Variability of Fixed-Point Speed Measurements

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One of the most vexing problems in the study of speed arises from the lack of background information needed in developing necessary sampling methods. In the most common method, speed measurements are taken at a fixed point. This raises the question of whether or not samples taken at different hours of the day, days of the week, or months of the year may be used to infer changes in speed behavior brought about by an artificially induced variable.

A study of vehicle speeds under normal conditions covering a period of six months suggests that a number of observations alone is insufficient as a measure of adequacy of sampling. Samples of vehicle speeds should be collected within fixed time intervals. The data for the study were collected in southern Wisconsin on a typical section of the rural state and Federal highway system.

Specific tentative conclusions reached are:

1. Hourly mean speeds show differences greater than chance would account for even after any possible effect produced by differences between days and months is eliminated.
2. The differences between Monday, Tuesday, Thursday, and Friday mean speeds are real and material.
3. The differences between monthly mean speeds are larger than can be accounted for by chance.
4. As sample sizes are increased without regard for the time interval involved, differences in the sample mean speeds provide estimates not only of the true changes in speed behavior but also changes in speed arising from differences in the hours, days, and months.
5. The quality of speed estimates can be improved by matching sampling periods by hour of day, day of week, and month of the year.

● MEASUREMENT of vehicle speeds in the traffic stream and subsequent interpretations have perhaps had too much attention since the motor car came to be a substantial factor in the national economy. The fact that most authorities on traffic accident prevention postulate a causal relationship between vehicle speeds and traffic accidents has been a major spur to investigations in this area.

Speeds in a traffic stream are most commonly measured with fixed-point measuring techniques, or spot speed studies, as they are usually called by traffic engineers. This method involves the measurement of an arbitrary number of vehicle speeds at a single point on a highway, using one of several mechanical or electronic speed-measuring devices currently available. From the speeds themselves, or from various computed statistics, certain inferences are drawn about the speed in the traffic streams under study.

This method is useful to the traffic engineer and others who seek to evaluate the effects of change in geometric design, signing, or increased law enforcement, for obvious and very practical reasons. Moreover it is applicable in research requiring an evaluation of speeds before and after modification of a single variable. The method requires a minimum capital investment in equipment and large amounts of data may be collected at a low per-unit cost as compared with other methods.

The measurement of vehicle speeds using fixed-point methods and subsequent interpretation would seem at first to be a relatively simple and straightforward problem.

Unfortunately, in spite of this deceptively simple appearance, the method has many and subtle shortcomings. Vehicle speeds occur within a time continuum bounded by an almost infinite number of dimensions, many of which directly affect speeds observed at a given point within the continuum. The problem of obtaining samples of vehicle speeds at a fixed point, which will adequately reflect the nature of the real world being studied, is complex. The more precise the measurement being attempted (as in the case of attempting to determine slight but real changes in speeds brought on by enforcement activity), the greater is the sampling problem.

A detailed inquiry into the development of sampling methods in speed measurements is necessary before experimental study of speed behavior on a large scale can be under taken. The nature of the real world and the theoretical considerations that underlie sar pling design suggest that sampling techniques now used for making fixed-point speed measurements are inadequate for refined experimental work. Before activity leading to real understanding of speed behavior can be undertaken, it is necessary to develop other, more satisfactory sampling techniques.

DEVELOPMENT OF THE STUDY

In September 1959, a series of exploratory samples of vehicle speeds were taken over a two-week period at several points on the state trunk highway system in southern Wisconsin. The purpose of these samples was to provide some initial insight into how vehicle speeds measured at a fixed location behave, and to provide the basis for assum tions necessary to develop a model for subsequent experimental investigation. The san ples thus gathered were deliberately varied so that data from different days of the week as well as different hours of the day would be available for study. No sampling period covered less than four continuous hours and many covered longer periods.

Evaluation of these exploratory samples gathered during September revealed statistically significant differences between the means of samples taken at the same location, under identical weather and light conditions, but on different days of the week or at different hours of the day. No systematic factors could be related to these variations, other than the times at which the samples were taken. Not only did the samples tested show statistica ly significant differences in regard to mean, variance, and proportion of vehicles over the speed limit, but the absolute value of the differences was sufficiently great to make the assignment of any, but extremely large changes to an experimental variable, impossit

As a result of this initial evaluation it was decided to undertake a study of vehicle speeds over an extended period of time under conditions as nearly normal as possible. The purpose of this study was to develop a theoretical model specifying a required sam pling design, and to test it by empirical methods. The development of the theoretical model and testing by empirical methods have proceeded concurrently, with modifications of both being made as the information available at a given time dictated.

