# Mechanical Properties of Granular Systems 

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The granular systems treated are devoid of interparticle rohesinn, and the individual grains are independent of each other except for frictional interaction and geometric constraints incidental to the existing type of packing. The component particles may be of any size from the finest sand to pebbles, cobbles, or even boulders.

Existing theoretical concepts are briefly presented, critically reviewed and checkedagainst a large number of available test data. Special attention is given to the macromeritic state concept developed in analogy with accepted theories concerning the solid, liquid and solution states of molecular assemblies.

It is shown that the macromeritic state concept is a valuable heuristic tool that not only yields new insights into the shear behavior of granular systems but also permits the derivation of simple equations agreeing closely with experimental results.
-SYSTEMATIC arrangement of spheres in connection with the flow of water through soil was first studied by Slichter (81). An understanding of the packing of grains is of great importance in many branches of science and technology, such as soil engineering, ceramics, concrete and asphalt technology, crystallography, cytology and even pure mathematics.

## Systematic Packing of Uniform Spheres

Three different systems may be formed by stacking square horizontal layers one above another (Fig. 1a).

Case 1. -The cubic system is obtained when each sphere in the next horizontal layer has its center vertically above that of the sphere below.

Case 2. - The orthorhombic system results when the center of the upper sphere is offset a distance $R$ in the direction of one of the rows ( $R$ is the radius of the spheres).

Case 3.-The rhombohedral system is formed when the center of the upper sphere is moved a distance $R \sqrt{2}$ in a direction bisecting the angle between two sets of horizontal rows.

In a similar manner, three packings may be formed by stacking simple rhombic layers one above another (Fig. 1b).

Case 4. - In the orthorhombic system, the spheres of the next rhombic layer are placed in such manner that the center of each sphere lies vertically above the sphere below it.

Case 5. - In the tetragonal-spheroidal system, each sphere in the next rhombic layer rests in the cusp between two spheres in the layer below.

Case 6. -The rhombohedral system (Fig. 1c) results when each sphere of the next rhombic layer is placed in the hollow formed by three spheres of the lower layer.

Cases 2 and 4 are identical except that they have different orientations in space. The same applies to Cases 3 and 6 .

The rhombohedral system is also referred to as the clusely packed hexaguilal system. Case 3 is sometimes called pyramidal or face-centered cubic (in analogy with crystal lattices), whereas Case 6 is termed tetrahedral.

[^0]

Figure l. Systematic arrangements of uniform spheres.

Some important properties of these packinge are given in Table 1.

The coordination number is the number of contacts that a typical sphere makes with its neighbors. The unit cell may be defined as that smallest portion of the system which gives a complete representation of the manner of packing. Graton and Frazer (32) discuss this for each type of packing. Reference is also made to other properties of these systems, such as the interfacial angles of the unit cells.

The rhombohedral system is the most important theoretically, and usually is the basis for calculations. It is also the most important from a practical viewpoint, because it is the densest possible state.

In the Case 3 type of rhombohedral packing, each sphere is in contact with four spheres below, four above, and four in the same layer. In Case 6, each sphere has contact with three below, three above, and six in the same layer.

In the rhombohedral system, there are two types of voids:

1. Concave-cube void formed by six spheres of which four form a square, one lies in the hopper produced, and the sixth lies vertically below it; and
2. Concave-tetrahedron void formed by four spheres, one of which lies in the hopperlike depression formed by the other three.

The number of concave-tetrahedron voids is twice that of concave-cube voids in the rhombohedral system, but the distribution differs in the two cases. All voids are interconnected in such a way that the largest sphere that can pass through the circuit has a radius of $(2 / \sqrt{3}-1) \mathrm{R}(39)$.

Graton and Frazer (32) show that there are several alternative arrangements of Case 6 packing analogous to multiple twinning in crystallography. This results from the fact that the unit of the rhombic layer (Fig. 1) has two hollows into either one of which a sphere may be placed.

The stability of the systems increases as the porosity decreases. The cubical system is the least stable, because each sphere is delicately supported at one point below and is only prevented from toppling by lateral support provided by the four neighboring spheres in the same layer. The orthorhombic and tetragonal-spheroidal systems have more stability, because each sphere has two points of support from below as it rests in the cusp formed by two adjacent spheres. A force having a component perpendicular to the plane of this cusp will tend to topple it. The rhombohedral system has complete stability: in Case 6 each sphere rests in the hopper-like depression formed by three spheres, whereas in Case 3 each sphere rests in the hopper-like depression formed by a square of four spheres. If Case 1 packing is disturbed slightly, it may pass through Case 2 to Case 3. This happens when each sphere leaves its position at the top of a lower sphere and falls into an adjoining cusp and thence into an adjacent hopper. In a similar manner, Case 4 may pass into the Case 5 state and finally into Case 6.

The cubical packing has the greatest potential energy, whereas the rhombohedral has the least. Because bodies try to attain the position of least potential energy, these systems tend to form the rhombohedral state, especially if a mechanical disturbance is applied to the system. However, the side walls of the container act against this tendency by preventing any lateral spreading of the system.

The question of the relative stability of a system is only one of the factors influencing the formation of a given packing. The relative stability concerns the vertical relationships, but there are other conditions which influence the horizontal relationships.

If equal spheres accumulate on a horizontal surface, there will be a strong tendency for a simple rhombic arrangement to show itself in the first layer (32). If this occurs there is considerably better than an even chance that this pattern will propagate upwards
to form Case 6 packing. It is, of course, highly unlikely that this will continue without interruption. There are bound to be many regions of haphazard packing, each such region being initiated by a sphere getting into the wrong place. Thus, in the building-up of the assemblage, there is a conflict between the tendency to form Case 6 packing and the tendency to produce haphazard packing. Haphazard packing may, purely by chance, result in a systematic or repetitive arrangement in some regions. Therefore, the probability is that the resulting assembly will consist of three-dimensional colonies of Case 6 packing and of intervening regions of haphazard arrangement.

There is an optimum rate of assembly to give maximum regularity of arrangement. Obviously, rapid dumping is unfavorable.

The angle which the side wall makes with the bottom influences the packing formation. A $90^{\circ}$ angle will favor Cases $1,2,4$, and 5 ; a $60^{\circ}$ or $120^{\circ}$ angle favors Cases 2 and 3 , whereas a $70^{\circ} 32^{\prime}$ or $109^{\circ} 28^{\prime}$ angle favors Case 6 . The packing is also influenced by the angles which the side walls make with each other. A $90^{\circ}$ angle favors formation of the square pattern and, hence, Cases 1,2 , and 3 . Intersection of the side walls at $60^{\circ}$ with themselves and at $90^{\circ}$ with the bottom favors Cases 4 and 6 . Intersection of the side walls at $60^{\circ}$ with themselves and at $70^{\circ} 32^{\prime}$ with the bottom strongly favors Case 6 packing.

The walls of the container give rise to a wall effect which causes the porosity in the vicinity of the wall to be greater than that in the body of the packing. This effect has been studied by Furnas (28) who obtained an expression for the voids, $\mathrm{V}_{\mathrm{W}}$, present in a ring at the wall of area $\pi \mathrm{d} D / 2$ :

$$
\begin{equation*}
\mathrm{V}_{\mathrm{w}}=\{\mathrm{V}+\mathrm{K}(1-\mathrm{V})\}\left(\frac{1+2 \mathrm{Kd}}{\mathrm{D}}\right)-\frac{2 \mathrm{Kd}}{\mathrm{D}} \tag{1}
\end{equation*}
$$

in which d is the diameter of the particles, D is the diameter of container, V is the voids present in the interior, and K is an experimental factor found to be 0.3. The wall effect increases as the ratio $d / D$ decreases.

If the wall effect may be neglected, the density of the system will be independent of the absolute size of the spheres. This is confirmed by the experimental studies of Westman and Hugill (94).

Kolbuszewski (49) Carried out experiments on the effect of the rate of pouring of sand on the resulting porosity. As the rate of pouring from a given height decreased, the porosity decreased to a limiting value. This value itself decreased with increasing height of drop up to a certain height.

## Systematic Packing of Spheres of Different Sizes

Horsfield (37) calculated the decrease in porosity resulting from the insertion into the voids of the rhombohedral system of successive spheres just large enough to fill the voids. The spheres filling the concave-cube voids are termed secondary spheres,

TABLE 2
EFFECT ON POROSITY OF SPHERES INSERTED IN VOIDS OF RHOMBOHEDRAL SYSTEM

| Sphere <br> Type | Radiusa | No. of <br> Spheres | Porosity |
| :--- | :--- | :---: | ---: |
| Primary | 1 | 1 | 25.95 |
| Secondary | 0.4142 | 1 | 20.69 |
| Tertiary | 0.2247 | 2 | 19.01 |
| Quaternary | 0.1766 | 8 | 1.74 |
| Quinary | 0.1163 | 8 | 14.81 |
| Filler | 0.0000 | - | 3.84 |

[^1] whereas those filling the concave-tetrahedron voids are tertiary spheres. The quaternary spheres are inserted in the largest voids left after the secondary and tertiary spheres are in place, and so on through the different types. The results are given in Table 2. Of course, it is impossible to attain practically a system packed in such a manner.

Hudson (39) imagined the voids of the rhombohedral system to be filled with $S$ spheres of equal radii, r, arranged in cubic symmetry. He calculated the density increment for values of S up to $27(22,39)$. The densest state was obtained when each concave-cube void contained 21 spheres with $r=0.1782 \mathrm{R}$ (in which R is the radius
of the primary spheres) and each concavetetrahedron void contained 4 spheres with $\mathrm{r}=0.1547 \mathrm{R}$.

Dense Random Packing of Unequal Spheres

Wise (101) studied mathematically a dense random packing of unequal spheres which is more representative of a real densely packed system than the preceding systematic models. This randompacking is obtained in the following manner.

A large sphere $A$ is taken and other smaller spheres are placed on its surface. The first and second spheres must touch each other and also A. Every new sphere must touch A and at least two others that touch each other. Figure 2 shows what is seen looking out from A. If the center of a sphere $D$ is joined to the centers of the spheres around D, a network of triangles is formed, each of which lies in a different plane. These triangles are the faces of a polyhedron with the centers of the spheres as its vertices. If these vertices are joined to the center of $A$, the whole space inclosed by the polyhedron is partitioned into tetrahedra associated with the given sphere D. Wise sets up a probability distribution function w for the four radii in each tetrahedron and deduces general equations for $w$. The properties of the packing are expressed in terms of $w$.

In the special case in which the logarithms of the radii of the spheres follow a normal distribution of standard deviation, $\sigma=0.4$, the mean radius is 1.08 and the mean por osity of the packing is 0.2 . The mean number of spheres in contact with a sphere of given radius can also be calculated and is shown in curve $C$ of Figure 3.


Figure 3. Dense random packing of spheres having radii obeying a log-normal distribution of standard deviation $\sigma=0.4$ (101).

Wise's main concept of tetrahedra is used by Kallstenius and Bergau (46) in determining by a graphical method the structure resulting from a random assembly of grains.

Another type of random packing was theorized by Brandt (7) and used in the calculation of the speed of a dilatational wave through a granular system. The primary spheres are packed randomly to a porosity, $n$, and smaller uniform spheres are packed to the same porosity in the voids of the primary system. Still smaller spheres are packed to the same porosity in the remaining voids and so on.

## REAL GRANULAR SYSTEMS

In a real granular system, the coordination numbers of the spheres vary according to the position of each sphere.
W. O. Smith (82) suggested a simple method to determine the average coordination number, N , of a system having a certain porosity, n . N is considered to decrease linearly as the porosity increases from 25.95 percent (rhombohedral system) to 47.64 percent (cubic system). N can, therefore, be obtained by interpolation between the corresponding values of 12 and 6 .



Figure 5. Average number of contacts per sphere, $\mathbb{N}$, as function of porosity n (83).

Later W. O. Smith et al. (83) determined experimentally the coordination numbers in a well packed aggregate of lead shot for various porosities. At a given porosity, the number of spheres having a given coordination number varied according to a Gaussian distribution (Fig. 4). At a porosity of 36 percent, the greatest number of spheres had a coordination number of 8 , and the values of $N$ ranged from 4 to 12 . As the porosity increased, there was a shift toward the lower coordination numbers, as may be expected. These authors thought that the actual system might for statistical purposes be treated as composed of separate clusters of rhombohedral or cubic arrangements, these being present in such a proportion as to give the observed porosity of the assembly. This consideration leads to the following expression for the average coordination number, N , in terms of the porosity, n :

$$
\begin{equation*}
\mathrm{N}=26.4858-\frac{10.7262}{1-\mathrm{n}} \tag{2}
\end{equation*}
$$

The curve representing this is shown in Figure 5 and agrees well with the observed experimental values.

