Application of Systems Engineering Methods to Traffic Forecasting

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Systems engineering techniques can be used in predicting trip distributions in urban road networks. To deal quantitatively with the interaction of components, each component must be describable mathematically and must be incorporated into the system in accordance with the requirements of linear graph theory. If the traffic system components can be identified, the traffic problem can be solved.

This paper discusses the principles of linear graph theory and the general requirements for using these methods. The most significant contribution comes from applying these techniques to traffic interchange. By using a hypothetical community, the techniques of systems engineering are compared with the gravity model and the electrostatic model for predicting the distribution of work trips.

• THE WORK described in this paper shows the application of systems engineering techniques to a traffic distribution problem. This method is a rigorous technique for computing the system of traffic flow in a road network. Precise and balancing results are obtained in one step by this method. Systems engineering techniques thus offer an extremely powerful tool for the analysis of systems when the system components and their measurements are adequately defined. This technique will be even more powerful when a computer solution for the routine matrix evaluation becomes available. It is excepted that this technique will not only be refined for application to traffic flow computation but will also be used for the analysis of other traffic engineering system problems.

BACKGROUND

The trend in the many studies of traffic engineering is toward a more mathematical and theoretical approach. This fact is evidenced by the writings of Herman, Schneider, Howe, Bevis, and others. One has only to refer to the bibliography of the special report by Haight (1) to conclude that traffic engineering is on the verge of a breakthrough. In spite of the as yet uncontrollable element of human choice or behavior, many phases of the traffic problem will evolve to a scientific level comparable to that of the physical sciences. The traffic engineering profession will probably have to settle for somewhat less replicability than the physical sciences because only two of their ingredients are physical, the vehicle and the facility, whereas the third, the user, presents different and still unsolved problems. This does not imply that the user problem is insurmountable. Although the individual has shown immunity to prediction, groups of many such individuals have shown that patterns can be observed.

Evolution has come to the other sciences by a slow but orderly progression. First, trial and error techniques were used and the effects were noted. The practitioner, who was faced with a series of difficult tasks, slowly added to his store of engineering judgment, and this he utilized on future problems as they developed. This was followed by a concentrated effort to collect and evaluate data. Analysis of data in the field of traffic flow has shown that groups of people are predictable and that mathematical formulas can be developed to express various travel habits and patterns. In the last stage of this evolution, the theory is established. In other fields, new theories have been developed by building on the terminology and theories defined previously and by critically analyzing the work of others in order to find a more direct approach.

So far the traffic theorists have concentrated their work on car-following theory, queuing and waiting line theory, and traffic simulation. Some work dealing with techniques such as the gravity model, the opportunity model and linear programing has been done in the area of theoretical origin and destination studies.

The work reported in this paper deals with development of a methodology for solving traffic flow in a road network by a mathematical model. The necessary preliminary theoretical testing of systems analysis as a technique for theoretical origin and destination studies has been accomplished during the study, but many simplifying assumptions were necessarily made. The theoretical results obtained with this technique are shown to compare encouragingly well with those available with other techniques. In the analysis of systems as a means for achieving a simple systematic procedure for formulating the system equations, the theory of oriented linear graphs, developed as an abstract mathematical topic, is valuable. Because the work with this technique is in its early stages, primary attention in this paper is given to the network description and only secondary consideration to the characteristics of the basic data.

The next step in the continuing research under way at Michigan State University will utilize these established concepts of systems engineering for predicting traffic flow in actual urban road networks.

SYSTEMS ENGINEERING THEORY

The techniques presented here have sometimes been referred to as "linear graph theory" and "network topology." These terms can be used interchangeably (2). A system can be defined as an orderly arrangement of interrelated elements acting together to achieve a specific purpose. Thus a system must have an avowed purpose, be free of extraneous or mathematically redundant parts, and have the elements or components joined in an orderly fashion. Discussion here is limited to systems made up of components having only two terminals, although there is no limit on the number of terminals the component may have in general systems theory.

For the computation of the system characteristics two steps are necessary:

1. To establish a mathematical model of the relevant physical characteristics of the system components expressed in terms of measurements.
2. To establish in mathematical form and in terms of measurements from a knowledge of the component characteristics and their mode of interconnection, the characteristics of the system; i.e., a mathematical model of the system.

Components are described mathematically by relating two measurements of the component in "isolation" from other components. (For a more detailed explanation see Appendix.) These measurements must be such that one is a "through" (or series) measurement called y, which when summed at each vertex must equal zero, and the other is an "across" (or parallel) measurement called x, which when summed around each circuit must equal zero. The relationship between measurements x and y is expressed mathematically and called the terminal equation of the component. The component is represented by an oriented line segment called the component terminal graph. The collection of component terminal graphs obtained by joining the vertices of each terminal graph in a one-to-one correspondence with the union of the physical components is called a system graph.