For the study, a highway location was sought which would meet certain minimal spe ifications. These were:

1. Marginal development, both at the measuring point and for at least 2 miles in either direction, should be minimum.
2. Enforcement activity, both present and past, must have been at a minimum.
3. Traffic volumes should be sufficiently high to assure an adequate number of observations and range of volume levels, but not high enough to produce congestion.
4. Speed limits should be the maximum permitted for rural hgihways throughout the state.
5. Highway design should be such as to minimize its effect on speed behavior.

In November 1958, a site was selected that approximated the desired specifications. The highway at this juncture consists of two 11-ft driving lanes with 4-ft gravel-packed shoulders. The surface is travel-worn asphalt in good condition. Road alignment approaching the sampling site is straight for 1.1 mi to the north and 1.8 mi to the south. The grade within 1 mi in either direction is negligible. The marginal development con sists of scattered farms and the legal speed limit is 65 mph in the daytime and 55 mph at night. The nearest zones requiring reduced speed are 4 mi to the north and 6 mi to the south.

Sampling was begun at this location during the first week of November. A continuous sample of vehicle speeds was taken for 16 hours each day, from 7 a.m. to 11 p.m., Monday through Friday. The procedure was repeated for each subsequent month with the exception of February, when weather conditions throughout the month made sampling impossible. At the present time sampling has been completed through October 1959. Sampling will be continued through November 1959, making a full year of representative samples available for evaluation.

FIELD METHODS

It is necessary to discuss briefly the method used to take measurements in the field. A recently published study (1) presents strong evidence that measurement techniques themselves may produce material bias in data. All measurements of vehicle speeds were made with an Electronic Radar Speed Meter consisting of a sensing unit, a standard indicating meter calibrated in miles per hour, and a 12-volt power source. The sensing unit is located adjacent to the highway in a specially constructed mailbox, with a 12-volt battery concealed in the base as a power source. The observer recording the data is housed in a station wagon placed well back in a farmyard concealed from the view of drivers passing along the highway. The indicating meter is connected to the sensing unit by means of a 500-ft cord. The entire unit is calibrated by a standard calibrating tuning fork before each day's sample, again after 8 hr of operation, and again at the completion of each day's sample.

Speeds are read from the indicating meter by an observer and recorded in 5-mph intervals on a specially prepared recording sheet. Samples are recorded by direction of travel and by hour to facilitate study of specific periods of the day. Vehicles are divided into three classes: passenger cars, light trucks, and heavy trucks. Unusual traffic units, such as vehicles pulling house trailers or road machinery, are not included in the general sample but are noted separately.

Periodic samples of the actual speeds of individual vehicles are taken at irregular intervals, usually covering all vehicles that pass in a given 2-hr period. The samples are used primarily for the purpose of analyzing the distribution of speeds for normalcy. Periodic tests are made for the purpose of determining sample bias by driving a vehicle with a calibrated speedometer past the sensing unit at a known speed. Normally, six northbound and six southbound tests are conducted. No bias has been detected in either direction.

The observers are off-duty highway patrolmen with extensive experience in the use of radar for enforcement purposes. Each observer spends a minimum of 2 hr with the field supervisor standardizing his reading techniques before actually making observations on his own. The reading accuracy of each observer is checked at periodic intervals by the field supervisor who takes a series of readings from the meter while seated directly behind the observer. Reading error has been determined to be within ± 1mph.

The speeds of all vehicles are measured regardless of whether they are in free flow. This was done because the interest is in studying the effects of volume, as well as other factors, on vehicle speeds. To avoid repetition, all comparisons are standardized as to weather and road conditions, unless otherwise noted. All samples used in the analysis were gathered in fair weather on a dry road surface.