## MECHANICAL BEHAVIOR OF GRANULAR SYSTEMS

A theoretical model for determining the mechanical behavior of a granular system is an arrangement of discrete spheres in direct elastic contact with one another.

## Contact Theory

The classical Hertz theory of contact predicts that when two elastic spheres in contact are compressed by a force, $N$, along their line of centers, there will be a plane circular area of contact (91). The radius, a, of this circle is assumed to be small compared to the radius, $\Omega$, of either sphere and is given by

$$
\begin{equation*}
a=(\theta N R)^{1 / 3} \tag{3}
\end{equation*}
$$

in which

$$
\begin{equation*}
\theta=3\left(1-\nu^{2}\right) / 4 \mathrm{E} \tag{4}
\end{equation*}
$$

and $\nu$ and E are Poisson's ratio and Young's modulus, respectively, of the spheres. The normal pressure $\sigma$ at distance, $\rho$, from the center of contact ( $\rho \leq a$ ) is given by

$$
\begin{equation*}
\sigma=\frac{3 N}{2 \pi a^{3}}\left(a^{2}-\rho^{2}\right)^{1 / 2} \tag{5}
\end{equation*}
$$

which is a parabolic distribution (Fig. 6). The relative approach $\alpha$ of the sphere centers is

$$
\begin{equation*}
\alpha=2\left(\frac{\theta \mathrm{~N}}{R^{1 / 2}}\right)^{2 / 3} \tag{6}
\end{equation*}
$$

from which the normal compliance C of the contact is

$$
\begin{equation*}
\mathrm{C}=\frac{\mathrm{d} \alpha}{\mathrm{dN}}=\frac{4}{3}\left(\frac{\theta^{2}}{\mathrm{RN}}\right)^{1 / 3}=\frac{1-\nu}{2 \mu \mathrm{a}} \tag{7}
\end{equation*}
$$

in which $\mu$ is the shear modulus of the spheres. As apparent from Eqs. 3 and 6, a and $\alpha$ do not vary linearly with N . This fact introduces mathematical difficulties.

If the normal force, N , is kept constant and a tangential force is applied in the plane of contact and gradually increased from zero to T, slip will start at the circumference of the circle of contact and progress radially inward covering an annular area. The inner radius, c , of the annulus of slip was found by Mindlin (56) to be


Figure 6. Distribution of normal, $\sigma$, and tangential, $\tau$, components of traction on contact surface of two like spheres subjected to normal force followed by monotonic tangential force (22).

$$
\begin{equation*}
c=a\left(1-\frac{T}{\mathrm{fN}}\right)^{1 / 3} \tag{8}
\end{equation*}
$$

in which $f$ is the coefficient of static friction. The relative tangential displacement, $\delta$, of the centers of the spheres is

$$
\begin{equation*}
\delta=\frac{3(2-v) \mathrm{fN}}{8 u \mathrm{a}}\left[1-\left(1-\frac{\mathrm{T}}{\mathrm{fN}}\right)^{2 / 3}\right] \tag{9}
\end{equation*}
$$

The tangential compliance of the contact is

$$
\begin{equation*}
\mathrm{S}=\frac{\mathrm{d} \delta}{\mathrm{dT}}=\frac{2-\nu}{4 \mu \mathrm{a}}\left(1-\frac{\mathrm{T}}{\mathrm{fN}}\right)^{-1 / 3} \tag{10}
\end{equation*}
$$

On the annulus of slip, the tangential component of traction, $\tau$, is assumed to be

$$
\begin{equation*}
\tau=\mathrm{f} \sigma \tag{11}
\end{equation*}
$$

As the tangential force $T$ approaches fN, Eq. 8 shows that $c$ tends to zero. When $\mathrm{T}=\mathrm{fN}$, rigid body sliding occurs.

It is important to note that the tangential compliance $S$ is of the same order of magnitude as the normal compliance $C$.

If the tangential force acting on the two spheres under consideration is gradually reduced from a peak value $\mathrm{T}_{1}\left(0<\mathrm{T}_{1}<\mathrm{fN}\right)$, an annulus of counter slip will be formed,
starting from the edge of the area of contact, and will gradually spread radially inward. Its inner radius was found by Mindlin et al. (58) to be

$$
\begin{equation*}
b=a\left(1-\frac{T_{1}-T}{2 f N}\right)^{1 / 3} \tag{12}
\end{equation*}
$$

The corresponding relative displacement of the centers of the spheres during unloading is

$$
\begin{equation*}
\delta_{u}=\frac{3(2-\nu) \mathrm{fN}}{8 \mu \mathrm{a}}\left[2\left(1-\frac{\mathrm{T}_{1}-\mathrm{T}}{2 \mathrm{fN}}\right)^{2 / 3}-\left(1-\frac{\mathrm{T}_{1}}{\mathrm{fN}}\right)^{2 / 3}-1\right] \tag{13}
\end{equation*}
$$

This is shown by curve PRS in Figure 7. When T decreases to zero, there will remain a residual displacement, $\delta_{\mathrm{R}}$, and a certain annulus of slip. (This can be seen by setting $\mathrm{T}=0$ in Eq. 12, $\mathrm{T}=\mathrm{T}_{1}$ in Eq. 8 and noting that $\mathrm{b}-\mathrm{c}>0$.) This indicates that the system is not elastic. The permanent set can be removed only by applying a tangential force in the reverse direction. When $\mathrm{T}=-\mathrm{T}_{1}, \mathrm{~b}=\mathrm{c}$; that is, the slip has been annulled. An oscillation of the applied tangential force between the values $T_{1}$ and - $T_{1}$ causes the closed hysteresis loop PRSUP to be followed.
The area inclosed in the loop represents the frictional energy $F$ dissipated in each cycle of loading:
$\mathrm{F}=\frac{9(2-\nu)(\mathrm{fN})^{2}}{10 \mu \mathrm{a}}\left\{1-\left(1-\frac{\mathrm{T}_{1}}{\mathrm{fN}}\right)^{5 / 3}-\right.$

$$
\begin{equation*}
\left.\frac{5 \mathrm{~T}_{1}}{6 \mathrm{fN}}\left[1+\left(1-\frac{\mathrm{T}_{1}}{\mathrm{fN}}\right)^{2 / 3}\right]\right\} \tag{14}
\end{equation*}
$$

For small amplitudes of loading (i.e., $\mathrm{T}_{1} / \mathrm{fN} \ll 1$ ) Eq. 14 reduces to

$$
\begin{equation*}
F=\frac{(2-\nu) \mathrm{T}_{1}^{3}}{36 \mu \mathrm{afN}} \tag{15}
\end{equation*}
$$

The conclusions of this theory have been verified experimentally except for Eq. 15 in which $\frac{\mathrm{F}}{3}$ was found to vary as $\mathrm{T}_{1}{ }^{2}$ instead of $\mathrm{T}_{1}{ }^{3}$. This discrepancy may be explained by supposing that at small values of $T_{1}$, energy is dissipated as a result of plastic deformation of a small portion of the contact surface (22).

The two spheres are generally subjected to varying normal and tangential forces (i.e., a varying oblique force) when the granular system of which they are a part is acted on by varying external forces or is in a state of internal vibration. In such cases, as mentioned by Deresiewicz (22), the relation between the instantaneous tangential forces and displacements depends not only on the initial state of loading but also on the entire history of normal and tangential forces.


Figure 7. Theoretical hysteresis loop showing relative tangential displacement, $\delta$, of centers of two spheres due to normal force (22).

Moreover, a variety of phenomena are involved which depend upon: whether either the normal or the tangential force is held constant while the other varies; whether they both vary and whether the sense of variation is such that one increases while the other decreases, both increase, or both decrease; whether the relative rate of change of the two forces is greater or less than the coefficient of friction; whether the immediately preceding history of loading was in the same or in the opposite sense as the current loading.

Deresiewicz (22) gives the solutions (i.e. , the tangential compliances) for oblique forces applied in a certain manner, including the oblique force which maintains a constant direction but oscillates in magnitude between equal values-a problem connected with a vibration of a granular system. Deresiewicz also outlines the solution to the case of a twisting couple applied about the line of centers of two spheres in contact. The spheres are compressed by a constant normal force. The results are similar to those presented previously for a tangential force applied to the two spheres at constant normal force.

## Velocity of Waves Through Granular System

To determine the velocity of compressional waves through a granular system, the grains have been assumed to be in elastic contact with each other and contact theory has been used.

Hara (34) studied the propagation of compressional waves of long wavelength in a system of equal spheres of cubic and rhombohedral (Case 3) arrangements. The direction of propagation was taken to be parallel to one of the edges of the unit lattice. The spheres were imagined to be replaced by mass-spring systems in series, in which the stiffness of each spring was computed from Hertz's theory of normal contact. If N is the normal force at each contact and $R$ is the radius of each sphere, the velocity, $C_{1}$, of the compressional waves was found by Hara to be proportional to $\left(N / R^{2}\right)^{1 / 6}$. If the system is acted on by its own weight only, $\mathrm{C}_{1}$ is proportional to the sixth root of the height of the system.

Gassmann (30) considered a rhombohedral (Case 6) system of equal spheres, each of radius, $R$. Taking as the origin the center of a sphere in the topmost layer (zerolayer) and taking the $z$-axis vertically downward, the following equations give the coordinates of the center of the sphere in the kth layer:

$$
\begin{gather*}
x=\frac{R}{2}\left[4 i+2 j+(-1)^{k}+1\right]  \tag{16}\\
y=\frac{\sqrt{3}}{4} R\left[4 j+(-1)^{k}+1\right]  \tag{17}\\
z=\frac{2 \sqrt{6}}{3} R k \tag{18}
\end{gather*}
$$

in which i and j are arbitrary integers.
Gassmann assumed there was zero pressure between spheres in the same layer and that a given sphere was subjected only to the weight of the spheres lying above it. The variations of the stress from its initial value are assumed to be small enough that the increments in the components of stress and strain are linearly related. If the number of layers and spheres is sufficiently large, the system at its initial state of stress can be considered a homogeneous anisotropic porous solid with the symmetry of Voigt's class No. 26 of crystals (53). In other words, the system has transversely isotropic symmetry (53), all directions perpendicular to the $z$-axis being equivalent with respect to elasticity. The matrix of the elastic constants is

| $\mathbf{C}_{1}$ | $\mathrm{C}_{1}-2 \mathrm{C}_{5}$ | $\mathrm{C}_{2}$ | 0 | 0 | 0 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{C}_{1}-2 \mathbf{C}_{5}$ | $\mathrm{C}_{1}$ | $\mathrm{C}_{2}$ | 0 | 0 | 0 |
| $\mathrm{C}_{2}$ | $\mathrm{C}_{2}$ | $\mathrm{C}_{3}$ | 0 | 0 | 0 |
| 0 | 0 | 0 | $\mathrm{C}_{4}$ | 0 | 0 |
| 0 | 0 | 0 | 0 | $\mathbf{C}_{4}$ | 0 |
| 0 | 0 | 0 | 0 | 0 | $\mathbf{C}_{5}$ |

in which $\mathrm{C}_{1}=\overline{\mathrm{C}}, \mathrm{C}_{2}=\mathrm{C}_{4}=4 \overline{\mathrm{C}}, \mathrm{C}_{3}=16 \overline{\mathrm{C}}, \mathrm{C}_{5}=0$ and

$$
\begin{equation*}
\overline{\mathrm{C}}=\frac{1}{24 \sqrt{2}}\left[\frac{6 \mathrm{E}^{2} \mathrm{~N}_{\mathrm{k}}}{\left(1-v^{2}\right)^{2} \mathrm{R}^{2}}\right]^{1 / 3} \tag{19}
\end{equation*}
$$

$N_{k}$ is the normal force on the contact between spheres of the $k$ th and $(k+1)$ th layers. If no load is assumed above the first layer

$$
\begin{equation*}
N_{\mathrm{k}}=\frac{\pi}{3} \mathrm{R}^{2} \rho \mathrm{gz} \tag{20}
\end{equation*}
$$

assuming the density of the air filling the voids to be negligible in comparison with the density, $p$, of the spheres; $g$ is the acceleration due to gravity. The partial differential equations of wave motion through the system can be obtained by substituting the elastic constants obtained previously into the general equations of motion of an elastic solid (53), assuming the wavelength to be large compared with R . The solution (30) gives the velocities of propagation of plane waves through the system. There exist in general three distinct wave velocities, corresponding to three different waves. Each velocity depends on the direction of wave propagation.