A "tree" is selected and the elements of the system graph are classified into either branches of the tree or chords. The "vertex postulate" or the "circuit postulate" is then applied to the system graph to establish the graph equations. The graph equations along with the terminal equations of the system components are defined as the system equations. The system equations represent a complete mathematical description of the system. These simultaneous equations are independent and can now be solved.

APPLYING TECHNIQUES TO TRAFFIC FLOW COMPUTATION

Although the techniques of system analysis were developed primarily in electrical network analysis, during the past several years this fundamental discipline of analysis has been applied usefully to many other areas, such as mechanical, hydraulic, and heat-transfer systems. Predicting traffic flow in an urban network also seems amenable to this technique, if the characteristics of the traffic problem can be defined in the form of suitable components and measurements which can be assembled into workable system graphs. The following discussion demonstrates how systems analysis can be applied to the traffic problem.

The selection of the units that will serve as components depends first on what type of system is being analyzed (i.e., transportation, sewage treatment, electrical, etc.), and second, on the specific question to be answered by the analysis of the system. For example, in the study of mass transit, the definition of components and measurements for a study of the system of service areas would probably be different than the definitions for a study of the effect of street capacities on the system.

In selecting the components for the system analysis of traffic flow in a road network it seemed at first feasible to consider the user, the vehicle, and the facility. "User" can be defined as one individual. An individual moves from place to place and this movement could be defined as flow or "through" variable (y). This movement or flow is related to a desire or "pressure" which can serve as the across variable (x). However, analyzing traffic flow on the basis of the individual user would lead to systems far too voluminous and too difficult to evaluate. Considering the vehicle as a basic component, a y measurement could be assigned to flow, but the x variable as desire or pressure is meaningless for a vehicle.

The dwelling unit and the family are the next possible basic components inasmuch as they combine the user and the vehicle. Both will afford the same x and y measurements as suggested for the individual. But even for the smaller urban areas, the number of dwelling units and families involved in the system would still be too large to provide a workable model, so that it will be more desirable to use even larger units, such as zones.

A zone, similar or identical to those used in origin and destination surveys, seems to be the best component evaluated to date. These zones should be defined so that the traffic characteristics within the zones are as homogeneous as possible. It is now assumed that the traffic characteristics of the entire zone can be computed from a limited number of parameters. For example, given a homogeneous residential zone containing a certain number of dwelling units or families and with known parameters such as residential density, income level, and car ownership, one should then be able to establish some value that would express the desire or pressure of that zone to generate a number of trips for a specific purpose. This pressure to make the trips from the given residential area to, say, the central business district (CBD) must be large enough to overcome the resistances against making these trips. The resistance is generated by the previous experience of traveling to the CBD, of parking difficulty and general congestion while in the CBD, and of returning home. If the pressure is not sufficient the trips will not be made. The number of trips that will be made is a function of pressure and resistance.

Research thus far has indicated that in the theoretical computation of traffic flow the classification of trip generation by trip purpose is essential. The percentage of trips assignable to each purpose seems well established. Further studies beyond those cited in the references (3) have proved that there are only very small variations in the percentages within classes of trips according to purpose. Accordingly, if it is assumed that in computing, say, shopping trips from a zone, one of the following three values can be determined from parameters, estimates, or in some other way:

1. Actual flow of shopping trips out of the zone in a given time period: $y(t)$ or
2. Demand or pressure for such shopping trips to be made from the zone: $x(t)$ or
3. Relationship between demand and flow; i.e., the function relating x and y:

$x = f(y(t))$ or $y = f(x(t))$,

then to solve the distribution problem of the trips generated in the zones, parts of the street network also must be defined as components with measurements suitable for systems analysis. In the preliminary stages only major arterials have been considered. Later developments might prove that, between any two zones, each group of streets of a specific type should be defined as a component with certain characteristics. Also, intersections might serve as separate components. The required x and y measurements could reflect such items as posted speed limit, parking, abutting land use, etc.

In the example to be discussed in the next section, simplifications are made. All routes between two zones are represented by two street components, one for each direction. No attempt was made to determine an x or y measurement, but rather an R value is used where x = Ry. In this equation R represents a parameter that is the mathematical relationship between x and y. It defines the relationship between flow and desire and might be visualized as a function of resistance or friction. The best physical measurement to be used as a basis for R was determined, at this stage only on an empirical basis, to be travel time. Preliminary work has led to a group of curves where the value R is a function of travel time and dependent on trip purpose (Fig. 1). Each street is assumed to be one-way for proper orientation of the terminal graph. The streets may also be viewed as two-way, but then each direction is represented as a separate component. The curve used has been obtained by substitution of a resistance scale and plotting the "frequency of trips" table as reported by Carroll (4). Further examination of travel time vs trip purpose as indicated in some of the recent comprehensive studies should reveal a stronger correlation than the curves presently used. This point, as well as many others, will require considerable research.