RESULTS

The data on which the results are based include samples taken through May 1959. It was decided at the outset to use time as a variable of classification. Each observed vehicle speed was classified by the hour of the day, day of the week, and month of occurrence. The method used for the analysis is known as "analysis of variance." Inasmuch as it was necessary to test the effect of all three variables of classification (hours, days, and weeks), and because the number of vehicles observed varies from hour to hour, it is necessary to use the method of fitting constants or some equivalent procedure. The usual technique of analysis of variance for equal numbers of observations in each cell will not work unless there are no effects associated with the variables of classification not being tested.

The method of fitting constants involves the construction of maximum-likelihood estimates of the effects associated with each variable of classification. Estimates of the mean speed for a given hour of a given day of a given month can then be made by adding the appropriate hour, day and month effect estimates to the grand average.

This particular analysis-of-variance test is based on the same assumptions and derived by the same methods as the more usual and familiar "t" test for the difference in means. If, for example, there were only two samples, the test used would reduce to a test operationally identical with the "t" test.

The data processed were confined to observations made between 7 a.m. and 4 p.m. on weekdays, Monday through Friday. The observations between 4 and 5 p.m. were omitted because previous tests had shown them to be heterogeneous. The 7 a.m. observations were omitted because the data for one of the hours were missing. It is possible to allow for the asymmetry due to the absence of these data, but in this instance the computations were eliminated by removing all 7-8 a.m. data from the sample.

For each set of data the following null and alternate hypotheses were tested:

Null	Alternate
All $c_k = 0$	Some $c_k \neq 0$
All $b_j = 0$	Some $b_j \neq 0$
All $a_i = 0$	Some $a_i \neq 0$

The first null hypothesis is that there is no month-to-month or seasonal shift in mean vehicle speeds. Similarly, the second and third null hypotheses imply that there are no day-of-the-week or hour-of-the-day effects.

In this case the likelihood ratio test can be reduced to an "F" test of the ratio of two independent estimates of the variance of e_{ijkL}. The denominator of this ratio is computed in the following manner:

The term $\sum_{ijkL} Y^2_{ijkL}$, which is the sum of the squared values of the observations, is calculated. From this is subtracted R (m + a, b, c), a function determined in the process of computing the estimated values of a_i, b_j, c_k, m. This difference, when divided by the appropriate degrees of freedom, gives an unbiased estimate of σ^2, allowing for the fact that the constants except those being tested may not all be zeros. This estimate of the mean square is called E, which is used in all three tests of hypotheses in each case. The numerator to test the hupotheses that $a_1 = a_2 = \ldots = a_R = 0$ is obtained as follows:

The constants for a model of the form $Y_{jkL} = m + b_j + c_k + e_{jkL}$, with the hour classification being ignored are estimated. A function R (m + c, b), similar to R (m + a, b, c) is derived. The difference between these two functions, R (m + a, b, c) - R (m + c, b), divided by the appropriate degrees of freedom, is an estimate of σ^2 if the null hypothesis is true. From construction this estimate is statistically independent of E. This estimate (corresponding to the null hypothesis that mean speeds are not related to time of day) is referred to as C_1. The ratio C_1/E, if the null hypothesis is true, has an F distribution with the appropriate degrees of freedom parameters. Values of $F\alpha$ may be picked from tables of the F distribution so that C_1/E will be more than $F\alpha$ only α percent of the time if the null hypothesis is true. In this instance values of α = 5 percent and α = 1 percent were chosen and, hence, F 0.05 and F 0.01 were selected. If, in fact, $a_1 = a_2 = \ldots = a_R = 0$, the null hypothesis is true; and if this procedure is adopted, the null hypothesis will be rejected, with the decision that not all $a_i = 0$ about one time in 20 or one time in 100, depending on whether F 0.05 or F 0.01 is used.

It is not the best possible technique to test several hypotheses on the same observations. However, these tests are orthogonal; that is, in principle the results of one test do not depend on the state of the real world with respect to the others. This appeared to be the best method for developing the background necessary for producing a satisfactory test-interpretation technique.

Table 1 is a symbolic outline of the analysis of variance just described for the null hypothesis, all $a_i = 0$, against the alternative hypothesis, some $a_i \neq 0$. The actual results are summarized in Tables 2 and 3.

