 4,  $\frac{3}{8}$ " and  $\frac{1}{2}$ " sizes, there are no limits drawn on the chart base. The reason for this is apparent when one studies the original data. Each of the aggregates is within specifications on the  $\frac{1}{2}$ " size; therefore, the limits cannot appear on the diagram because no matter what combination is used, the amount of this size will be within specifications.

The same thing is true of the  $\frac{3}{8}$ ", No. 4, and No. 8 sizes. Although the amounts vary between the aggregates, in no case would they be out of specifications as none of the aggregates has a limit below or above the specified amounts. These amounts were deliberately chosen to illustrate this point.

It should also be noted that the upper and lower specification limit lines for any particular size are parallel. A little reflection upon this will reveal the reason. Remembering that the relationship between aggregates of any single size specification is represented by a plane, and that the specification limits are represented by a line on that plane, and all points on the line being at the same elevation above the base plane, the limit line can be likened to a contour line. A contour line on a plane surface is a straight line. Contour lines on a plane surface representing different elevations would have to be a constant distance apart; therefore, they must be parallel. The fact that these lines will always be parallel provides a check on the plotting of the data. When the lines do not come out parallel, a check should be made to find the obvious error. Because it is not probable that one would make a mistake of the same magnitude in two unrelated calculations or plottings, the fact that the lines all come out with the desired relationship is a satisfactory check on the plotting, and repetition of the work would normally be unnecessary.

It should also be pointed out that even though the solution indicated here is the most economical as far as the cost of the combined aggregate is concerned, it may not be the best solution from the technical standpoint. The best quality of final product would result when the solution point was chosen as near as possible to the centroid of the area of possible solutions, if the specifications are written so that the middle of the allowable range is the best possible quantity for that size. To elaborate on this particular subject is beyond the scope of this paper; however, these items should be kept in mind. True economy for the final product might result from a combination which will give a mixture of the greatest density or the greatest workability. Evaluation of these factors would be on the basis of what the final material is to be used for and what other costs are associated with the work other than the cost of the aggregate.

A diagram of this sort not only indicates the possible solutions, but also the factors which are most apt to cause the material to fall outside the specifications. If this were aggregate for portland cement concrete rather than asphaltic cement concrete, the fact that the No. 200 sieve minimum is controlling the blend would have a different meaning because it is commonly specified that the lower limit on this sieve size will be zero. If such were the case here, the area of possible solution would be extended to the A-B edge of the base diagram, making an entirely different picture. First it would mean that no aggregate C would be necessary, and that the range of percentages for the combination of A and B would be 31 to 55 percent of A and 45 to 69 percent of B.

In other words, any time part of the area of possible solution has the boundary of the base as a border of the area, the two aggregates represented by that side of the base triangle can be combined to produce a satisfactory mixture. If one of the corners of the base is included in the area of possible solution, the aggregate represented by that corner will itself meet the specifications. If it turns out that there is no area of possible solution, the diagram is still of use because it is possible to determine what size needs to be corrected to bring the mixture within the specifications. This is done by examining the lines and determining which line, or lines, must be moved to provide an area of possible solution. When these lines are determined, and the new position is located, back plotting will reveal which aggregate must be changed, and how much.

This same diagram can be used to specify the size components for a third aggregate when it is found that two aggregates cannot be combined to meet specifications. Example III shows how this can be done.

### Example III

Given the following specifications and gradations of available aggregates:

#### Asphalt Institute Type VIIa (percent passing by wt)

<u>Sieve Size</u>	<u>Specifications</u>	<u>Aggr. A</u>	<u>Aggr. B</u>	<u>Aggr. C</u>
$\frac{3}{8}$ "	100	100	100	100
No. 4	85 - 100	30	85	100
No. 8	80 - 95	25	80	100
No. 16	70 - 89	20	50	100
No. 30	55 - 80	15	35	95
No. 50	30 - 60	10	30	70
No. 100	10 - 35	8	24	30
No. 200	4 - 14	3	24	18

Given the following schedule of costs: Aggregate A, \$3.00 per ton; B, \$6.00 per ton; and C, \$5.00 per ton. Washing to remove fines costs \$2.00 per ton for an expected reduction of material finer than No. 200 sieve to 2 percent.