Other possible components of the system are shopping areas, work or employment centers, the CBD, recreational areas, and others. Once again the component must be described mathematically by x or y, or by some relationship between x and y. Early trials have been limited to representing work trips to shopping areas and industrial areas, where a flow y has been estimated on the basis of number of employees. Other measures of an area's attraction will be considered in future research. A possible relationship might come from comparing the specific area with some accepted standard area, just as a resistor is calibrated by comparing it with some standard. This type of approach is similar to McGrath's work in New Haven, where trip attraction based on an effective acre is established as a standard (5). Other possible component measurements will be tried as the research progresses.

SAMPLE PROBLEM

As an example, a small hypothetical community where an origin and destination survey is not available is assumed. The specific problem to be solved here by means of linear graph theory is one of determining the distribution of work trips from each residential zone to each employment zone. Initially, the community is separated into residential zones and zones of employment. The labor force of each nearly homogeneous residential zone might be determined from planning studies or from records of the local Chamber of Commerce. Work trips might be approximated by a correlation equation using driving time to the CBD and an estimate of car ownership. The number of auto driver trips could then be estimated from an average car occupancy value. The number of auto driver work trips arriving at each employment zone could be established by some similar procedure or by a sampling of the known number of employees in that zone. An estimate of auto driving time between every pair of zones in question could be prepared by actual observation or from a travel time map. The validity of this approach is not the concern of this paper.

Regardless of the techniques use, it may be assumed that the following information has been obtained concerning the small hypothetical community. There are four residential zones with a specified number of auto driver work trips as given in Table 1. The number of work trips that arrive at each of the three employment zones is also given in Table 1. The number of trips destined for employment zone 7 (Element 1) is not defined, although it is clear that 3,000 trips must be destined for this zone unless there is an unbalance in the work trip generation vs attraction. The number has been

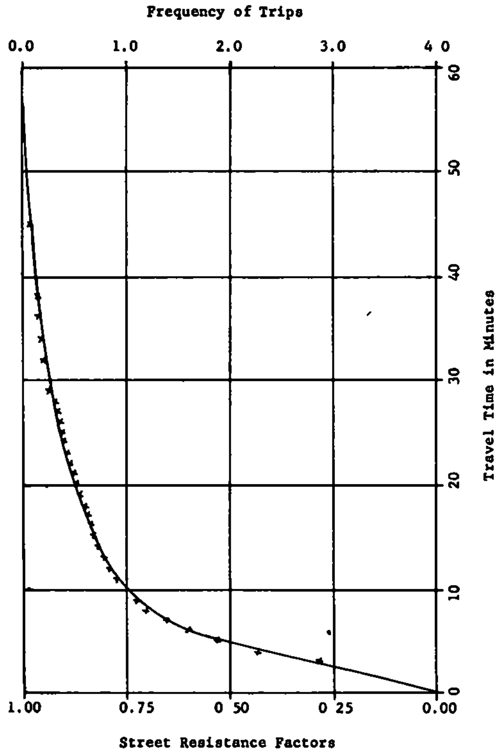

Figure 1. Travel time resistance for work trips.

TABLE 1

AUTO DRIVER WORK TRIPS TO AND FROM ZONES

Element	Zone	From	To
26	Res-1	4,000	--
27	Res-2	3,000	--
28	Emp-4	--	3,000
29	Emp-5	--	5,000
30	Res-3	2,000	--
31	Res-6	2,000	--
1	Emp-7	--	?

TABLE 2

TRAVEL TIMES BETWEEN ZONES IN MINUTES

1	28	29	Zones
10	10	14	26
17	14	10	27
10	17	14	30
14	20	10	31

left open in the table because the method of solution requires that this number be computed. The correct answer serves as a check on the computation.

The estimated travel times between the centroids of each zone are given in Table 2. The element number given for each zone is the number used for identification in the following analysis procedure.

The flow of trips completes a cycle in one day. Thus the time for which this analysis is made is a 24-hr period. Trips to work and trips home are identical in number but occur on different route components during different times of the day. The trips emanating from one zone also return to that zone during the study period.

Component Representation

Each zone is represented as a component part of the urban network and its measurements are shown in Figure 2.

Each residential zone, employment zone, and the system of routes that connect these zones can similarly be illustrated as components. Each component is represented as an oriented line segment which is referred to as the terminal graph of the component. For example, in Figure 2, a residential zone is shown schematically and as a terminal graph.

System Graph

With these components, the system graph can now be drawn in accordance with the following rules:

1. Components are joined in the system graph according to the manner in which the components are combined in the physical system.

2. The direction of flow is indicated by the direction of the line representing the components.

3. A "tree" is selected. This tree is a subgraph of the system graph containing all vertexes but no circuits. The tree is used in formulating the systems equations.