TABLE 1

SYMBOLIC* ANALYSIS OF VARIANCE TABLE
(Null Hypothesis: All $a_i = 0$; Alternate Hypothesis: Some $a_i \neq 0$)

Variance Due to	Degrees of Freedom	Sum of Squares	Mean Square
Fitting $m + c_k, b_j$	S + T - 1	$R(m + c_k, b_j)$	
Difference	R - 1	(by subtraction)	C_1
Fitting $m + a_i, b_j, c_k$	R + S + T - 2	$R(m + a_i, b_j, c_k)$	
Remainder	N - R - S - T + 2	(by subtraction)	E
Total	N	$\sum_{ijkL} Y^2_{ijkL}$	

*In this case, the symbols have the following definitions: R = number of hours within which the data are classified, S = number of days within which the data are classified, T = number of months within which the data are classified, N = total number of observations made, and $\sum_{ijkL} Y^2_{ijkL}$ = sum of the squares of the observed values.

If $C_1/E > F_\alpha (R - 1, N - R - S - T + 2)$, then the probability of this value of C_1/E being observed is less than α if the null hypothesis is true. For values of C_1/E which exceed F_α for some predetermined level of α, the null hypothesis that all $a_i = 0$ is rejected and the alternate hypothesis that some $a_i \neq 0$ is accepted.

If, in fact, the a_i are not all zero, then C_1 is no longer an estimate of σ^2. It can be shown that C_1 will estimate some number larger than σ^2. The amount by which $E(C_1)$ (the expected value of C_1) exceeds σ^2 depends both on the magnitude of the a_i and the distribution of the number of observations. With the number of observations fixed, the greater the magnitude of the a_i, the more likely it becomes that a C_1/E greater than F_α will be observed.

In a similar manner tests of the null hypotheses $b_1 = b_2 = \ldots b_S = 0$ and $c_1 = c_2 = \ldots = c_T = 0$ can be constructed by obtaining the function $R(m + a, c)$ and $R(m + a, b)$. The appropriate degrees of freedom may be obtained as follows: If there are R hours, S days, and T months of observations in the data, and in this period N observations have been made, the degrees of freedom for $R(m + a, b, c,) = N - R - S - T + 2$. For $R(m + a, b)$, $R(m + a. c)$, $R(b + m, c)$ appropriate degrees of freedom are $(T - 1)$, $(S - 1)$ and $(R - 1)$, respectively.

The first four sets of data consist of vehicle speeds collected Mondays through Fridays in November and December 1958, and January, March, April, and May 1959. Each of these sets contains data from 30 separate days. The hours of each day during which data were collected for each set were as follows:

Set 1 passenger cars — days: 8 a.m. - 4 p.m., total 240 hr.
Set 2 passenger cars — nights: 5 p.m. - 11 p.m., total 180 hr.
Set 3 trucks — days: 8 a.m. - 4 p.m., total 240 hr.
Set 4 trucks — nights: 5 p.m. - 11 p.m., total 180 hr.

For each set the following hypotheses were tested:

1. That the variation between all hours, allowing for variation between months and days, was not great enough to be significant.
2. That the variation between all days, allowing for variation between months and hours, was not great enough to be significant.
3. That the variation between all months, allowing for variation between hours and days, was not great enough to be significant.

The results of these tests are summarized in Table 2.

TABLE 2

SUMMARY OF DIFFERENCES IN MEAN SPEEDS

Data Set	Test	C_1/E Ratio	Significance Level
Cars — 8 a.m. - 4 p.m.	Hourly mean speeds	4.67	0.01
	Daily mean speeds	19.17	0.01
	Monthly mean speeds	9.28	0.01
Cars — 5 p.m. - 11 p.m.	Hourly mean speeds	29.58	0.01
	Daily mean speeds	24.36	0.01
	Monthly mean speeds	69.64	0.01
Trucks — 8 a.m. - 4 p.m.	Hourly mean speeds	2.17	0.05
	Daily mean speeds	3.13	0.05
	Monthly mean speeds	58.33	0.01
Trucks — 5 p.m. - 11 p.m.	Hourly mean speeds	14.30	0.01
	Daily mean speeds	14.39	0.01
	Monthly mean speeds	35.01	0.01

From an examination of the data in Table 2 it may be concluded that, with possible exception of the daily and hourly mean speeds for trucks in the daytime, the difference in mean speeds are larger than can be accounted for by chance. The differences between hourly mean speeds, even after any possible effect produced by differences between days and months is removed, is too great to be attributed to mere chance fluctuations. The same thing may be said about daily mean speeds and monthly mean speeds.