Scalp screening to remove aggregate size greater than:

No. 4 -- \$1.50/ton	No. 16 -- \$3.00/ton
No. 8 -- \$2.00/ton	No. 30 -- \$5.00/ton

Find: A blend of these aggregates as they are, or processed under the foregoing price schedule, which will have a safety margin of 2 percent, and will be the least cost for any possible combination.

Solution: The solution of this problem will involve some elements of trial and error, but if one were to attempt to get such a solution from the start by such a method, only the element of chance would give the proper solution without a great many cuts at the problem. Also, there would be no assurance that the minimum had been reached without several other trials. It is believed this problem points out the advantage of the system developed in this paper.

1. Plot the information given in the problem on the trilinear chart (Fig. A4) in the manner outlined in the previous example. The results of this plotting (dotted lines) indicate that there is no possible combination of three aggregates which can be combined to meet the specifications.

At first glance, it would appear that a solution could be achieved by regulating the No. 200 size to make the upper limit less than the lower limits of the No. 4 and No. 8 sizes. It should be noted, however, that the only manner in which this can be accomplished is by washing aggregate C to remove the excess fine material. Further inspection reveals that aggregate C makes up the greatest part of the blend and processing this aggregate would mean very high costs. This may or may not be the most economic solution.

#### 2. Calculations:

##### A. Wash aggregate C.

Washing aggregate C will change the cost of that aggregate to:

$$\frac{\$5.00}{0.84} + \$2.00 = \$7.95 \text{ per ton}$$

Removal of the fine material will change the gradation of the aggregate to:

$\frac{3}{8}$ ", No. 4, No. 8, and No. 16 ---100%	No. 100 ---25%
No. 30 --- 80%	No. 200 --- 2%
No. 50 --- 59%	

The solution with the modified aggregate is plotted with specially marked

lines, and area M indicates all possible solutions. It should be noted that it is now possible to eliminate aggregate A from the combination, and also that the remaining aggregates are the most expensive of the three. Two points should be investigated. These are points 1 and 2 on the chart.

Point 1: 0.14A, 0.355B, 0.505C

Cost:  $0.14(\$3.00) + 0.355(\$6.00) + 0.505(\$7.95) = \$0.47 + \$2.13 + \$4.01 =$   
\$6.61 per ton

Point 2: 0.00A, 0.55B, 0.45C

Cost:  $0.55(\$6.00) + 0.45(\$7.95) = \$3.30 + \$3.58 = \$6.88$  per ton

B. Scalp screen aggregate A to eliminate sizes No. 4 and No. 8:

Scalp screening aggregate A will change the cost to:

$$\frac{\$3.00}{0.90} + \$2.00 = \$5.34 \text{ per ton}$$

The new gradation will be:

$\frac{3}{8}$ " , No. 4, and No. 8 ---100%	No. 50 ---11%
No.16 --- 22%	No.100 --- 9%
No.30 --- 17%	No.200 --- 3%

The new gradation and solution are plotted on the chart in still another specially marked set of lines.

Analyzing area N in the light of the new prices, it is now possible to eliminate aggregate B from the combination. Aggregate C is now the least costly; therefore, the most economical solution will be at point 3 on the chart, where 29 percent of C and 71 percent of A make up the most economical blend for this solution. The cost here will be  $0.29(\$5.00) + 0.71(\$5.34) = \$1.45 + \$3.79 =$   
\$5.24 per ton.