4. Specified desire (x values) are placed in the branches of the tree (B-1) (there are no specified x values here).

5. Specified flow (y values) are placed in the chord set, (C-2).

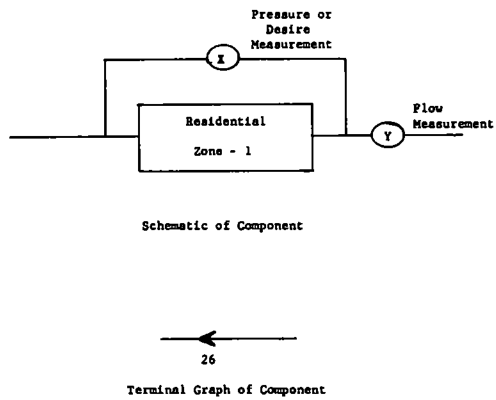

Figure 2. Component representation.

Figure 3a shows the system graph and how it is built; Figure 3b shows the tree chosen for this example. A tabulation of the values used as given measurements for the establishment and solution of this system is given in Table 3. In the matrix solution shown later, only R and y are used. The other parameters are given only for information. The relation between travel time and R was discussed and shown earlier in Figure 1.

Circuit Equations

The numerical solution of the system requires the writing of a set of equations (in matrix form) which is done here in accordance with the circuit postulate of linear graph theory.

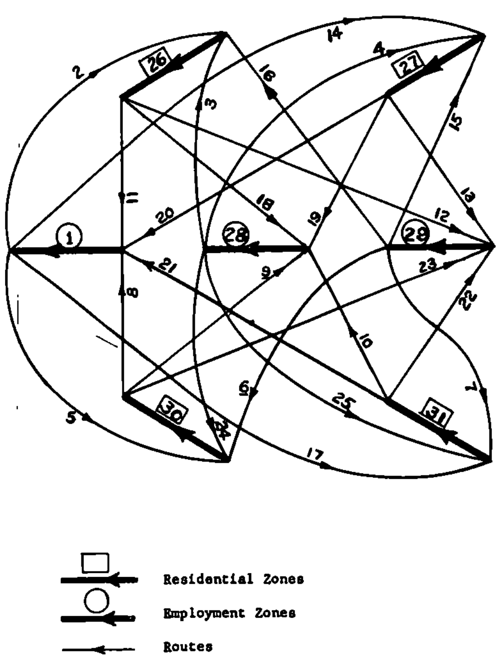

Figure 3a. System graph development.

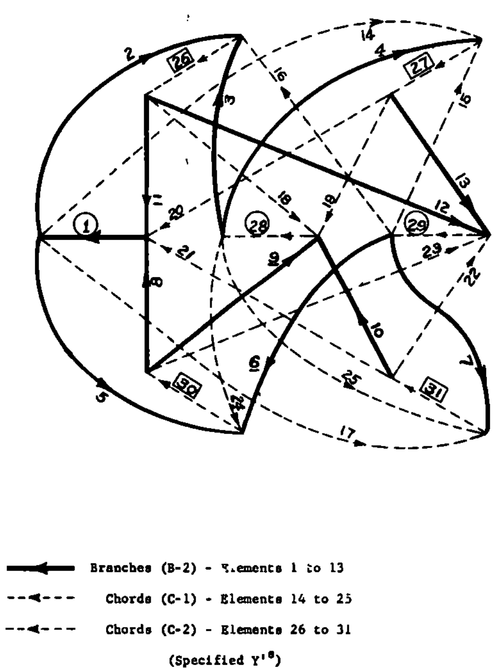

Figure 3b. System graph and tree.

TABLE 3

COMPONENT INFORMATION NECESSARY TO SOLVE THE LINEAR GRAPH

Element No.	Component	Travel Time (min)	Resistance Factor	Y (work trips)
1	Employment (I-7)	--	--	?
2	Street	10	0.7500	
3	Street	10	0 7500	
4	Street	14	0.8125	
5	Street	10	0.7500	
6	Street	14	0.8125	
7	Street	10	0 7500	
8	Street	10	0.7500	
9	Street	17	0.8450	
10	Street	20	0.8675	
11	Street	10	0.7500	
12	Street	14	0.8125	
13	Street	10	0.7500	
14	Street	17	0.8450	
15	Street	10	0.7500	
16	Street	14	0.8125	
17	Street	14	0 8125	
18	Street	10	0.7500	
19	Street	14	0.8125	
20	Street	17	0.8450	
21	Street	14	0.8125	
22	Street	10	0.7500	
23	Street	14	0.8125	
24	Street	17	0.8450	
25	Street	20	0.8675	
26	Residential (R-1)	--	---	4,000
27	Residential (R-2)	--	---	3,000
28	Employment (I-4)	--	---	3,000
29	Employment (I-5)	--	---	5,000
30	Residential (R-3)	--	---	2,000
31	Residential (R-6)	--	---	2,000

1. In accordance with the circuit postulate, the general equation for the k^{th} circuit is:

$$\sum_{j=1}^{e} b_j x_j = 0$$

in which

b_j = 0 if the j^{th} element is not included in the k^{th} circuit;

b_j = 1 if the orientation of the j^{th} element is the same as the orientation for the k^{th} circuit;

b_j = -1 if the orientation of the j^{th} element is opposite to the orientation of the k^{th} circuit.