At the same time that the data described were processed, and before the results were known, two additional sets of data were processed. These data consisted of the following:

Set 5 passenger cars — 8 a.m. - 4 p m., Monday, Tuesday, Thursday, Friday.
Set 6 passenger cars — 8 a.m. - 4 p.m., Tuesday, Thursday.

The purpose of these tests was to determine whether there was any combination of days for which some of the differences previously noted were not material. The results of this series of tests are summarized in Table 3.

The data summarized in Table 3 suggest the following conclusions:

1. The difference in monthly mean speeds is great enough to be material in all cases.
2. The differcnces between Monday, Tuesday, Thursday, and Friday mean speeds are real and material.

TABLE 3

SUMMARY OF DIFFERENCES IN MEAN SPEEDS

Data Set	Test	C_1/E Ratio	Significance Level
Cars—8 a.m. - 4 p.m. Mon., Tues., Thurs., and Fri.	Hourly mean speed	1.55	N.S
	Daily mean speed	12.36	0.01
	Monthly mean speed	8.30	0.01
Cars—8 a.m. - 4 p.m. Tues. - Thurs.	Hourly mean speed	2.52	0.05
	Daily mean speed	-	N.S
	Monthly mean speed	3.56	0.01

3. The differences between hourly mean speeds for these days are not material if allowance is made for differences arising from changing days of the week and month.
4. The variation in daily mean speed between Tuesday and Thursday is not material if allowance is made for differences arising from changing days of the week and month.

It should be noted that the conclusions reached for set 5 in regard to differences in hourly mean speed differ substantially from the conclusions reached using set 1. The differences are attributed to the fact that set 1 contained a Wednesday sample taken under abnormal weather conditions. It is quite possible that the effect of weather on the Wednesday samples was sufficient to override the effects of the normal days included in the sample, and thus produced the significant hourly effect apparent in set 1. This point will be explored more fully when complete data are available.

Evaluating the results in total it would appear that the differences noted between months is sufficiently great to preclude the possibility that it will disappear even with matching of equivalent months. Monthly mean speeds consistently showed the highest C_1/E ratios and all differences in monthly mean speeds showed significance at the 0.01 level of probability.

There is some evidence that certain blocks of hours are homogeneous and may be treated as equivalent if daily and monthly variation are taken into account. There is also some hope that certain days of the week may be treated as equivalent. In both cases conclusive results must await testing of total data, "cleaned" of bad weather influence, to find which (if any) hours and days may be considered homogeneous.

DISCUSSION

The work completed thus far clearly indicates that the measurement and interpretation of vehicle speeds are extremely complex. Even when such obvious factors as weather, light conditions, and location of measurement are standardized, other variable factors still tend to make comparative evaluations difficult. The data evaluated so far strongly suggest that samples taken at different time periods, even though weather, light and location conditions are standardized, cannot be used to interpret the significance of a difference in the mean of samples. Samples taken at the same location a month or a year apart will tend to be unreliable as a measure of changes produced by a newly introduced variable.

The quality of sampling can undoubtedly be improved by matching hours of the day, days of the week, and months. In the collection of fixed-point speed samples there appears to be no great difficulties in matching sampling time in terms of hour, day, and month. However, there is considerable question at present whether even such matching will completely eliminate the inherent variability.

It seems clear that in any event speed samples should be collected within fixed time intervals, as opposed to simply taking some minimum "satisfactory" number of observations. The danger in relying on the number of observations alone as the criterion of the adequacy of the sample is that as the sample size grows, the differences in samp mean speed reflect not only true changes in speed behavior over the interval between sampling periods, but also the changes in mean speed arising from differences in the hourly, daily, and monthly variables. Larger samples involving unmatched sampling periods insure the rejection of the typical null hypothesis that there has been no real change in speed behavior even in those cases where no change exists.