#### Recapitulation and Final Solution:

Comparing the three cost figures found from the chart (and these are the only possible minimum cost figures), it is noted that the solution for point 3 is the least; however, the problem called for a 2 percent safety margin, making the chosen blend 31 percent of C, 69 percent of A, and 0 percent of B, for which the cost will be  $0.31(\$5.00) + 0.69(\$5.34) = \$5.23$  per ton.

It will be noted that no consideration was given to the adjustment of aggregate B. Examination of the chart should make the reasons for this obvious. Washing the fines out of aggregate B would pivot the No. 200 maximum line clockwise from the left end to some point on the B—C base. Unless aggregate C is also washed, there is no point where interference with the No. 4 and No. 8 minimums can be eliminated. A change in the larger sizes would not produce a suitable combination.

#### Example IV

##### Aggregate Blending on the Basis of the Probability of Test Results

Routine testing of aggregates for control will provide data which may be used to determine the range and frequency of certain test results. The trilinear chart may be used to determine combinations most likely to result from certain proportions of the components, as well as the proportions to be used if the mean value of the test results is used. The use of probability in economic selection leads to a wider variety of economic studies than does the single value of the mean.

If there have been enough tests to establish the pattern for the aggregates to be used, these data can give certain statistical constants which will determine the probability of a certain combination resulting in a proper gradation, as follows:

1. It is not likely that further testing will extend the range of the results for any particular size; therefore, the range should approximately equal the range of  $3\sigma$  either side of the mean, or  $\sigma = R/6$ .

2. If  $2\sigma$  probability limits are used, 95 percent of all test results should fall between these limits.

When ranges rather than means are plotted on the chart, there will result areas of probability rather than a single area of possibility. The manner in which the areas fall one on top of the other will determine the probability of a combination chosen within the area meeting the specifications.

Table A1 data was made up to illustrate how calculations for the most probable gradation of the combined aggregate can be determined from the range and frequency of the results of tests on the separate aggregates. As an illustration: 22 percent of

TABLE A1  
FREQUENCIES (BY PERCENT) OF RESULTS OF ALL SAMPLE TESTS PERFORMED

% Pass.	$\frac{3}{4}$ "			$\frac{1}{2}$ "			No. 4		No. 8		% Pass.	$\frac{4}{8}$	$\frac{8}{50}$		$\frac{50}{200}$			C	
	A	B	C	A	B	C	B	C	B	C		4C	A	C	A	B	C		
100	100	100	100		100	100	1				50	19		1					
99							1				49	12		1					
98							1				48	6		2					
97							1				47	4		2					
96							1				46	3		3					
95							2				45	2		3					
94							3				44	2		4					
93							4				43	1		5					
92							8				42	1		7					
91							15				41	1		10					
90							22				40	1		13					
89							16				39			11					
88							9				38			8			1		
87							4				37			6			1		
86							3				36			5			2		
85							2				35			4			3		
84							2				34			4			4		
83							1				33			3			5		
82							1				32			2			7		
81							1				31			1			13		
80							1				30			1			18		
79							1				29			1			14		
78									1		28			1			12		
77									1		27						7		1
76									1		26						5		2
75									1		25						3		3
74									2		24						2		4
73									4		23						1		6
72									6		22						1		8
71									14		21								10
70									20		20							1	20
69									15		19							1	11
68				1					12		18							1	8
67				1					8		17							2	5
66				2					5		16							2	4
65				2					3		15							3	3
64				4					2		14							4	2
63				7					2		13							5	1
62				10					1		12							8	1
61				13					1		11							11	
60				23			1				10							14	
59				12			1				9							11	
58				9			1				8							9	
57				5			2				7							8	
56				4			3				6							6	
55				3			4				5							4	
54				2			5				4							3	
53				1			7				3							3	
52				1			10		1		2							2	
51							14		1		1							1	
											0	0	0				1	0	0

all samples of material tested on the No. 4 sieve for aggregate B show 90 percent passing. Likewise, no more than 1 percent of the samples show 100 percent passing and no more than 1 percent of the samples show less than 79 percent passing.