Gassmann calculated the variation of the largest velocity with depth using the elastic constants of granite and taking the z -axis as the direction of wave travel. The velocity varied as the sixth root of the pressure.

Deresiewicz (22) discusses the experimental work on the relation between the velocity, the pressure and the percentage of water filling the voids.

Brandt (7) considers a system of several sizes of spheres randomly packed. His model is closer to an actual granular system than are the regular models assumed by others. The largest (primary) spheres, of number, $\mathrm{K}_{1}$, and radius, $\mathrm{R}_{1}$, are assumed to be randomly packed to a porosity, $n$. The number, $\mathrm{K}_{1}$, is taken to be large enough that the wall effect of the inclosure may be neglected. The volume V of the inclosure is

$$
\begin{equation*}
\mathrm{V}=\frac{4 \pi \mathrm{~K}_{\mathbf{1}} \mathrm{R}_{\mathrm{n}}{ }^{3}}{3(1-\mathrm{n})} \tag{21}
\end{equation*}
$$

Secondary spheres ( $\mathrm{K}_{2}$ in number and of radius, $\mathrm{R}_{2}$ ) are assumed to be packed randomly in the voids of the primary system to the same porosity, n. In a similar manner, tertiary spheres are packed in the remaining voids, and finally quaternary spheres are randomly packed in the spaces that are left. If $R_{i} / R_{i+1}$ is large enough

$$
\begin{equation*}
\mathrm{V}_{1}: \mathrm{V}_{2}: \mathrm{V}_{3}: \mathrm{V}_{4}=1: \mathrm{n}: \mathrm{n}^{2}: \mathrm{n}^{3} \tag{22}
\end{equation*}
$$

in which $V_{i}$ is the total true volume of the ith set of spheres. This relation has been shown experimentally by Furnas (29) to be approximately true when $R_{i} / R_{i+1} \geq 5$.

If the volume of the inclosure containing the granular system is decreased so that the spheres deform, and $\Delta_{1}$ is the decrease in the radius of each primary sphere, the new volume of the inclosure is

$$
\begin{equation*}
\mathrm{V}_{\mathrm{n}}=\frac{4 \pi \mathrm{~K}_{1}\left(\mathrm{R}_{1}-\Delta_{1}\right)^{3}}{3(1-\mathrm{n})}=\frac{4 \pi \mathrm{~K}_{1} \mathrm{R}_{1}^{3}}{3(1-\mathrm{n})}-\frac{4 \pi \mathrm{~K}_{1} \mathrm{R}_{1}^{2} \Delta_{1}}{(1-\mathrm{n})} \tag{23}
\end{equation*}
$$

neglecting terms involving $\Delta_{1}{ }^{2}$ as is done in the Hertz theory of contact. The decrease in the inclosure volume is

$$
\begin{equation*}
\mathrm{V}_{\mathrm{d}}=\mathrm{V}-\mathrm{V}_{\mathrm{n}}=\frac{4 \pi \mathrm{~K}_{1} \mathrm{R}_{1}{ }^{2} \Delta_{1}}{(1-\mathrm{n})} \tag{24}
\end{equation*}
$$

The decrease in the volume of the spheres is of the order of $\Delta_{1}{ }^{2}$ and may be neglected. Hence, $\mathrm{V}_{\mathrm{d}}$ may be taken as the decrease in void volume, that is, the decrease in the bulk volume of the secondary system of spheres resulting in a decrease $\Delta_{2}$ in the radius of each secondary sphere. Repetition of this reasoning gives the relation

$$
\begin{equation*}
\frac{\Delta_{2}}{\Delta_{1}}=\frac{1}{n U} \tag{25}
\end{equation*}
$$

in which $U=R_{1} / R_{2}$. Assuming $U=R_{i} / R_{i+1}(i=1,2,3,4)$, this procedure yields

$$
\begin{equation*}
\frac{\Delta_{i}}{\Delta_{1}}=\frac{1}{(n U)^{i-1}} \tag{26}
\end{equation*}
$$

Brandt then considers a case where the flexible inclosure of the granular system is subjected to an all-round pressure, $p_{0}$, while a liquid is introduced at a pressure $P_{L}$. An energy balance may then be set up in which the energy, $\mathrm{E}_{\mathrm{T}}$, required to decrease the bulk volume of the system is equated to the sum of the energy, $\mathrm{E}_{\mathrm{S}}$, used in deforming the spheres and the energy, $\mathrm{E}_{\mathrm{L}}$, employed to compress the interstitial liquid:

$$
\begin{equation*}
\mathrm{E}_{\mathrm{T}}=\frac{4 \pi \mathrm{~K}_{1} \mathrm{R}_{1}^{2}}{1-\mathrm{n}} \int_{0}^{\Delta_{1}} \mathrm{p}_{\mathrm{o}} \mathrm{dx} \tag{27}
\end{equation*}
$$

and

$$
\begin{equation*}
E_{S}=\sum_{i=1}^{4} N K_{i} \int_{0}^{\Delta_{i}} F_{i} d x \tag{28}
\end{equation*}
$$

in which N is the average coordination number of the spheres and is assumed to be the same for the four sets, which are packed to the same porosity, $n$, and $\mathrm{F}_{\mathrm{i}}$ is the average force at the contact of a sphere in the ith set. The energy balance gives rise to a cubic equation in $F_{1}$. The approximate solution obtained by Brandt after insertion of the experimental value $\mathrm{N}=8.84$ is

$$
\begin{equation*}
\mathrm{F}_{1}=\frac{2.34 \mathrm{R}_{1}{ }^{2}\left(\mathrm{p}_{\mathrm{o}}-\mathrm{P}_{\mathrm{L}}\right)}{\mathrm{C}_{\varphi \rho}\left[1+\frac{30.75 \mathrm{~B}^{3 / 2}\left(1-\nu^{2}\right)}{\mathrm{E}\left(\mathrm{p}_{\mathrm{O}}-\mathrm{P}_{\mathrm{L}}\right)^{1 / 2}}\right]} \tag{29}
\end{equation*}
$$

in which $B$ is the bulk modulus of the liquid, $\nu$, is the Poisson's ratio of the spheres and $\left(C_{0}\right)^{1 / 3}=2 / 3 n^{1 / 2}$, approximately. From this value of $F_{1}, \Delta_{1}$ and the new bulk volume, $\mathrm{V}_{\mathrm{n}}$, of the system can be determined. $\mathrm{V}_{\mathrm{n}}$, can in turn be used to find the speed $\mathrm{C}_{\mathrm{d}}$ of a dilatational wave in the system from the known equation

$$
\begin{equation*}
\mathrm{C}_{\mathrm{d}}=\left[\frac{3 \mathrm{~g}}{\rho}\left(-\mathrm{V} \frac{\mathrm{dp}}{\mathrm{dV}}\right)\left(\frac{1-v}{1+v}\right)\right]^{1 / 2} \tag{30}
\end{equation*}
$$

in which $\rho$ is the density of the total system including the interstitial fluid.
For a system with air in the voids (its density being neglected) Brandt obtains

$$
\begin{equation*}
\mathrm{C}_{\mathrm{d}}=\left[\frac{2 \mathrm{~g}}{3 \rho \mathrm{n}(1-\mathrm{n})}\left(\frac{1-\nu}{1+\nu}\right)\right]^{1 / 2} \frac{\mathrm{P}_{\mathrm{o}}^{1 / 6}}{\mathrm{~K}^{1 / 3}} \tag{31}
\end{equation*}
$$

at low pressures, K being a constant.
Experimental data support the theoretical result of the proportionality of $\mathrm{C}_{\mathrm{d}}$ to $\mathrm{P}_{\mathrm{O}}^{1 / 6}$ at low pressures but depart from it at high pressures. The point of transition between the behavior at low and high pressures probably represents the upper limit of applicability of the classical Hertz theory of contact (22).

Although the theoretical results of Hara, Gassmann and Brandt are in qualitative agreement with these experimental data, they predict values of the velocity of wave propagation less than those obtained by experiment because they consider only the normal
components of the forces at the points of contact and neglect the tangential components. Because the tangential stiffness of a contact has the same order of magnitude as its normal stiffness, it must be considered. This has been done for certain idealized granular systems.

## Stress-Strain Solutions for Idealized Granular Systems

Tangential forces or twisting moments at the contacts between the grains of a granular system cause the load-displacement relations to be nonlinear and inelastic. Therefore, the mechanical response of the system depends not only on the initial loading but also on the history of loading and the stress-strain relations at any point of the system must be expressed as increments of stress related to increments of strain.

Duffy and Mindlin (23) obtained a solution for a rhombohedral system (Case 3) of spheres subjected to a certain simple program of loading. The system is assumed to be initially subjected to an isotropic compressive stress, $\sigma_{0}$. An arbitrary incremental stress, small in relation to $\sigma_{0}$, is then applied. The initial contact forces are all purely normal and equal to each other. Also the compliances are nearly the same for all the contacts. The incremental stress-strain relations obtained are

$$
\begin{align*}
& d \sigma_{11}=C_{11} d \epsilon_{11}+C_{12} d \epsilon_{22}+C_{12} d \epsilon_{33} \\
& d \sigma_{22}=C_{12} d \epsilon_{11}+C_{11} d \epsilon_{22}+C_{12} d \epsilon_{33} \\
& d \sigma_{33}=C_{12} d \epsilon_{11}+C_{12} d \epsilon_{22}+C_{11} d \epsilon_{33}  \tag{32}\\
& d \sigma_{23}=2 C_{44} d \epsilon_{33} \\
& d d_{13}=2 C_{44} d \epsilon_{13} \\
& d \sigma_{12}=2 C_{44} d \epsilon_{12}
\end{align*}
$$

in which

$$
\begin{equation*}
\mathrm{C}_{11}=\frac{2(4-3 \nu)}{\nu} \mathrm{C}_{12}=2 \mathrm{C}_{44}=\frac{4-3 \nu}{2-\nu}\left(\frac{3 \mu^{2} \sigma_{0}}{2(1-\nu)^{2}}\right)^{1 / 3} \tag{33}
\end{equation*}
$$

The equations are referred to rectangular coordinate axes parallel to the edges of the unit cube of the system. The form of these relations corresponds to those in a crystal with cubic symmetry. From them, the velocity of wave propagation may be derived, assuming that the variations in the stress accompanying the propagation are small compared to the initial stress in the system. Strictly speaking, integration of Eq. 32 should be limited to increments in which the stress remains isotropic. When this is not the case, the compliances will vary from contact to contact and it will become necessary to consider the history of loading of each contact.

Duffy and Mindlin (23) have also obtained solutions for the case when the system is subjected to a loading giving rise to an isotropic pressure, $\sigma_{0}$, and a uniaxial pressure, $\sigma_{\mathrm{a}}$, parallel to one of the edges of the unit cube. The subsequent increments in loading are supposed to be variations in either $\sigma_{\mathrm{O}}$ or $\sigma_{\mathrm{a}}$. The differential stress-strain relations have the same form as those of a tetragonal crystal with six independent elastic moduli. In this case, there are two types of contacts, each with a different loading history.

## MECHANICAL RESISTANCE PROPERTIES OF GRANULAR SYSTEMS

As a result of Coulomb's studies, the shearing strength, $s$, of a granular system is commonly expressed as $s=\sigma \tan \phi$ in which $\sigma$ is the normal pressure on the failure plane and $\phi$ is the so-called angle of internal friction. The resistance to shear is due to a combination of effects, including sliding friction, rolling friction and interlocking (88). Tan $\phi$, the shearing strength per unit normal stress, is not constant but depends on a number of factors. Methods of measurement of $\tan \phi$ and their comparative merits are discussed in various references (9, 12, 41, $\underline{51}, \underline{70}, \underline{74}, \underline{80}, \underline{87}, \underline{93}$ ).