Conclusions already drawn indicate that the classical statistics of measurement cannot be applied indiscriminately to problems of traffic measurement. It would seem that, before statistical tests can be used to evaluate traffic measurements, care must be taken to insure that the test selected is applicable. In the case presented here the classical method of using the number of observations as the sole criteria for determining the adequacy of a sample can result only in erroneous and misleading results.

With the variability found in the data, fixed-point speed measurements have a limited value until such time as corrective measures can be used. Unfortunately, the question of an adequate sampling method cannot at the moment be answered in a positive manner. When the complete data now being gathered in this study are available two things may be accomplished:

1. Limits within which true speed values may be expected to fall can be established, thus serving to specify limits within which fixed-point measuring techniques may be used.

2. Correction factors may be developed which can be used to eliminate the bias that occurs because of daily, hourly, and monthly variation.

From study of currently available data it would appear that both these objectives are within the realm of feasibility.

REFERENCES

1. Crowther, R.F., Shumate, R.P., and Smith, R.D., "The Effect of Pneumatic Road Tubes on Vehicle Speed." The Traffic Institute, Northwestern University.
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Appendix

BACKGROUND FOR ANALYSIS OF VARIANCE

Assumptions

1. The speeds of cars observed within an hour constitute a random sample from a normal population.
2. Although the mean of the normal population may vary between hours, days, and locations, the variance of the underlying distribution remains unchanged.
3. The effects of the variables of classification are simply additive.

Notation

Let a_i be the effect associated with the ith hour, i = 1, 2 ... R.
Let b_j be the effect associated with the jth day, j = 1, 2 ... S.
Let c_k be the effect associated with the kth day, k = 1, 2 ... T.

Also, $\sum_{i=1}^{i=R} a_i = \sum_{j=1}^{j=S} b_j = \sum_{k=1}^{k=T} c_k = 0$

$Y_{ijkL} = m + a_i + b_j + c_k + e_{ijkL}$ where m is a constant, independent of the variables of classification and e_{ijkL} is a non-observable realization of a normal random variate with mean zero and variance σ^2. If we adopt the standard notation E (x) = the expected value of a random variable x, then our assumptions imply the following (in addition to the linear form):

1. $E(e_{ijkL}) = 0$

2. $E\,(e_{ijkL})^2 = \sigma^2$

3. $E\left[(e_{ijkL})\,(e_{wxyZ})\right] = 0$ unless $\begin{matrix} i = w \\ j = x \\ k = y \\ L = Z \end{matrix}$

et N_{ijk} be the number of cars observed in the ith hour of the jth day of the kth month.

ypotheses to Be Tested

For each set of data we wish to test the following null hypothesis against the cor-esponding alternative hypothesis.

Null Hypothesis		Alternative Hypothesis
1. $a_1 = a_2 \ldots = a_R = 0$	All $a_i = 0$	Some $a_i \neq 0$
2. $b_1 = b_2 \ldots = b_S = 0$	All $b_j = 0$	Some $b_j \neq 0$
3. $c_1 = c_2 \ldots = c_T = 0$	All $c_k = 0$	Some $c_k \neq 0$

hese null hypotheses may be restated as:

1. The mean speed of cars varies from hour to hour.
2. The mean speed of cars varies from day to day.
3. The mean speed of cars varies from month to month.

We wish to be able to test each of the null hypotheses irrespective of whether or not he remaining null hypotheses not immediately under test are true. Thus the question e wish to ask about the variations in hourly mean speeds is: if the effects on average peed associated with days and months are allowed for, does average speed change rom hour to hour? The questions we ask about the day of the week and month of the ear variables of classification are analogous.

eriving the Test Procedures

Under our assumptions a test may be derived from the likelihood ratio test proce-ure. It may be shown that in all the cases of the linear hypothesis the test criterion ecomes an F test of the minimum mean squared deviation under the null against the inimum mean squared deviation under the alternative hypothesis (2).