### Plotting

The means ( $\bar{X}$ ) and the range limits from Table A2 are plotted on the trilinear chart (Fig A5) in the same manner as the means were plotted in the preceding problems. The specification limits are plotted on the connecting lines in the rectangular portions of the chart for means as well as for the upper and lower probability limits. Projections of the specification limits to the base triangle are connected across the base triangle in the same manner as previously. In this problem there are three lines for each specification limit on each sieve size, rather than one line as in the previous problems.

TABLE A2  
CONSOLIDATION OF DATA, PERCENT PASSING

Aggr.	Size	$6\sigma$	$\sigma$	$\bar{X}$	$2\sigma$	Range
A	$\frac{3}{4}$ "	0	0	100	0	0
	$\frac{1}{2}$ "	16.2	2.7	59.9	5.4	54.5 - 65.3
	No. 4	23.4	3.9	20.0	7.8	12.2 - 27.8
	No. 8	0	0	0	0	0
B	$\frac{3}{4}$ "	0	0	100	0	0
	$\frac{1}{2}$ "	0	0	100	0	0
	No. 4	23.4	3.9	90.2	7.8	82.4 - 98.0
	No. 8	18.0	3.0	70.2	6.0	64.2 - 76.2
	No. 50	16.2	2.7	29.9	5.4	24.5 - 35.3
C	No. 200	16.2	2.7	19.7	5.4	14.3 - 25.1
	$\frac{3}{4}$ "	0	0	100	0	0
	$\frac{1}{2}$ "	0	0	100	0	0
	No. 4	19.8	3.3	50.1	6.6	43.5 - 56.7
	No. 8	23.4	3.9	39.8	7.8	32.0 - 47.6
	No. 50	19.8	3.3	10.0	6.6	3.4 - 16.6
	No. 200	0	0	0	0	0

Any point chosen above an upper probability limit on an upper specification limit line, or below a lower probability limit on a lower specification line, will have a probability of meeting the specifications of less than 0.05. There will be many regions on the chart where points could be chosen in the probability ranges compounded by several of these intersecting areas, but all less than 0.05. These are not significant to the problem and are not individually designated by symbol.

If there is any area of the chart where a point could be chosen below the upper probability limit of the lowest upper specification limit and above the lower probability limit of the highest lower specification limit, that point will represent a combination of aggregates which will meet the specifications with a greater than 0.95 probability. This description is difficult to follow. By analogy, this is similar to the area below upper specification limits and above lower specification limits on a single-line chart such as those in the previous problems. In this case, the lines have width (determined by probability) so the area in question must fall between the edges of the areas rather than between lines. It is highly probable that such an area will not occur on a chart; however, in this example area R does represent such a situation.

Around the area previously described will be other areas where the probability reduces, due to one or more of the sizes being in the range of  $2\sigma$  probability. A point in an area overlain by only one band of probability will represent a combination having a range of probability of meeting the specifications from  $(2\sigma)^2$  to  $2\sigma$ . A point

in an area overlain by two probability bands will represent a combination where the probability of meeting the specifications falls in the range between  $(2\sigma)^3$  and  $(2\sigma)^2$ . This extends to lower and lower probabilities at an exponential rate equal to the number of probability bands overlying the area where the point is chosen. The theory of compound probabilities is the basis for this statement.

In the sample problem, the probabilities described are differentiated by various symbols as shown in Figure A5.

**A Note of Caution:** The method of arriving at the statistical constants, and the data from which the constants were derived for this problem, were for the purpose of producing an illustration. Anyone attempting to use this system should be confident that the statistical material is reliable by using whatever methods are justified by the situation. It is recommended that a thorough background in statistical methods of quality control be achieved before going on to base decisions on the results of such a demonstration as this.