2. Each circuit will have one and only one chord, and the circuit equations will be written in such sequence that a unit matrix results for the entries C-1 and C-2.

3. Equations using chords (C-1) are written first and chords (C-2) written last.

4. The x's are arranged in the column matrix in the following order X_{B-1}, X_{B-2}, X_{C-1}, and X_{C-2}.

5. The resulting matrix product is

$$
\left[\begin{array}{ccccccccccccc|cccccccccccc|cccccc}
0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\begin{bmatrix} x1 \\ x2 \\ x3 \\ x4 \\ x5 \\ x6 \\ x7 \\ x8 \\ x9 \\ x10 \\ x11 \\ x12 \\ x13 \\ x14 \\ x15 \\ x16 \\ x17 \\ x24 \\ x25 \\ x18 \\ x19 \\ x20 \\ x21 \\ x22 \\ x23 \\ x26 \\ x27 \\ x28 \\ x29 \\ x30 \\ x31 \end{bmatrix} = 0
$$

Column blocks: B-2 (columns 1 2 3 4 5 6 7 8 9 10 11 12 13); C-1 (columns 14 15 16 17 24 25 18 19 20 21 22 23); C-2 (columns 26 27 28 29 30 31)

General Solution of Equations in Symbolic Form

$$
\begin{bmatrix} B_{11} & B_{12} & U & 0 \\ B_{21} & B_{22} & 0 & U \end{bmatrix}
\begin{bmatrix} X_{B-1} \\ X_{B-2} \\ X_{C-1} \\ X_{C-2} \end{bmatrix} = 0
$$

$$
\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}\begin{bmatrix} X_{B-1} \end{bmatrix} + \begin{bmatrix} B_{12} & U \\ B_{22} & 0 \end{bmatrix}\begin{bmatrix} X_{B-2} \\ X_{C-1} \end{bmatrix} + \begin{bmatrix} 0 \\ U \end{bmatrix}\begin{bmatrix} X_{C-2} \end{bmatrix} = 0
$$

Substituting $RY = X$ for X_{B-2} and X_{C-1}:

$$
\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}\begin{bmatrix} X_{B-1} \end{bmatrix} + \begin{bmatrix} B_{12} & U \\ B_{22} & 0 \end{bmatrix}\begin{bmatrix} R_{B-2} & 0 \\ 0 & R_{C-1} \end{bmatrix}\begin{bmatrix} Y_{B-2} \\ Y_{C-1} \end{bmatrix} + \begin{bmatrix} 0 \\ U \end{bmatrix}\begin{bmatrix} X_{C-2} \end{bmatrix} = 0
$$

One of the key advantages of this type of analysis is the possibility of replacing certain unknown variables in an equation with a relation of known values. In the next substitution, the unknown Y_{B-2} are replaced by the known values Y_{C-2} (2).

$$\begin{bmatrix} Y_{B-2} \\ Y_{C-1} \end{bmatrix} = \begin{bmatrix} B_{12}^{T} & B_{22}^{T} \\ U & 0 \end{bmatrix} \begin{bmatrix} Y_{C-1} \\ Y_{C-2} \end{bmatrix}$$

which, when inserted, gives the main equation:

$$\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} \begin{bmatrix} X_{B-1} \end{bmatrix} + \begin{bmatrix} B_{12} & U \\ B_{22} & 0 \end{bmatrix} \begin{bmatrix} R_{B-2} & 0 \\ 0 & R_{C-1} \end{bmatrix} \begin{bmatrix} B_{12}^{T} & B_{22}^{T} \\ U & 0 \end{bmatrix} \begin{bmatrix} Y_{C-1} \\ Y_{C-2} \end{bmatrix} + \begin{bmatrix} 0 \\ U \end{bmatrix} \begin{bmatrix} X_{C-2} \end{bmatrix} = 0$$

In this main expression, the bottom line of equations, dealing only with known values, does not contribute to the solution. With the top equation from the preceding, the general mesh form or circuit equation can be written:

$$B_{11} \cdot X_{B-1} + \left[B_{12} \cdot R_{B-2} \cdot B_{12}^{T} + U \cdot R_{C-1} \right] \left[Y_{C-1} \right] +$$

$$\left[B_{12} \cdot R_{B-2} \cdot B_{22}^{T} \right] \left[Y_{C-2} \right] + 0 \cdot X_{C-2} = 0$$

The first term is nonexistent in this case, because no X_{B-1} values exist in this problem. Furthermore, because the last term is multiplied by zero, it vanishes. The resulting equation for this problem is therefore

$$\left[B_{12} \cdot R_{B-2} \cdot B_{12}^{T} + R_{C-1} \right] \left[Y_{C-1} \right] + \left[B_{12} \cdot R_{B-2} \cdot B_{22}^{T} \right] \left[Y_{C-2} \right] = 0$$