Thus we wish to construct estimators of a_i, b_j, c_k, and m, ($\hat{a}_i$, $\hat{b}_j$, $\hat{c}_k$, $\hat{m}$, respec-vely), which minimize $\sum_{ijkL} (Y_{ijkL} - \hat{a}_i - \hat{b}_j - \hat{c}_k - \hat{m})^2 = \sum_{ijkL} \hat{e}^2$. For a given set of bservations, $\sum_{ijkL} \hat{e}^2_{ijkL}$ can be regarded as a function of $\hat{a}_i$, $\hat{b}_j$, $\hat{c}_k$, $\hat{m}$ the estimates f the parameters, a_i, b_j, c_k, m. Let this function be represented by F ($\hat{e}$). Differ-ntiate F ($\hat{e}$) with respect to the variables $\hat{a}_i$, $\hat{b}_j$, $\hat{c}_k$, $\hat{m}$, where ($i = 1, 2 \ldots R$, $j = 1$, $\ldots S$, $k = 1, 2 \ldots T$), and set the resulting partial derivatives equal to zero:

hus if $\frac{\partial F\,(\hat{e})}{\partial\,\hat{m}} = 0$ then $\sum_{ijkL} (Y_{ijkL} - \hat{a}_i - \hat{b}_j - \hat{c}_k - \hat{m}) = 0$*.

ince N_{ijk} is the number of observations in the ith hour, of the jth day, of the kth onth we have the first normal equation:

$$a \sum_{jk} N_{1jk} + \hat{a}_2 \sum_{jk} N_{2jk} + \ldots + \hat{a}_R \sum_{jk} N_{Rjk} + \hat{b}_1 \sum_{ik} N_{i1k} + \hat{b}_2 \sum_{ik} N_{i2k} + \ldots + \hat{b}_S \sum_{ik} N_{iSk} +$$

$$a \sum_{ij} N_{ij1} + \hat{c}_2 \sum_{ij} N_{ij2} + \ldots + \hat{c}_T \sum_{ij} N_{ijT} + \hat{m} \sum_{ijk} N_{ijk} = \sum_{ijkL} y_{ijkL}$$

It may be shown that under our assumptions the second order conditions which require he value of F ($\hat{e}$) to be a minimum are also met.

Similarly we may derive a normal equation for a_r from the equation

$$\frac{\partial F(\hat{e})}{\partial \hat{a}_r} = 0$$

This implies for i = r

$$\sum_{jk1} (Y_{rikL} - \hat{a}_r - \hat{b}_j - \hat{c}_k - \hat{m}) = 0$$

Therefore

$$\hat{a}_r \sum_{jk} N_{rjk} + \hat{b}_1 \sum_k N_{r1k} + \hat{b}_2 \sum_k N_{r2k} + \ldots + \hat{b}_S \sum_k N_{rSk} + \hat{c}_1 \sum_j N_{rj1} + \hat{c}_2 \sum_j N_{rj2} + \ldots + \hat{c}_T \sum_j N_{rjT} + \hat{m} \sum_{jk} N_{rjk} = \sum_{jk1} Y_{rjk1}$$

In this manner a system of linear equations can be developed. Any non-trivial solutic provides a set of weights which will make orthogonal tests of hypotheses possible. W employed the additional constraints that

$$\sum_i \hat{a}_i = \sum_j \hat{b}_j = \sum_k \hat{c}_k = 0$$

in order to impose a specific solution.

Part of the total variation in Y_{ijkL} is accounted for by the fact that not all a_i, b_j, c_k, = 0. We wish to eliminate this part of the variation from the analysis. Let us de fine the corresponding sum of squares as R(m + a, b, c). Thus, R(m + a, b, c) =

$$\sum_i (\hat{m} + \hat{a}_i) \sum_{jkL} Y_{ijkL} + \sum_j \hat{b}_j \sum_{ikL} Y_{ijkL} + \sum_k \hat{c}_k \sum_{ijL} Y_{ijkL}$$

There are R + S + T - 2 degrees of freedom associated with this function. A similar analysis is possible under the null hypothesis that $a_1 = a_2 \ldots = a_R = 0$. The sum of squares accounted for in fitting the constants b_j, $c_k + m$ is denoted by R (b_j, $c_k + m$). There are S + T - 1 degrees of freedom associated with R (b_j, $c_k + m$).

It should be noted that by construction the estimates of a_i, b_j, c_k, m are maximum likelihood estimates under our assumptions. If the assumption that the e_{ijkL} have a normal distribution is relaxed, while the other assumptions are maintained, the esti-

mates $\hat{a}_i$, $\hat{b}_j$, $\hat{c}_k$, $\hat{m}$ are still the best linear unbiased estimates.