Dependence of Tan $\phi$ on Void Ratio
At a constant normal stress, $\tan \phi$ (i.e., peak value) increases as the initial void

## CASE 6

## CASE 3



Figure 8. Units of generalized rhombohedral systern.
ratio decreases. When a dense granular system is sheared, it expands, because the closely interlocked grains need sufficient space to be able to roll or slide over one another. The increase in volume means that work is done against the normal pressure which reflects itself in a higher $\tan \phi$. In a loose system containing many arches, the volume will decrease during shear as the arches are broken down and the grains rearrange themselves to a denser state. Eventually the critical void ratio is attained at which shear continues at constant volume (14, 88). Winterkorn suggests that in shear, granular systems behave as macromeritic liquids (98). The comparison with ordinary molecular liquids gave rise to the following formula (100):

$$
\begin{equation*}
\tan \phi=\frac{C^{\prime}}{V-V_{\min }} \tag{34a}
\end{equation*}
$$

which is equivalent to

$$
\begin{equation*}
\tan \phi=\frac{C}{e-e_{\min }} \tag{34b}
\end{equation*}
$$

in which $\mathrm{C}^{\prime}$ and C are constants, e is the void ratio, V is the volume of the system in the given state, and $e_{\min }$ and $V_{\min }$ are the corresponding values with the system in the densest possible state. Winterkorn's formula is verified by comparing its predictions with experimental data obtained by various workers.

Idel (42) considered a generalized rhombohedral system (Case 6) of equal spheres. In this system every three spheres forming a hollow to support a fourth sphere were generally not in contact with each other (Fig. 8). This resulted in a higher porosity than the 26 percent of the normal rhombohedral system in which the three supporting spheres are in contact with each other. Idel defined the contact angle $\theta$ as the angle made with the horizontal by the normal at the point of contact between the upper and a lower sphere (Fig. 8). The following relation was obtained:

$$
\begin{equation*}
\mathrm{n}=1-\frac{\pi}{9 \sqrt{3}} \frac{1}{\sin \theta \cos ^{2} \theta} \tag{35}
\end{equation*}
$$



Figure 9. Wittke's solution for $\tan \phi$ of rhombohedral system as function of porosity for different values of coefficient of friction $f$, i.e., $\tan \psi(\underline{102})$.
in which $n$ is the fractional porosity. For the usual (densest) rhombohedral system (Case 6), $\theta=54^{\circ} 43^{\prime}$. For the Case 3 generalized rhombohedral system (four supporting spheres) Wittke obtained

$$
\begin{equation*}
\mathrm{n}=1-\frac{\pi}{12} \frac{1}{\sin \theta \cos ^{2} \theta} \tag{36}
\end{equation*}
$$

In this case $\theta=45^{\circ}$ for the densest state when $n=26$ percent. The least possible value of $\theta$ for either case is $30^{\circ}$.

From statical considerations, Wittke (102) obtained the following relation for the angle $\phi$ of the general rhombohedral system (either case) under the stress conditions of the triaxial test:

$$
\begin{equation*}
\tan ^{2}\left(\frac{\phi}{2}+45^{\circ}\right)=2 \tan \theta \tan (\theta+\psi) \tag{37}
\end{equation*}
$$

in which $\tan \psi=f$, the coefficient of friction between the spheres. Because $\theta$ is related to n by Eqs. 35 or $36, \tan \phi$ may be related by Eq. 37 to n . Curves of $\tan \phi$ against $n$ can thus be drawn for given values of $\tan \psi$ (Fig. 9). These curves differ for Cases 3 and 6 of the rhombohedral system, because Eqs. 35 and 36 are different.

Wittke performed experiments on glass beads packed in the generalizedrhombohedral state. The measured $\tan \phi$ values were about 50 percent lower than the theoretical values according to Eq. 37. The major part of this discrepancy was attributed to friction developed between the end layers of the system (top and bottom) and the bounding plates.

The theory developed by Wittke is based on a particular idealized granular system. Because, as indicated by Figure 9, $\tan \phi$ depends considerably on the arrangement of the system, Wittke's solution for $\tan \phi$ as a function of n cannot be considered applicable directly to a general granular system, such as a sand. In the shear zone of a dense granular system, there is a loosening up as compared to the rest of the system (93, 102). The over-all volume changes taking place during shearing are considered by Newland and Allely (65) and by Poorooshasb (69).

## Dependence of Tan $\phi$ on $f$

A number of expressions have been derived relating $\tan \phi$ to the coefficient of friction between the grains, f. Caquot (13) considered an irregular system composed of grains of various sizes and shapes in random distribution and found that $\tan \phi=(\pi / 2)$ f.

Bishop (2) obtained approximate expressions for $\phi$ from energy considerations. In the case of triaxial compression in which $\sigma_{2}=\sigma_{3}<\sigma_{1}$,


Figure 10. Normal unit vectors at 12 tangent points of sphere in Case 6 rhombohedral system (20).

$$
\begin{equation*}
\sin \phi=\frac{15 \mathrm{f}}{10+3 \mathrm{f}} \quad\left(\sigma_{1}=\text { normal stress }\right) \tag{38}
\end{equation*}
$$

For plain strain where $\sigma_{2}=\frac{\sigma_{1}+\sigma_{3}}{2}$,

$$
\begin{equation*}
\sin \phi=3 / 2 \mathrm{f} \tag{39}
\end{equation*}
$$

The expressions of both Bishop and Caquot give $\tan \phi=0$ when $\mathrm{f}=0$ (Fig. 12). This cannot be true for void ratios below the critical value, because in such cases a finite amount of work must be done against the normal forces as the system expands.

Considering a rhombohedral (Case 6) system of equal spheres, Dantu (20) obtained relations between $\tan \phi$ and $f$ in the following manner. Each sphere has 12 points of contact with neighboring spheres and the direction cosines of the unit normal vectors (Fig. 10) are given in Table 3.

Assuming the system is subjected to normal stresses, $\sigma_{\mathrm{x}}, \sigma_{\mathrm{y}}, \sigma_{\mathrm{z}}$, and to shear stresses, $\tau_{\mathrm{xy}}, \tau_{\mathrm{yz}}, \tau_{\mathrm{zx}}$, the usual notation for the subscripts being used, the normal reactions between the grains can be calculated from considerations of statics.

Assuming zero friction between the
TABLE 3
DIRECTION COSINES OF UNIT NORMAL VECTORS ${ }^{\text {a }}$

| Normal <br> Vector | $\alpha$ | $\beta$ | $\delta$ |
| :---: | :---: | :---: | :---: |
| $\overline{1}$ | 1 | 0 | 0 |
| $\frac{1}{2}$ | $-1 / 2$ | $+\sqrt{3 / 2}$ | 0 |
| $\frac{3}{4}$ | $-1 / 2$ | $-\sqrt{3 / 2}$ | 0 |
| $\frac{4}{5}$ | $+1 / 2$ | $+\sqrt{3 / 6}$ | $-\sqrt{6 / 3}$ |
| $\frac{0}{6}$ | $-1 / 2$ | $-\sqrt{3 / 3}$ | $-\sqrt{6 / 3}$ |

${ }^{2} \alpha, \beta$, and $\delta$ are the direction
cosines referred to the $x, y$, and $z$ axes, respectively (Fig. 10).


Figure ll. Normal reactions acting on sphere in Case 6 rhombohedral system when system is suljected orly to nomal shresies, $\sigma_{x}, 0_{y}$ and $\sigma_{z}$ (20)

1. If $\tau_{\mathrm{xy}}=\tau_{\mathrm{yz}}=\tau_{\mathrm{zx}}=0$,

$$
\begin{equation*}
\mathrm{F}=\mathrm{R}^{2} \sqrt{2} \sigma_{\mathrm{z}}, \mathrm{G}=\frac{\mathrm{R}^{2} \sqrt{2}}{3}\left(4 \sigma_{\mathrm{y}}-\sigma_{\mathrm{z}}\right), \mathrm{H}=\frac{\mathbf{R}^{2} \sqrt{2}}{3}\left(6 \sigma_{\mathrm{x}}-\sigma_{\mathrm{z}}-2 \sigma_{\mathrm{y}}\right) \tag{40}
\end{equation*}
$$

in which $R$ is the radius of the spheres and $F, G, H$ are the normal reactions between the spheres (Fig. 11);
2. If $\tau_{\mathrm{xy}}=\tau_{\mathrm{yz}}=\tau_{\mathrm{zx}}=0$, and $\sigma_{\mathrm{x}}=\sigma_{\mathrm{y}}=\sigma_{\mathrm{r}}$, as in the triaxial test

$$
\begin{equation*}
\mathbf{F}=\mathbf{R}^{2} \sqrt{2} \sigma_{\mathrm{Z}}, \mathrm{G}=\mathrm{H}=\frac{\mathbf{R}^{2 \sqrt{2}}}{3}\left(4 \sigma_{\mathrm{r}}-\sigma_{\mathrm{z}}\right) \tag{41}
\end{equation*}
$$

3. If $\tau_{\mathrm{xy}}=\tau_{\mathrm{yz}}=\tau_{\mathrm{zx}}=0$, and $\sigma_{\mathrm{x}}=\sigma_{\mathrm{y}}=\sigma_{\mathrm{z}}=\sigma$, corresponding to application of a hydrostatic pressure to the system, then

$$
\begin{equation*}
F=G=H=R^{2} \sqrt{2} \sigma \tag{42}
\end{equation*}
$$

In Case 2, failure of the system will occur when $\mathrm{G}=\mathrm{H}=0$, i. e., when $\sigma_{\mathrm{Z}} / \sigma_{\mathrm{r}}=4$. This corresponds to tan $\phi=0.750$ or $\phi=36^{\circ} 52^{\prime}$. Assuming a coefficient of friction, f, between the spheres, Dantu finds by statics for this case that

$$
\begin{equation*}
\mathrm{F}=\frac{2 \mathrm{R}^{2}}{\sqrt{2}+\mathrm{f}} \sigma_{\mathrm{Z}}, \quad \mathrm{G}=\mathrm{H}=\frac{8 \mathrm{R}^{2} \sqrt{2}}{6}\left(\sigma_{\mathrm{r}}-\sigma_{\mathrm{z}} \frac{\sqrt{2}}{4} \frac{1-\sqrt{2} \mathrm{f}}{\sqrt{2}+\mathrm{f}}\right) \tag{43}
\end{equation*}
$$

The condition of limiting equilibrium corresponds to $\mathrm{G}=0$ when

$$
\begin{equation*}
\frac{\sigma_{\mathrm{z}}}{\sigma_{\mathrm{r}}}=4 \frac{\sqrt{2}+\mathrm{f}}{\sqrt{2}-2 \mathrm{f}} \tag{44}
\end{equation*}
$$

from which

$$
\begin{equation*}
\sin \phi=\frac{\sigma_{Z}-\sigma_{r}}{\sigma_{z}+\sigma_{r}}=3 \frac{\sqrt{2}+2 f}{5 \sqrt{2}+2 f} \tag{45}
\end{equation*}
$$

The curve representing $\tan \phi$ against $f$ is compared in Figure 12 with curves obtained


FRICTION COEFFICIENT $f$
Figure 12. Different theoretical solutions for tan $\phi$ of regular system of spheres in terms of coefficient of friction $f$ of spheres.
by others. Tan $\phi$ tends to infinity $\left(\phi=90^{\circ}\right)$ when $\mathrm{f}=\sqrt{2} / 2=0.71$. Physically this means that for values of f greater than 0.71 , the granular system can have vertical sides and at the same time support a vertical load without any lateral support. Failure occurs by actual crushing of the grains when the load has attained a sufficiently high value.

Considering an ideal packing of equal spheres, Spencer (84) obtained a relation between $\varnothing, f$ and $n$.

In his study of the generalized rhombohedral system (Case 6), inclosed in a cylinder as in a triaxial test, Idel (42) obtained from statical considerations the following expression:

$$
\begin{equation*}
\sin \phi=\frac{1.5 \tan (\psi=\theta)-1}{1.5 \tan (\psi+\theta)+1} \tag{46}
\end{equation*}
$$

in which $\tan \psi=\mathrm{f}$. Because n is related to $\theta$ by Eq. 35, this expression gives the variation of $\tan \phi$ with $\mathbf{f}$ at different porosities. Eq. 46 is different from Eq. 37 from which Wittke's curve is drawn.