Arithmetic Solution

Numerical computations with the previous equation will give the desired values for Y_{C-1}, the flow on the streets connecting the zones. Replacing the symbols by the actual matrix and solving the triple matrix product gives the resulting matrix equation in terms of R. Inserting the proper R values and reducing the matrix size where possible, gives the two matrix equations:

$$\begin{bmatrix} +3.1575 & +2.3125 & +0.7500 & +0.0000 & -1.5000 & -1.5000 \\ +2.3125 & +4.6250 & +2.3125 & -1.5625 & -2.2500 & -3.0625 \\ +0.7500 & +2.3125 & +3.1250 & -1.5625 & -1.5000 & -2.3125 \\ +0.0000 & -1.5625 & -1.5625 & +3.1250 & +0.7500 & +2.3125 \\ -1.5000 & -2.2500 & -1.5000 & +0.7500 & +3.0950 & +2.2500 \\ -1.5000 & -3.0625 & -2.3125 & +2.3125 & +2.2500 & +4.6800 \end{bmatrix} \begin{bmatrix} Y14 \\ Y15 \\ Y16 \\ Y17 \\ Y24 \\ Y25 \end{bmatrix} + 0 \cdot \begin{bmatrix} Y18 \\ Y19 \\ Y20 \\ Y21 \\ Y22 \\ Y23 \end{bmatrix} + \begin{bmatrix} -5437.50 \\ -8625.00 \\ -6187.50 \\ +1687.50 \\ +3750.00 \\ +4687.50 \end{bmatrix} = 0$$

$$0 \cdot \begin{bmatrix} Y14 \\ Y15 \\ Y16 \\ Y17 \\ Y24 \\ Y25 \end{bmatrix} + \begin{bmatrix} +3.0950 & +2.3450 & +0.7500 & -1.5950 & -2.3450 & -1.5000 \\ +2.3450 & +4.7200 & +2.3125 & -1.5950 & -3.1575 & -2.3125 \\ +0.7500 & +2.3125 & +3.1575 & +0.0000 & -1.5625 & -1.5625 \\ -1.5950 & -1.5950 & +0.0000 & +3.2750 & +2.4625 & +0.7500 \\ -2.3450 & -3.1575 & -1.5625 & +2.4625 & +4.7750 & +2.3125 \\ -1.5000 & -2.3125 & -1.5625 & +0.7500 & +2.3125 & +3.1250 \end{bmatrix} \begin{bmatrix} Y18 \\ Y19 \\ Y20 \\ Y21 \\ Y22 \\ Y23 \end{bmatrix} + \begin{bmatrix} -1595.00 \\ -2220.00 \\ -2125.00 \\ -1640.00 \\ -1765.00 \\ -0875.00 \end{bmatrix} = 0$$

These equations can now be solved directly by solving first for Y_{14} to Y_{25} and then for Y_{18} to Y_{23}. The solutions are given in Table 4.

TABLE 4

WORK TRIPS OR FLOW THROUGH EACH ELEMENT

Element No.	Work Trips
1	3,000.00
2	1,199.10
3	1,215.40
4	825.08
5	536.13
6	973.54
7	1,044.91
8	536.13
9	490.33
10	469.19
11	1,199.10
12	1,585.50
13	1,396.05
14	778.87
15	1,396.05
16	1,585.50
17	485.90
18	1,215.40
19	825.08
20	778.87
21	485.90
22	1,044.91
23	973.53
24	490.33
25	469.19

Discussion of Systems Solution

The matrix manipulations and the arithmetic solution have only been sketched. A more detailed discussion of the steps taken and of the validity of the computation would have been too extensive for this paper, but these matters are explained in detail in the texts on this subject (2).

It might be argued at this point that a rather cumbersome algebraic procedure was used to solve a relatively trivial problem and that systems analysis for larger and more complex systems would be impossible to do manually. This is correct, but the extensive algebraic manipulations are an easy task for an electronic computer. The only reason for manually computing such a small example here is to demonstrate how such a problem can be worked. Michigan State University is presently developing a computer program which will solve these steps and produce the final answers directly from the given terminal and system equations.

A flow diagram of the solution showing the flow of trips to work only is given in Figure 4.

Comparison with Gravity Model

In this and the following sections the same sample problem is worked using the iterative processes called "gravity model" (6) and "electrostatic model" (7), respectively. Only the principal formulas used and the correction factors are repeated here.