Problems and Solutions Using the Probability Chart

For the purpose of the subsequent problems, cost data have been assumed as follows:

Aggregate Costs:	Cost of Rejected Batch:
A --- \$2.00/ton	(Aggregate Cost + \$1.00)n,
B --- \$2.50/ton	where n is the number of tons
C --- \$5.00/ton	in the batch.

Determine the cost per ton of combined aggregate for the several points shown on the chart.

**Solution:**

By the coordinates of the triangular chart the percentages of each of the constituent aggregates can be determined. The constituent percentage times the constituent cost gives the proportionate cost for the combined. The sum of the proportionate costs determines the total cost of aggregate in the combination. Because this chart gives the probability of the combination falling within the specifications, one minus this probability is the probability that a batch of the combination will be rejected. (This would be the maximum probability of rejection, but the method of inspection might make rejections equal the maximum.) The cost of a rejected load is more than the cost of the constituents due to the labor that has gone into the product; the rejection cost times the probability of rejection gives the cost per ton of production to cover the cost of rejection. This amount should be added to the production cost as insurance that the rejection of a batch will not result in a money loss. Table A3 gives the cost of the several aggregate mixtures as designated by Points A through L in Figure A5.

TABLE A3  
CALCULATION OF MIXTURE COSTS FOR EXAMPLE IV

Coordinates (%)				Material Cost (\$)				Rejection		
Pt.	A	B	C	A	B	C	Total	Max. Prob.	Cost (\$/ton)	Total Cost (\$)
A	10%	26%	64%	0.20	0.65	3.20	4.05	0.05	0.25	4.30
B	13%	29%	58%	0.26	0.725	2.90	3.885	0.095	0.465	4.35
C	13%	68%	19%	0.26	1.70	0.95	2.91	0.265	1.035	3.945
D	12%	56%	32%	0.24	1.40	1.60	3.24	0.226	0.96	4.20
E	12%	44%	44%	0.24	1.10	2.20	3.54	0.226	1.06	4.60
F	13%	40%	47%	0.26	1.00	2.35	3.61	0.185	0.85	4.46
G	0%	7%	93%	0	0.175	4.65	4.825	0.143	0.835	5.66
H	0%	27%	73%	0	0.675	3.65	4.325	0.095	0.565	4.99
I	0%	37%	63%	0	0.925	3.15	4.075	0.143	0.725	4.80
J	0%	45%	55%	0	1.125	2.75	3.875	0.185	0.905	4.78
K	0%	53%	47%	0	1.325	2.35	3.675	0.226	1.055	4.73
L	13%	37%	50%	0.26	0.925	2.50	3.425	0.143	0.635	4.06



It includes not only the costs of the aggregates in the combination, but also the cost of insuring that the rejected material will not result in a money loss to the manufacturer. In this example, the cost of the combination represented by point C, even with the high probability of rejection, is the the most economical.

No attempt has been made here to recommend the combination to use, but only to present the data. The use of such data in preparing bids for furnishing material should be considered in the light of the specifications and the quality control methods to be used by the owner.

### SOURCE OF GRAPH FORM

Some of the practices outlined in this paper have been used for some time by various people; however, the methods outlined have seldom been written up. The method outlined for combining three aggregates, is believed to be unique; at least, the writer has never encountered such a method either in use or in the literature.

The method for combining three aggregates was conceived in the fall of 1956 and a brief paper explaining the method was presented to the Fairbanks Branch of the Alaska Chapter of the American Society of Civil Engineers. Although the idea was developed from its basic elements, it is not claimed that the principle is original. After completing the paper mentioned, the author found a German printed graph form of precisely the same type as used for the solution of this problem. It is not known, however, for what use the German graph was intended.

Editor's Note: To reproduce the charts in the form used herein, the basic graph was traced from the originals submitted by the author; a lithographic plate was made; and copies were printed. The examples discussed in the paper were then individually drawn on the basic grid and reduced to one-half size in the final printing.