Extending the theoretical calculations of Thurston and Deresiewicz (90) on the mechanism of failure of a rhombohedral system of equal spheres, Scott (75) obtained

$$
\begin{equation*}
\tan \phi=\frac{\sqrt{3}+(4 \sqrt{2}) \mathrm{f}}{2(\sqrt{6}-\mathrm{f})} \tag{47}
\end{equation*}
$$

Sjaastad (79) considered a Case 6 system and ignored the contribution to shear strength of rolling friction as compared to that of sliding friction. Failure of the system occurs when one distinct layer of spheres, together with all those above it, slides over the layer immediately below, which remains at rest together with all the lower layers. Considering a typical sphere resting in the hollow formed by the three spheres below it, it is apparent that there are two extreme modes of failure: (a) type 1 in which


Figure 13. Two extreme failure paths in Case 6 rhombohedral system (79).
the top sphere slides directly over the top of one of the bottom spheres, and (b) type 2 in which the top sphere slides through the cleavage between two of the lower spheres (Fig. 13). Two conditions must be considered in each mode of failure: (a) the static case in which failure is about to occur, and (b) the kinetic case existing after a very small displacement when the top sphere has lost contact with one or two of the underlying spheres. The expressions obtained by Sjaastad are as follows:

1. Type 1, static case

$$
\begin{equation*}
\tan \phi=\frac{1}{\sin \theta-\mathrm{f} \cos \theta}\left[\cos \theta+\frac{\mathrm{f}}{3(\sin \theta+\mathrm{f} \cos \theta)}\right] \tag{48}
\end{equation*}
$$

and kinetic case

$$
\begin{equation*}
\tan \phi=\frac{\cos \theta+\mathrm{f} \sin \theta}{\sin \theta-\mathrm{f} \cos \theta} \tag{49}
\end{equation*}
$$

in which $\theta=54^{\circ} 43^{\prime}$.
2. Type 2, static case

$$
\begin{equation*}
\tan \phi=\left[\frac{2 \mathrm{f}}{3(\sin \theta+\mathrm{f} \cos \theta)}+\sin \alpha\right] /\left[\cos \alpha-\frac{\mathrm{f} \sin \alpha}{\sin 60}\right] \tag{50}
\end{equation*}
$$

and kinetic case

$$
\begin{equation*}
\tan \phi=\frac{\sin \alpha \sin 60^{\circ}+\mathrm{f} \cos \alpha}{\cos \alpha \sin 60^{\circ}-\mathrm{f} \sin \alpha} \tag{51}
\end{equation*}
$$

in which $\theta=54^{\circ} 43^{\prime}$ and $\alpha=19^{\circ} 30^{\prime}$.
For each type a mean curve is obtained (Fig. 14) by taking the average of the kinetic and static values. Assuming equal probability for any sphere to move along either failure path, Sjaastad obtained the curve for 26 percent porosity shown in Figure 12. Another assumption was that of equal partition of energy between the two extreme failure paths. Because less energy is required for type 2 failure, the probability ratio


Figure 14. Sjaastad's theoretical solutions for $\tan \phi$ of Case 6 rhombohedral syster in terms of friction coefficient $f$ (79).
is about 2 to 1 in favor of it. However, very little difference was found between the curve resulting from this assumption and that based on equal probability.

Sjaastad assumed that for the regular packing at the upper limit of porosity, the cubic, $\tan \phi$ is equal to f. Therefore, curves at intermediate porosities could be obtained (Fig. 15) by assuming further that $\phi$ is proportional to the relative density. The latter assumption is based on data published by Burmister.

Some of the results obtained experimentally by various workers on systems with nearly perfect and equal spheres are plotted in Figure 15 in which there is satisfactory agreement with Sjaastad's equal partition solution in the range $f=0.1$ to 0.5 . In particular, although Sjaastad's data on glass beads at 38 percent porosity are somewhat greater than theoretical values, the line joining them is almost parallel to the theoretical curve at $\mathrm{n}=38$ percent. On the other hand, Sjaastad's data do not show such an agreement with the solutions of Spencer, Idel and Wittke. Considering all the data shown in Figure 15, they fit Spencer's solution best at the lower values of f, but Sjaastad's over a wider range.

None of these solutions are completely verified by the data now available. To determine which is the most accurate, values of $\tan \phi$ at $f \geq 1$ must be measured ongranular systems approaching the ideal. Because the solutions diverge considerably at the higher values of $f$, such data should easily show which is the most valid.

All solutions except Caquot's show a rapid increase of $\tan \phi$ with f . An increase in f is obtained when the system is subjected to a vacuum and some heat is applied, so that the adsorbed layers of gas are mostly removed and there is more intimate contact between the particles. Sjaastad's experiments (79) have proved this increase in f.


Figure 15. Experimental data on nearly ideal systeras vs Sjaastad's Equal Partition of Energy solution (79).

## Effect of Grain Size on Tan $\varnothing$

According to the expressions for $\tan \phi$ obtained by various workers on the basis of idealized granular systems, $\tan \phi$ is independent of the absolute grain size of the system. To determine conclusively whether $\tan \phi$ depends on the grain size, controlled tests on systems with regular arrangements of spherical grains of uniform size and surface characteristics should be performed. The results obtained until now are inconclusive, because the systems tested have generally departed more or less from uniformity in size, shape and condition of surface of their grains. Also care was not generally taken to obtain the same initial void ratios for the samples tested so as to afford a true basis for comparison. Wittke's (102) experiments are an exception, but the grain sizes used by him do not differ widely. Sjaastad (79) used a wider size range and also maintained the same initial void ratio, but his system was less ideal than that of Wittke.

Hennes (35) measured the shear strength of samples of rounded gravel of approximately uniform size. Tan $\phi$ was found to increase appreciably with grain size up to $1 / 4 \mathrm{in}$. , beyond which there was little variation (Fig. 16). The shapes of the grains varied considerably within each sample and from sample to sample. The results, therefore, do not show the influence of grain size alone. The samples were densified before testing in the same manner by tamping and vibration, causing widely differing values of the initial void ratio. However, Table 4 indicates that the largest size ( $3 / 4$ to $1 / 2 \mathrm{in}$.) had a considerably greater initial void ratio than the smaller sizes. Because $\tan \phi$ increases as the void ratio decreases, the largest size may be expected to show a value of $\tan \phi$ greater than 1.17 at the same void ratios as the smaller sizes. This indicates a tendency of $\tan \phi$ to increase with grain size in the tested range, but it is not conclusive owing to nonuniformity of the samples in size and shape. The resistance


Figure 16. Hennes' experimental curve showing effect of grain size on tan $\phi$ for rounded gravel (35).
to shear depends considerably on the state of packing, particularly along the plane of sliding, and the packing is influenced by the gradation and shape characteristics.

From tests on crushed quartz in loose state, Parsons (68) found that tan $\phi$ had a minimum value at a grain size of 0.75 mm (Fig. 17). The samples were placed in the shear box by pouring from the same height. The results are inconclusive, because the degree of angularity or sharpness of the grains varied with the grain size.

Kjellman and Jakobson (48) obtained a higher value of $\phi$ at a larger grain size (Table 5) using samples of round pebbles. The initial void ratio, $e_{0}$, of the coarser samples was larger (i.e., smaller unit weight), so that if corrected to the same $e_{0}$ the difference between the values of $\phi$ for the fine and coarse material would become greater. Here again the results are inconclusive, because secondary effects may have played a part.

Using samples of sand, Wu (103) obtained decreasing values of $\phi$ with increasing grain size in the range from 0.1 to 3 mm . This is at least partly due to the greater nonuniformity of the coarser samples (42). For systems at the same void ratio, $\tan \phi$ decreases as the nonuniformity increases.

From triaxial tests on glass beads and quartz sand, Idel (42) concluded that tan $\phi$ was independent of grain size. The size of the glass beads tested varied from 3 to 35 mm , whereas that of the

TABLE 4
HENNES' EXPERIMENTAL DATA (35) ${ }^{\text {a }}$

| Grain Size | Direct <br> Shear <br> Tan $\phi$ | Init. Void <br> Ratio | Triaxial <br> Tan $\phi$ | Init. Void <br> Ratio |
| :--- | :--- | :--- | :--- | :--- |
| Nos. $8-16$ | 0.903 | 0.698 | 0.91 | 0.61 |
| Nos. $3-4$ | 1.12 | 0.622 | 0.93 | 0.61 |
| $3 / 4-1 / 2$ in. | $\mathbf{1 . 1 7}$ | 0.85 | 1.03 | 0.58 |

[^2]

Figure 17. Parsons' experimental curve showing effect of grain size on tan $\phi$ for

quartz sand varied from 0.4 to 5 mm . Idel's results, however, showed a considerable scatter in the measured values of $\tan \phi$. For instance, in the case of the glass beads at a porosity of 39.5 percent, the measured values of $\tan \phi$ ranged from 0.55 to 0.70 .

Wittke (102) developed an ingenious experimental technique for obtaining a system of practically uniform glass beads in a rhombohedral packing (Case 3 or 6) and for testing it triaxially in that state. With a porosity of 26 percent, the value of $\tan \phi$ for three sizes of glass beads ( $15.0,14.8$ and 11.85 mm ) stayed nearly constant at 0.37 in the Case 3 state. With Case 6 packing (porosity also 26 percent), $\tan \phi$ varied slightly from 0.78 to 0.86 for the three sizes. However, Wittke's results do not permit generalization, because the sizes used varied only over a narrow range.

Bishop's measurements (3) on Chesit Bank pebbles and on Ham River sand showed that the two materials had nearly the same $\tan \phi$ at equal porosities. The ratio of the two sizes was 1:60. Casagrande (15) reports a similar constancy of $\tan \phi$ with grain size.

Sjaastad (79) performed direct shear tests on five nearly uniform samples of glass beads. Each sample passed one U.S. standard sieve and was retained on the next sieve. The mean sizes varied from 0.46 to 3.1 mm , and the initial void ratio was kept constant at 0.608 . The results (Fig. 18) support the argument that $\tan \phi$ is independent of grain size. The considerably higher $\phi$ value shown by the $1.1-\mathrm{mm}$ sample was attributed to its greater uniformity compared with the others, shown by microscopic observation.

In summary, there is some evidence indicating an increase of $\tan \phi$ with grain size. However, the evidence is inconclusive, because this increase may well be due solely to secondary effects such as shape and nonuniformity of the component particles and variation in their surface characteristics. There exists more and better evidence suggesting that $\tan \phi$ is independent of grain size.

## Other Factors Influencing Tan $\varnothing$

It has been established experimentally


Figure 18. Sjaastad's experimental data showing effect of grain size on $\phi$ for glass beads (79).
for granular soil systems that $\tan \phi$ decreases with the applied normal pressure (88). It is also found that $\tan \phi$ increases with increasing angularity of the grains, i.e., with decreasing sphericity ( $15,16,35,88$ ).

A uniform granular system with constant void ratio and a certain friction coefficient, f, may have different values of $\tan \phi$ depending on its mode of packing. Wittke (102) found theoretically that a system of uniform spheres in the Case 3 packing had about half the value of $\tan \phi$ as the same system in Case 6 packing. This was also verified experimentally. Because Case 6 may be turned into Case 3 by a suitable rotation in space, Wittke's findings indicate anisotropy. That is, the value of $\tan \phi$ depends on the direction of the applied stresses.

Wittke's theory also shows that the stress and strain conditions affect $\tan \phi$. Thus, for example, his theory gives a value of 1.0 for a Case 6 system at an $f 0.1$ when the conditions are those of a plane-strain test, whereas for the same system under the conditions of a triaxial test, $\tan \phi$ equals 0.9 . This finding provides an explanation for the higher experimental values of $\tan \phi$ of a granular soil obtained by the box shear test as compared with those obtained by triaxial testing (35, 88).

Whitman (96) has shown experimentally that $\tan \phi$ is independent of the rate of strain in the triaxial test when the failure-time is varied from 5 msec to 5 min .

Certain experimental variables, such as the ratios of the dimensions of the sample tested and the ratio of its diameter to that of the inclosing cylinder, may affect the measured $\tan \phi(102)$. These are not discussed here, because they do not pertain to the system as such but only to the experimental technique.

## Granular Systems with Different Grain Sizes

Figure 19 shows the porosities obtained for granular systems containing two grain sizes when the systems are subjected to equal amounts of compactive effort. For a system of given grain size ratio, there is a certain composition that gives the least porosity. The further the system is from uniformity, i.e., the smaller the ratio $d_{m i n} /$ $\mathrm{d}_{\text {max }}$, the easier it is to obtain smaller porosities. Comparison of the shear strengths of different granular systems at the same porosities shows that uniform systems have the greatest shear strength, other factors such as grain shape and roughness being equal ( $16,35,100,105$ ).