The results of the second estimate of the gravity model and of the fourth assignment of the electrostatic model are compared with the results obtained with systems engineering methods in Table 5. The difference of each flow line computation by the three methods is shown and expressed in percentages, assuming the systems solution as a basis.

$$T_{ij} = \frac{T_i \times T_j / (D_{ij})^x}{\sum_{1}^{j} T_j / (D_{ij})^x}$$

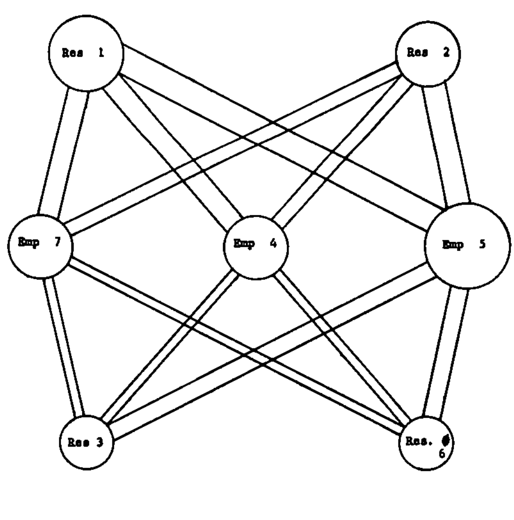

Figure 4. Flow diagram of work trips from four residential zones to three employment zones.

in which

T_i = number of auto driver work trips made from each residential area (i);

T_j = size of attractor represented by number of auto driver trips made to each industrial area (j);

D_{ij} = distance factor expressed in terms of travel time;

x = empirically determined exponent, assumed as 0.5.

Corrections in the first computation are made as follows:

1. A correction factor is computed for each value of j which equals

$$\frac{T_j}{\sum_{i}^{i} T_{ij}}$$

2. The first estimate is multiplied by the appropriate correction factors.

Comparison with Electrostatic Model

$$V_{P_i Q_j} = \frac{\dfrac{Q_j}{R_{ij}} \times P_i}{\sum_{j=1}^{m} \dfrac{Q_j}{R_{ij}}} \qquad (i = 1, 2, \ldots, n)$$

in which

$V_{P_i Q_j}$ = probability of movement from i to j;

P_i = number of workers living in Zone i;

Q_j = number of jobs available at Zone j;

R_{ij} = straight-line distance from i to j if the field contains no physical barriers. Where such barriers exist, R would have to be the straight-line distance from i to the point of passage across the barrier plus that from the point of passage to j.

TABLE 5

COMPARISON OF FLOW COMPUTATIONS

To Zone	Model[a]	From Zone 26			From Zone 27			From Zone 30			From Zone 31			Total to Zones
		Flow	Δ	%	Flow	Δ	%	Flow	Δ	%	Flow	Δ	%	
1	S	1,199 1			778 9			536 1			485 9			3,000.0
	G	1,161 5	- 37 6	- 3 1	695 0	- 83 9	-10 8	623 5	+ 87 4	+16 3	519 9	+34.0	+7 0	2,999 9
	E	1,217 6	+ 18 5	+ 1 5	583 7	-195.2	-25 0	707 8	+171 7	+32 0	490.9	+ 5.0	+1.0	3,000 0
28	S	1,215 4			825 1			490 3			469.2			3,000.0
	G	1,231 9	+ 16 5	+ 1.4	812 1	- 13 0	- 1 6	507 2	+ 16.9	+ 3.5	461.4	- 7 8	- 1 6	3,012 6
	E	1,359.8	+144.4	+11 9	791 6	- 33.5	- 4 1	464.9	- 25 4	- 5 2	383 7	-85 5	-18 2	3,000 0
29	S	1,585.5			1,396 1			973 5			1,044.9			5,000.0
	G	1,620 5	+ 35 0	+ 2 2	1,494.1	+ 98 0	+ 7 0	869 5	-104.0	-10 7	1,014.7	-30 2	- 2 9	4,998 8
	E	1,423.5	-162 0	-10 2	1,624.2	+228 1	+16 3	827.4	-146 1	-15 0	1,124.8	+79 9	+ 7.6	4,999 9
Total	S	4,000 0			3,000.1			1,999 9			2,000 0			11,000.0
	G	4,013 9			3,001 2			2,000.2			1,996 0			11,011 3
	E	4,000.9			2,999 5			2,000.1			1,999 7			11,000 1

[a]S = systems engineering, G = gravity model, E = electrostatic model

Corrections in this computation are based on the following equation:

1. First assignment is multiplied by the appropriate correction factor C_j for j = 1, 2, ..., m

$$C_j = \frac{Q_j}{\sum_{i=1}^{n} V_{P_i Q_j}}$$

2. The second assignment is multiplied by the appropriate correction factor C_i for i = 1, 2, ..., n

$$C_i = \frac{P_i}{\sum_{j=1}^{m} V_{P_i Q_j}}$$

3. The first and second steps are repeated for each successive assignment.

COMMENTS

The example problem, although extremely small, should have served to demonstrate the great potential of systems analysis by linear graph techniques for theoretical traffic flow problems. Many simplifying assumptions have been made. Further research is required to establish the technique to the point where it can be used to its full potential. The next step in the research will be a test of the theory against actual origin and destination surveys.

Future work on establishing other system components must be tried to evolve finally the best possible components or component systems, with proper x and y measurements and terminal representation. It is possible that a heterogeneous zone might be separated into distinct parts, each of which may be defined by a component and described mathematically.

The components used to date require a more complete study. For example, the curves used to predict the resistance factor on the street components must be studied more completely.