Assuming that $\tan \phi$ varies linearly with n between 26 percent and 43 percent (justifiedaccording to Wittke's (102) solution), Idel (42) comes to the conclusion that all granular systems of two grain sizes should have the same $\tan \phi$ when in the densest possible state, no matter what their percentage composition or grain size ratio (of course, assuming


Figure 19. Porosities of systems composed of grains of two sizes and subjected to equal amounts of compactive effort (42).
equal shapes and f values). This conclusion is expected by Idel to hold for systems of more than two sizes. The same would apply to the systems in their loosest state. For an intermediate porosity, Idel assumes that the corresponding $\tan \phi$ value can be found by linear interpolation. The nearer the system is to uniformity, the larger is the rate of change of $\tan \phi$ with $n$.

The term relative density is used to indicate the position of the system between its loosest and densest states, these latter being obtained by arbitrarily fixed procedures (10, 49). Relative density $=\left(\mathrm{n}-\mathrm{n}_{\mathrm{d}}\right) /$ $\left(\mathrm{n}_{\ell}-\mathrm{n}_{\mathrm{d}}\right)$ at a porosity n in which $\mathrm{n}_{\ell}, \mathrm{n}_{\mathrm{d}}$ are the loosest and densest porosities. A comparison in importance has been made of $n_{\ell}$ and $n_{d}$ to the liquid and plastic limits of a clay. The relative density determines, among other things, the supporting value of the system and its compressibility (10). Burmister (10) gives results showing $\overline{\mathrm{a}}$ linear variation of $\phi$ with the relative density of a given system. Wu (103) obtains similar results.

Hennes' (35) experimental results show that increasing angularity (i.e., nonsphericity of the grains) causes a greater increase in the shear strength of a graded system than in that of a system with uniform grain size.

## TESTING OF MACROMERITIC SYSTEMS THEORY

According to the macromeritic theory, a granular system of identical spheres is in a potentially liquid state when it has a void ratio of 0.62 (the critical value) or higher (98, 100). The critical void ratio (CVR) thus corresponds to the melting point of simple chemical substances. Winterkorn (100) derived Eq. 34a on the basis of Batschinski's formula for the viscosity of simple molecular liquids at different temperatures. However, Eq. 34a can be expected to hold strictly only for the macromeritic liquid state, i. e., for systems above the CVR. It is, however, equivalent to Eq. 34b, whose predictions have been compared with data obtained from shear tests (direct or triaxial) on cohesionless soils (25).

Data from direct shear tests on Ottawa standard sand (88, p. 349) produced a curve for a normal pressure of 3 tons/sq ft in which $\phi=32.3^{\circ}$ when $\mathrm{e}=0.57$, and $\phi=29.5^{\circ}$ when $e=0.61$. Substitution of these values in Eq. 34b yields values for $C$ and $e_{\text {min }}$ of 0.226 and 0.212 , respectively. The formula can then be used to calculate values of $\tan$ $\phi$ at any void ratio, and the same normal pressure, as in Table 6 which shows good agreement between values. Whenever the experimental data at high void ratios are higher than the calculated ones, partial collapse to a lower void ratio during the testing may be suspected. Table 7 shows the values of $e_{\min }$ and $C$ calculated at different normal pressures by means of Eq. 34b. At each normal pressure, two values of e were

TABLE 6
CALCULATED VS EXPERIMENTAL TAN \& VALJESS ${ }^{\text {a }}$

| Void Ratio <br> e | Tan \% |  | Dev. <br> (\%) |
| :---: | :---: | :---: | :---: |
|  | Cal. | Exp. |  |
| 0.56 | 0.650 | 0,649 | $+0.2$ |
| 0.59 | 0.598 | 0,596 | +0.3 |
| 0. 62 | 0.554 | 0.554 | 0.0 |
| 0. 64 | 0.528 | 0,539 | -2.0 |
| 0.66 | 0,505 | 0.531 | - 4.9 | applied together with the corresponding measured $\phi$ values. The table indicates that at different normal pressures $C$ does not deviate by more than 12 percent. These calculations have been made using data from experimental curves arbitrarily drawn as mean curves through points showing considerable scatter (89, Fig. C-5). Thus, a 12 percent deviation of C is not excessive.

The values of $\mathrm{e}_{\min }$ in Table 7 are rea-
sonable when compared with the value 0.35 for the void ratio of the densest (rhombohedral) state of an assembly of equal spheres. The samples on which the data were obtained were only approximately uniform in size and their grains were not spherical. Therefore, they may be expected to have a "densest state" of lower void ratio than 0.35 .

Data from triaxial tests on washed Fort Peck sand (88, p. 350) at a minor principal stress, $\bar{\sigma}_{3}$, of 34 psi , produced experimental values of $\phi=41.6^{\circ}$ at $\mathrm{e}=$ 0.62 , and $\phi=34.0^{\circ}$ at $\mathrm{e}=0.825$. From these and Eq. 35, $\mathrm{e}_{\min }=-0.028$ and $\mathrm{C}=$ 0.575 giving the calculated values of $\tan$ $\phi$ in Table 8. Though a negative $\mathrm{e}_{\mathrm{min}}$ value is used in the formula, it gives good predictions.

Data from direct shear and triaxial tests on Sand B (87, p. 1061) are summarized in Table 9. For each type of test the values of C show good agreement. A comparison of the two types, however, shows that the value of $C$ depends on the kind of test used for measuring the shear strength. Table 9 shows that the triaxial test gives higher values of C and this is confirmed by calculations with other experimental data (25). This variation of C with method of test is probably due to the different stress conditions imposed on the system.

De Beer (21, p. 281) obtained data by means of triaxial tests on sand at constant $\sigma_{3}$. The experimental curves show the variation of $\phi$ with $n$ at various values of $\sigma_{3}$ from $0.02 \mathrm{~kg} / \mathrm{cm}^{2}$ to $50 \mathrm{~kg} / \mathrm{cm}^{2}$. The curve at $\sigma_{3}=1 \mathrm{~kg} / \mathrm{cm}^{2}$ gives the following values: (a) at $\mathrm{n}=0.39(\mathrm{e}=0.640), \varnothing=$ $41.7^{\circ}$; and $(\mathrm{b})$ at $\mathrm{n}=0.45(\mathrm{e}=0.818), \phi=$ $34.0^{\circ}$. These values give $\mathrm{e}_{\min }=0.083$ and $\mathrm{C}=0.496$ from Eq. 34b which is then used at other void ratios to obtain the calculated $\tan \phi$ values in Table 10. The agreement with the experimental data is very good. Table 11 gives the values of $C$ and $e_{\text {min }}$ calculated from the data at different $\sigma$ values.

It is seen that in the large stress range from 0.02 to $50 \mathrm{~kg} / \mathrm{cm}^{2}$, C is practically constant, as it should be. As $\sigma_{3}$ increases, $e_{\text {min }}$ shows a consistent decrease.

Burmister's data from direct shear tests performed on Ottawa standard sand (9, pp. 1073-75) have been applied to Eq. 34b to give Table 12. Here again the constancy of C is confirmed.

Wu's data (103) from triaxial tests on three specimens of sand referred to as 134 , 121,133 , respectively, give Table 13. For each specimen, the value of C is approximately the same at the two values of $\sigma_{3}$.

TABLE 11
VALUES OF C AND $\mathrm{e}_{\min }{ }^{1}$

|  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: |
| Cunfining <br> Stress $\sigma_{3}$ <br> $\left(\mathrm{~kg} / \mathrm{cm}^{2}\right)$ | e | $\mathrm{e}_{\min }$ | C | Dev. ${ }^{2}$ <br> $($ (b) |
| 0.02 | 0.640, | 0.818 | 0.132 | 0.533 |
| 0.1 | 0.640, | 0.818 | 0.118 | 0.520 |
| 0.2 | 0.640, | 0.818 | 0.110 | 0.515 |
| 1 | 0.640, | 0.818 | 0.083 | 0.496 |
| 2 | 0.640, | 0.818 | 0.051 | 0.497 |
| 5 | 0.613, | 0.785 | 0.000 | -1.8 |
| 10 | 0.613, | 0.785 | -0.504 | -2.0 |
| 50 | 0.613, | 0.754 | -0.085 | 0.501 |

${ }^{1}$ From De Beer (21, p. 281).
${ }^{2}$ Of $C$ from mean.

TABLE 12
VALUES OF C AND $\mathrm{emin}^{1}$

| VALUES OF C AND $\mathrm{emin}^{1}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| Normal <br> Pressure <br> $\left(\mathrm{kg} / \mathrm{cm}^{2}\right)$ | e | emin | C | Dev. $^{2}$ <br> $(\%)$ |
| 2.033 | 0.483, | 0.637 | 0.208 | 0.278 |
| 1,593 | 0.501, | 0.551 | 0.201 | 0.204 |
| 1.033 | 0.506, | 0.663 | 0.264 | 0.261 |
| 0.533 | 0.499, | 0.656 | 0.274 | 0.262 |

${ }_{2}^{1}$ From Burmister (9, pp. 1073-1075).
${ }^{2}$ Of C from mean.

TABLE 13
VALUES OF C AND $\mathrm{e}_{\mathrm{min}}{ }^{1}$

| VALUES OF C AND $\mathrm{e}_{\mathrm{min}}{ }^{1}$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Sand <br> Specimen <br> $\left(\mathrm{kg} / \mathrm{cm}^{2}\right)$ | e | $\mathrm{e}_{\text {min }}$ | C |  |  |
| 121 | 1.40 | 0.452, | 0.637 | -0.805 |  |
|  | 2.80 | 0.445, | 0.605 | -0.770 |  |
| 133 | 1.40 | 0.371, | 0.553 | -0.881 |  |
|  | 2.80 | 0.352, | 0.550 | -0.670 |  |
| 134 | 1.40 | 0.600, | 0.823 | -0.460 |  |
|  | 2.80 | 0.606, | 0.805 | -0.525 |  |
| 1 |  |  |  |  |  |

${ }^{1}$ From Wu (103).

TABLE 14
CALCULATED VS EXPERIMENTAL TAN $\phi$ VALUES ${ }^{\text {a }}$

| Porosity <br> n | Void Ratio <br> e | Tan $\varnothing$ |  | Dev, <br>  |
| :--- | :---: | :--- | :---: | :---: |
|  | Calc. | Exp, |  |  |
| 0.39 | 0.640 | 0.860 | 0.863 | -0.3 |
| 0.41 | 0.695 | 0.785 | 0.776 | +1.2 |
| 0.43 | 0.754 | 0.719 | 0.707 | +1.7 |

aexperimental data from Nash (64, p. 163, Fig. 6).

TABLE 15
CALCULATED VS EXPERIMENTAL TAN $\phi$ VALUES ${ }^{\text {a }}$

| Porosity <br> $n$ | Void Ratio <br> e | Tan $\phi$ |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  |  | Calc. <br> Dev. <br> $(\phi)$ |  |  |
| 0.40 | 0.667 |  | Exp. |  |
| 0.42 | 0.725 | 0.821 | 0.827 | -0.7 |
| 0.46 | 0.851 | 0.665 | 0.774 | -1.3 |

${ }^{\text {a }}$ Experimental data from Bjerrum et al, (5, p.33, Fig. 7).

TABLE 16
CALCULATED VS EXPERIMENTAL TAN $\phi$ VALUES ${ }^{\text {a }}$

| Void Ratio <br> e | $\operatorname{Tan} \phi$ |  | Exp. |
| :---: | :---: | :---: | :---: |

a Experimental data from Rutledge (74, p. 56, Table 2).

TABLE 17
CALCULATED VS EXPERIMENTAL TAN $\phi$ VALUES ${ }^{\text {a }}$

| Porosity <br> n | Void Ratio <br> e | Tan $\phi$ |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  |  | Dev. <br> (\%) |  |  |
|  | 0.470 | 0.897 | 0.936 | -4.2 |
| 0.40 | 0.667 | 0.619 | 0.674 | -8.2 |

${ }^{\text {a }}$ Experimental data from Bishop (3).

Nash's triaxial tests on a nearly uniform sand (64, p. 163, Fig. 6) yield the following data: (a) at $\mathrm{e}=0.667(\mathrm{n}=0.40), \phi=39.4^{\circ}$; and (b) at $\mathrm{e}=0.818(\mathrm{n}=0.45), \phi=33.3$. These give $C=0.498$ and $\mathrm{e}_{\mathrm{min}}=0.061$ from which the calculated values of $\tan \phi$ in Table 14 are obtained.

The following are data taken from the results of drained triaxial tests on sand made by Bjerrum et al. (5, p. 33, Fig. 7): (a) at $\mathrm{e}=0.613(\mathrm{n}=0.38), \phi=41.4^{\circ}$; and (b) at $\mathrm{e}=0.785(\mathrm{n}=0.44), \phi=35.5^{\circ}$. These give $C=0.640$ and $e_{\min }=-0.112$ which are used for Table 15.

Rutledge's data obtained by triaxial tests on Sardis dam sand (74, p. 56, Table 2) yield (a) at $\mathbf{e}=0.55, \bar{\phi}=36.1^{\circ}$; and (b) at $\mathrm{e}=0.80, \phi=29.7^{\circ}$. These give $\mathrm{C}=0.656$ and $\mathrm{e}_{\min }=-0.352$, used in Table 16.

Bishop's data obtained by direct shear tests on Walton gravel (3) yield: (a) at $\mathrm{e}=0.428(\mathrm{n}=0.30), \phi=44.8^{\circ}$; and (b) at $\mathrm{e}=$ $0.545(\mathrm{n}=0.353), \phi=37.5$. These give $\mathrm{C}=0.394$ and $\mathrm{e}_{\min }=0.031$, used in Table 17 .

Eq. 34b has also been tested against additional data (25) published by Taylor (87, 88), De Beer (21), Burmister (9, 10, 12), Wu (103), Nash (64), Bjerrum et al. (5), Rutledge (74), Bishop (3), and Caquot and Kerisel (13). The results are similar to those preceding.

## GENERAL CONCLUSIONS

Eq. 34 b reproduces rather accurately the $\tan \phi$ values of granular systems within a given range of strain energy, expressed as normal pressure or minor principal stress. The constants $C$ and $e_{\min }$ are derived from $\tan \phi$ determinations at two different void ratios, with the system being at the same level of strain energy as in the contemplated use.

The value of C has been found to be approximately constant at different strain energy levels provided that the same type of test--plane shear or triaxial-is employed. Physically, the C-factor is of a composite nature. It can be conceived as expressing primarily particle-particle interaction; hence, C should also express the ease with which a system of high void ratio will collapse to the CVR and of the ease or difficulty with which pores or "holes" from other parts of the system will migrate to the shear failure planes or zones during the shear process. Furthermore, because C is of the essence of a "free energy consuming" factor, its value should express also the various mechanisms for energy consumption, such as translation and translation plus rotation, mobilized in the system during the shearing process.

The values for $e_{\min }$ generally decreased as the strain energy of the system increased. The strain energy is of the nature of a free energy; shear in the type of system considered is an irreversible process and the main characteristic of such processes is the production of entropy. Hence, increase in strain energy of the system shouldfavor the shear process and be reflected in either the $\mathbf{C}$ or $\mathrm{e}_{\min }$ values or both. These considerations indicate the direction which must be followed for theoretical refinement of the simple formula for $\tan \phi$. The need for such refinement is indicated by the fact that at high strain energies the calculated $\mathrm{e}_{\mathrm{min}}$ values may become negative. It is remarkable, however, and important from a practical point of view, that even negative $e_{\min }$ values did not prevent the simple formula from yielding rather accurate data.

The relative constancy of factor C and the rather accurate predictions of $\tan \phi$ that can be made by means of the acknowledgedly oversimplified Eq. 34b show the essential validity of the assumption of physical analogy between macromeritic and molecular systems and their respective conditions of state-solid and liquid with relatively low internal friction for pure systems of identical component particles and liquids with wide range of internal friction for systems with particles of different sizes, shapes and character. There exists, however, a definite need for more exact evaluation of the physical significance of factor $C$ and of the apparent variation of $e_{\min }$ with increasing strain energy of the granular system. Such evaluation may even make important contributions to our theoretical understanding of the behavior of true, i. e., molecular, liquids.

From this point of view of the macromeritic systems theory, the CVR at constant volume represents the volumetrically defined melting point of the system. Identification of the various types of CVR, is made by Taylor (88) and more recently by Geuze (31) and Fahmy (24). Winterkorn (98) has shown that the CVR (any type) decreases linearly as $\log \sigma_{3}$ increases. This relationship was checked and confirmed by Farouki (25) using other data given by Taylor ( $87, \underline{88}$ ) and Burmister (9). Hence for any given system one may write

$$
\begin{equation*}
C V R=a-b \log \sigma_{3} \tag{52}
\end{equation*}
$$

in which a and b are constants. In this equation, the minor principal stress, $\sigma_{3}$, may be replaced by the normal stress, $\sigma_{1}$.

In normal molecular liquids, the melting point increases with increasing pressure. Where the opposite is the case, as with normal ice and water, this is due to a structural arching effect arising from the directional nature of the H -bonds of the $\mathrm{H}_{2} \mathrm{O}$ molecule. The decrease of the volumetrically defined melting point of macromeritic systems with increasing strain energy makes their behavior more comparable to that of water than that of normal molecular liquids. This may be one of the reasons why the laws of macromeritic assemblies, developed originally for noncohesive granular systems, maintain their intrinsic validity even in typical cohesive soil-water systems.

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## Discussion

ARPAD KEZDI, Professor, Technical University, Budapest, Hungary. - Every investigation of the behavior of granular systems used to start with the examination of ideal packings of spheres. This is an idealization of nature and not directly applicable in practice; however, these studies may lead to a better understanding of nature, the realization of some important facts and the establishment of some, at least qualitative, statements. The concept of grain assemblies as macromeritic liquids, introduced by Winterkorn, throws a new light on the investigations of packings of equal spheres because the laws of physics, with respect to liquids, can thereby be applied to grain assemblies.

When comparing the characteristics of different states of matter, Winterkorn includes the grain assemblies as a separate state along with the solid, liquid and gaseous states (113). Sands and gravels may also be listed with liquids. The variation of the coefficient of latexal pressure, $\mathrm{K}_{0}$ (known in soil mechanics as coefficient of earth pressure at rest), shows clearly the fields for every condition of state. If a solid body displays a very great cohesion ( $\mathrm{c} \rightarrow \infty$ ), $\mathrm{K}_{\mathrm{O}}$ tends to zero. With decreasing binding forces between the elementary particles (caused by an increase of temperature, by vibration, or electric effects ), the $\mathrm{K}_{\mathrm{O}}$ value increases to 0.4 to 0.5 , which is characteristic for grain assemblies. Greater values occur with the viscous liquids, in which the internal shearing resistance is much smaller and finally $\mathrm{K}_{\mathrm{O}}=1$ is reached for ideal liquids with zero internal friction. In the case of gases $\mathrm{K}_{0}$ exceeds 1 , due to the atomic particle movements. The increase of $\mathrm{K}_{\mathrm{O}}$ can be achieved by transmitting energy to the system, either as mechanical energy forcing particles out of the solid, as vibration energy decreasing the number of contacts between the particles, thus reducing the inner resistance in a transient manner, or as thermal energy which enlarges the distances between the particles and increases their speed, thus transforming a solid to a liquid or a liquid to a gas. By applying a common yardstick for the different kinds of energy, it will be possible to plot the $\mathrm{K}_{\mathrm{O}}$ values as a function of the stored energy in the system, thus giving the range for the different conditions of state (Fig. 20). The greater the stored energy, the easier the migration of the "holes" in the system and, therefore, the greater the coefficient of lateral pressure.

Some Properties of Packings of Uniform Spheres
The first systematic treatment of packings of uniform spheres was given by Slichter (81). He established the different arrangements of the spheres and gave formulas to


Fieure 20. Relation between condition of state and pressure coefficient.
calculate their density based on the following assumptions: (a) the spheres are ideally rigid and undeformable; (b) the system of spheres is of infinite extent; and (c) there are no adhesive forces between the spheres; frictional forces will be mobilized by movements only.

A packing of spheres can be either continuous or discontinuous, compressible or incompressible. It is continuous, if, by starting on the surface of a sphere, selected at random, any sphere may be reached by crossing on surface contact points only. Dry granular soils obviously form a continuous packing.

A packing is compressible if a hydrostatic pressure, applied on plane surfaces, limiting a closed volume of the packing, can cause a compression of the system. Compressible systems are stable in unloaded state only if adhesive forces act between the spheres, as in clay and silt soils. Packings suffering no compression on the application of hydrostatic pressure are incompressible; this state is best approached by natural sands and gravels. This does not imply, however, that the simultaneous application of hydrostatic and shearing stresses cannot cause any compression of the system. In incompressible systems, there is at least one point of contact on every half sphere. The following investigations are limited to continuous and incompressible packings.

The density of a packing may be either uniform or variable. Uniform density results when an infinite system may be formed from finite unit cells without gaps and voids between the cells. In a more rigorous definition of uniformity, in the same-congruentposition every sphere must be with respect to its neighbors.

To construct uniform packings, a certain configuration in the plane is chosen such that the plane may be covered completely with an infinite number of unit cells without overlapping. Points on the borderlines that serve as centers of spheres are fixed so that the spheres do not intersect each other. The plane of the spheres' centers is the middle plane of the layer. An identical layer is placed on the first with middle planes parallel. A third similar layer is placed. The distance between the first and second layers, and the second and third layers, respectively, may be different. The whole system is composed of layers constructed in this manner. The fourth layer is in every respect identical with the first one; therefore, only three layers need be investigated. One sphere is selected in the middle layer and the number of spheres in contact with it in the lower, middle and upper layers are designated $u, m, \ell$. The symbol [ $u, m, \ell$ ] is used to characterize the construction procedure. The coordination number of the system is given by $N=u+m+\ell$. Cubic packing, the loosest state of uniform incompressible packings, has the symbol [1, 4, 1] and rhombohedral packing, the densest state, has the symbol [3, 6, 3] (Fig. 21). The symbol of the orthorombic system is $[2,4,2]$ and that of the tetragonal-spheroidal is $[2,6,2]$. It is evident that packings

## Elevation

(a)

plan


$$
\alpha=90^{\circ} ; \quad N=6
$$

$$
[1,4,1]
$$

## Elevation

(b)

$\alpha=60^{\circ} ; \quad N=12$
$[3,6,3]$

Figure 21. Cubic (a) and rhombohedral (b) packing of spheres.
for every coordination number between 6 and 12 can be constructed in several ways and the densities of packings with the same coordination number are not necessarily the same (107).

A continuous, incompressible, and uniform packing with coordination number as low as 4 is constructed as follows: The base figure is a hexagon, with every second corner the center of a sphere. The length of the side is smaller than D ; the diagonals are greater than D. The spheres are not in contact; therefore, $\mathrm{m}=0$. The lower layer is constructed so that every sphere contacts three spheres in the middle layer. The upper layer is constructed by establishing one point of contact with the sphere of the middle layers. Then $u=1$, and the symbol of the system is [1, 0, 3] (Fig. 22).

The porosity of this system is given by the relation

$$
\begin{equation*}
n=1-\frac{D^{3} \pi}{6} \frac{2 \nu}{A\left(m_{1}+m_{2}\right)} \tag{53a}
\end{equation*}
$$

in which $D$ is the diameter of the spheres, $\nu$ is the number of spheres on the base figure given by normal projection of the packing to the base plane, A is the area of the base figure, and $m_{1}$ and $m_{2}$ are the distances between the three planes. For example, in the packing [3, 6, 3]

$$
\begin{equation*}
\mathrm{n}=1-\frac{\pi}{3 \sqrt{2}}=0.259 \tag{53b}
\end{equation*}
$$

when

$$
\begin{aligned}
v & =1+6 \times 1 / 3=3, \\
\mathrm{~m}_{1} & =\mathrm{m}_{2}=2 \mathrm{r} \sqrt{2} / \sqrt{3}, \text { and } \\
\mathrm{A} & =6 \mathrm{r}^{2} \sqrt{3} .
\end{aligned}
$$

The porosity of packings with the same coordination number depends on the distance between the parallel planes used in the construction; it varies, therefore, between given limits. Data on different packings are listed in Table 18. As can be seen, there are continuous and incompressible systems with porosities above 70 percent; the value, $n=47.46$ percent, generally taken as the maximum, is smaller than the porosity of many packings.