Although only a hypothetical case has been presented here, the results, when compared with the gravity model or the electrostatic model, show promise for eventual acceptance of this method.

It must be recognized, too, that changes in parameters in the given system can be made easily without disturbing the principles of the technique. Thus, any refinements in component definition can be entered as they become available without need for development of a new technique. Eventually this technique will permit research into the parameters.

Appendix

DETAILS OF LINEAR GRAPH

Measurements

In the mathematical analysis of any given type of physical system (electrical, mechanical, thermal, etc.) the tie between the mathematics and the system is generally accomplished through the use of two basic measurements; the across (or x) and the through (or y) measurements. The x and y measurements used to date are

1. Electrical.—x is voltage and y is current flow.
2. Mechanical translation.—x is displacement and y is force.
3. Thermal systems.—x is temperature and y is heat flow.
4. Hydraulics.—x is pressure and y is flow.

Definitions

Terminal Graph. —The terminal graph of an n-terminal component is defined as a collection of (n - 1) oriented line segments which forms a connected graph with no circuits and includes exactly one vertex for each terminal of the component.

In general, the terminal graph serves to identify the variables in the terminal equations with a unique set of measurements and establishes a vital link between the physical component and its mathematical description. The graph or terminal equation are each incomplete in themselves.

Terminal Equations. —The mathematical equations relating the measurements represented by the through and across variables of the terminal graph are called the terminal equations of the component.

Terminal Representation. —The terminal graph plus the terminal equations are called the terminal representation of a system component.

System Graph. —A system graph is a collection of component terminal graphs obtained by uniting the vertices of the terminal graphs in a one-to-one correspondence with the union of the physical components. When the fundamental operational concept of the linear graph is adopted, the system graph follows directly from the prescribed manner in which the components of the system are connected. If the characteristics of the system components can be determined—and they must be if the system is to be analyzed—there is never any question as to the form of the system graph.

System Equations. —The graph equations along with the terminal equations of the system components are defined as the system equations. The fundamental cut-set equations, stating that the sum of all through measurements at the vertices equal zero, and fundamental circuit equations, hereafter referred to as the graph equations, serve to establish a set of independent equations among the through and among the across variables used in presenting the characteristics of the system components.

The system equations represent a complete mathematical description of the system. When the terminal equations of the system components are linear, a partial solution to these simultaneous equations can be effected without the necessity of calculating an inverse. The partial solutions obtainable depend on the given forms of component terminal equations.

Tree. —If a connected graph G contains v vertices, connected subgraph of G containing all v vertices and no circuits is defined as a tree.

Branches. —The elements in the tree are appropriately called branches. Although there may be many different trees in any graph, a tree is easily identified by simply allowing one and only one element to join any pair of the v vertices. The tree which has all elements incident at one vertex is called a Lagrangian tree.

Chords. —The elements of a connected graph G which form the complement of a tree are defined as chords.

Specified Values. —The tree will then be further subdivided into those elements for which one has specified x or across variables, symbolically referred to as (B-1), and those for which no variable is known (B-2). The chords are also subdivided into (C-1) which are the unknowns and (C-2) for which the y or through variable has been specified.

Postulates

The graph equations can be established when the vertex and circuit equations are satisfied. The system must be such that the x or across variables will sum to zero around the circuits of the systems graph and further that the y or through variables sum to zero at the vertices of the linear graph. The fundamental across and through variables used to represent measurements in the various types of physical systems all have these important and fundamental properties. The mathematical formulation of these properties (the vertex and circuit equations, together with the component terminal equations) forms the basis for the analysis of physical systems. If the through and across variables of the system graph are defined so that they sum to zero at the vertices and around the circuits, then these techniques of formulation will apply to that physical system. These criteria can be formally stated as postulates.

Vertex Postulate.—If the system graph of a physical system contains e-oriented elements and if y_j represents the fundamental through variable of the j^{th} element, then at the k^{th} vertex of the graph

$$\sum_{j=1}^{e} a_j y_j = 0$$

in which

a_j = 0 if the j^{th} element is not incident at the k^{th} vertex;

a_j = 1 if the j^{th} element is oriented away from the k^{th} vertex;

a_j = -1 if the j^{th} element is oriented toward the k^{th} vertex.

Circuit Postulate.—If the linear graph of a physical system contains e-oriented elements and if x_j represents the fundamental across variable of the j^{th} element, then for the k^{th} circuit

$$\sum_{j=1}^{e} b_j x_j = 0$$

in which

b_j = 0 if the j^{th} element is not included in the k^{th} circuit;

b_j = 1 if the orientation of the j^{th} element is the same as the orientation chosen for the k^{th} circuit;

b_j = -1 if the orientation of the j^{th} element is opposite to that of the k^{th} circuit.

When the circuits used in writing the independent equations are chosen so as to include the branches first and chords last, a general and convenient form can be obtained. This is also applicable when writing the vertex or segregate equations except that one and only one branch is included when summing around the vertices.

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