The cubic system is the simplest packing and has a porosity of 47.46 percent. This system may be transformed to an orthorhombic and to a rhombohedral one. In the first case, every sphere of a given layer glides on a sphere of the lower layer, with the direction of movement parallel to the straight line connecting the centers of the given row of spheres. The unit cell, consisting of eight spheres, is transformed from a cube to a rhombohedron (Fig. 21). The amount of movement can be given by the variation of the orientation angle, $\alpha$, between $60^{\circ}$ and $90^{\circ}$ (Fig. 23). At $\alpha=60^{\circ}$, the system is orthorhombic.

The porosity of the system in terms of $\alpha$ is

$$
\begin{equation*}
\mathrm{n}=1-\frac{\pi}{6 \sin \alpha} \tag{54}
\end{equation*}
$$

The volume of the unit cell varies with $\sin \alpha$. The same relation applies to the height of the unit cell. The variation of V and n is shown in Figure 24. If $\mathrm{n}=$ $\left(\mathrm{V}-\mathrm{V}_{\mathrm{S}}\right) / \mathrm{V}$, and $\mathrm{V}_{\mathrm{S}}$ is the volume of solids in the cell

$$
\begin{equation*}
V_{S}=V(1-n)=\pi / 6 \tag{55}
\end{equation*}
$$

The volume of the spheres in the unit cell remains constant during the movement. This case may be considered that of plane deformation, the unit cell being deformed in one direction only.

In the second basic case, several types of movement must be applied to the system. There must be additional movement in the direction normal to that of the plane deformation. The movement is carried out uniformly; i.e., the deformation and compression of the unit cell occur at a uniform rate. The angle of orientation (that is, the angle between two edges of the same side on the unit cell) is the same for every two edges
(Fig. 25). Assuming this, movement may be described with the help of $\alpha$, and the volume of the unit cell and the porosity can be given as functions of $\alpha$.

The formula for the porosity has been given by Slichter as early as 1889:

$$
\begin{equation*}
n=1-\frac{\pi}{6(1-\cos \alpha) \sqrt{1+2 \cos \alpha}} \tag{56}
\end{equation*}
$$



Figure 23. Orthorhombic packing of spheres.

The volume of the unit cell is

$$
\begin{equation*}
V=(1-\cos \alpha) \sqrt{1+2 \cos \alpha} \tag{57}
\end{equation*}
$$

and the volume of solids is given by Eq. 55. The void ratio is, therefore,

$$
\begin{equation*}
\mathrm{e}=\frac{\mathrm{n}}{1-\mathrm{n}}=\frac{6 V}{\pi}-1 \tag{58}
\end{equation*}
$$

The height of the unit cell varies with $\alpha$ (Fig. 26): $\mathrm{h}=\mathrm{D} \sin \delta$ with $\mathrm{D} \cos \delta=\mathrm{D} \cos \alpha / \cos$ $(\alpha / 2)$ which yields

$$
\begin{equation*}
\mathrm{h}=\mathrm{D} \sqrt{1-\frac{\cos ^{2} \alpha}{\cos ^{2} \alpha / 2}}=\mathrm{D} \frac{(1-\cos \alpha) \sqrt{1+2 \cos \alpha}}{\sin \alpha} \tag{59}
\end{equation*}
$$

Figure 27 shows the variation of $\mathrm{V}, \mathrm{F}$ and h for $\mathrm{D}=1$ in terms of $\alpha$.
The coordination number of the system does not vary continuously during the movement; it takes the final value only after performing the described movement. Smith et al. (83), with the intention of applying the results derived for uniform spheres to actual particle systems, assumed that the actual system may for statistical purposes be treated as composed of separate clusters of rhombohedral or cubic arrangements, these being present in such a proportion as to give the observed porosity of the assembly. This consideration leads to the following expression for the average coordination number, $N$, in terms of the porosity, $n$ :

$$
\begin{equation*}
N=26.4858-\frac{10.7262}{1-n} \tag{60}
\end{equation*}
$$

The curve representing this (Fig. 28) agrees well with the observed experimental values. From the relationship between the angle of orientation and the porosity, the curve $\mathrm{N}=\mathrm{f}(\alpha)$, i. e., the relationship between the angle of orientation and the coordi-


Figure 24. Relative volume of unit cell vs angle of orientation.


Figure 25. Rhombohedral packing.


Figure 26. Height of unit cell.
nation number (Fig. 28b), may be constructed. The plot of N as a function of the relative volume of the unit cell (Fig. 29 ) is a straight line that represents, most likely, Eq. 60 in another form. The equation of the straight line is:

$$
\begin{gather*}
\mathrm{N}=6+6(2+\sqrt{2})(1-\mathrm{V})= \\
6(4.41-3.41 \mathrm{~V}) \tag{61}
\end{gather*}
$$

Coordination number and porosity may also be related by plotting the values given in Table 18. Porosities and the relative volumes of the unit cell are shown in Figure 30. The plot consists, of course, of isolated points for the integers N ; the connecting dotted line is given only to show the trend of variation. It is interesting to show that Figure 31, giving $N=f(\alpha)$ and constructed on the base of the curves in Figure 32, does not differ much from the data shown in Figure 28b.
${ }^{\circ}$ Extending the interpretation of $\alpha$ beyond $60^{\circ}$ results in decreasing values of $n$, and the spheres intersect each other. The rate of decrease is considerable, because the volumes of the intersecting parts have to be considered twice. When this volume equals the volume of the remaining voids, theoretically $n=0$. This occurs at $\alpha=49^{\circ}$, where the volume of the unit cell is equal to $V=0.523=\pi / 6$. The variation of $n$ with $\alpha$ for the range $90^{\circ} \geq \alpha \geq 49^{\circ}$ is given in Figure 33. The part $60^{\circ}>\alpha \geq 49^{\circ}$ of the curve may be used to determine the angle of orientation for packings of nonuniform spheres with porosities greater than 25 percent. This angle may be taken as a characteristic of the substitute packing of uniform spheres. Efforts to find a physical meaning for values of $\alpha$ greater than $90^{\circ}$ have not been successful. (At $\alpha=120^{\circ}$, $n$ again equals zero; however, the part of the curve for values between 90 and $120^{\circ}$. is not the same as that between 60 and $90^{\circ}$.

## Shearing Resistance of Packings

The following is an attempt to determine the stresses necessary to bring the cubic system into the rhombohedral system. The shearing resistance of the densest packing ( $\mathrm{N}=12$ ), filling the entire space, will be infinitely great, thus forming a closed system. An increase of the volume cannot take place, even if the shearing stresses increase to infinity. The porosity of a certain arrangement in the infinite space can decrease only as the effect of shearing stresses; any combination of stresses causes a tendency of densification.

As shown in Figures 24 and 27, the unit cell suffers a compression and a distortion during the movements that transform a cubic system ( $\mathrm{N}=6$ ) into a rhombohedral one $(\mathrm{N}=12)$. The decrease of volume can be achieved by the application of a uniform all-


Figure 27. Relative volume of unit cell vs angle of orientation.


Figure 28. Coordination number according to assumption of Smith.


Figure 29. Coordination number as function of relative volume.


Figure 30. Coordination number and relative volume.


Figure 31. Coordination numbers for packings given in Table I.
round pressure; the decrease of the angle of orientation may be caused by uniform shear (Fig. 33). The relation between volume change and shear strain and the respective stresses may be determined by use of the general laws related to liquids.

As a first approximation, a linear relationship is assumed to exist between volume change and hydrostatic pressure. The volume in consideration-the volume of the unit cell-is occupied partly by voids and partly by sphere parts. For the present, a substitute liquid filling the unit cell is assumed. The volume change from $\alpha=90^{\circ}$ to $\alpha=$ $60^{\circ}$ is given by

$$
\begin{equation*}
\frac{V-V_{0}}{V_{0}}=-\frac{\Delta V}{V_{0}}=\frac{\sigma}{C} \tag{62a}
\end{equation*}
$$



Figure 32. Variation of porosity with angle of orientation.
and


Figure 33. Packing of spheres loaded by hydrostatic and shearing stresses.

$$
\begin{array}{r}
\sigma=\frac{\mathrm{c}}{\mathrm{~V}_{0}}\left(\mathrm{~V}-\mathrm{V}_{0}\right)= \\
\quad \mathrm{C}_{1}\left(\mathrm{~V}-\mathrm{V}_{0}\right) \tag{62b}
\end{array}
$$

in which $C$ is the bulk modulus of compressibility (depending on the surface properties of the spheres ) and $V_{0}$ is the volume at $\alpha=60^{\circ}\left(V_{0}=\pi / 6\right)$.

The deformation caused by shearing stresses consists of the change of $\alpha$ accompanied by the decrease of porosity. In the theory of liquids (108), the shearing stress is given by

$$
\begin{equation*}
\tau=-\frac{\partial \mathrm{F}}{\partial \beta} \tag{63}
\end{equation*}
$$

in which $\beta=90^{\circ}-\alpha$ and $\mathrm{F}(\beta, \mathrm{T})$ is the free energy of the lattice referred to one sphere. It is a function of variation of the height of the unit cell and of the temperature. Because these are isothermic processes, this term can be taken as the potential energy of one sphere with respect to the sphere immediately below. It is, then, given by $h-h_{0}$, where $h$ is the height of the unit cell at $\alpha$ and $h_{0}$ is the minimum value at $\alpha=60^{\circ}$ or $D \sqrt{2 / 3}$ (Fig. 26, Eq. 59). Assuming again, as a first approximation, a linear relationship for $\mathrm{h}=\mathrm{h}(\alpha)$ (connecting $\mathrm{h}=1$ for $\alpha=90^{\circ}$ and $\mathrm{h}=\mathrm{h}_{\mathrm{o}}$ for $\alpha=60^{\circ}$ ), $\mathrm{h}=1-\mathrm{C}_{2} B$ or

$$
\begin{equation*}
\frac{\partial \mathrm{F}}{\partial \beta}=\frac{\partial \mathrm{h}}{\partial \beta}=\mathrm{C}_{2} \tag{64}
\end{equation*}
$$

and, therefore, $\tau=$ const. $=\mathrm{C}_{2}$.
The shearing resistance of the medium can be given (Fig. 34) as

$$
\begin{array}{r}
\tan \phi=\tau / \sigma=\frac{\tau}{\sigma} \frac{\mathrm{C}_{2}}{\mathrm{C}_{\mathbf{1}}\left(\mathrm{V}-\mathrm{V}_{0}\right)}= \\
\frac{\mathrm{C}}{\mathrm{~V}-\mathrm{V}_{0}} \tag{65}
\end{array}
$$

which is in complete agreement with the equation of Winterkorn suggested by the analogy between liquids and grain assemblies (106).

Values, calculated on the basis of the concept of the solid and liquid state of macromeritic systems (Eq. 65) have


Figure 34. Determination of angle of friction. been compared with experimental data obtained by various dependable workers on the friction properties of granular materials (26).

It must be emphasized that this formula represents the first approximation. Besides the substitution of the relations $V=f(p)$ and $h=h(\beta)$ with straight lines-which actually, as it can be seen from Figures 24 and 27, may be considered justified-it neglects an important factor. This approximation is also involved in Batschinski's formula: namely, the activation energy for the diffusion of holes in the liquid and in the particle assembly has been disregarded. It should be borne in mind also that the 'application of the equation to higher pressures can hardly give exact values, because the dependence of the volume on the pressure and of the energy on the volume deviates from a linear law in this region.

Eq. 65 can be used to verify the difference between plane shear and shear in three dimensions. Considering Figures 5 and 8, respectively, with the term $V_{0}$ (volume of solids) the same in both cases, the difference in $\tau$ or h must be taken into account.

$$
\begin{equation*}
\tau \text { plane }=\frac{1-0.866}{\pi / 6}=0.255 \tag{66a}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau \text { space }=\frac{1-0.815}{\pi / 6}=0.353 \tag{66b}
\end{equation*}
$$

yield

$$
\begin{equation*}
\frac{\tan \phi \text { plane }}{\tan \phi \text { space }}=\frac{0.255}{0.353}=0.725 \tag{67}
\end{equation*}
$$

This may explain the discrepancies in the value of $\tan \phi$ as determined by direct shear or by triaxial test, respectively. It means that if the latter amounts, for instance, to $\phi=40^{\circ}$, the direct shear is likely to give $\phi=\arctan (0.725 \tan \phi)=32^{\circ}$. Many test results in the literature show similar deviations (111).

To arrive at a better approximation in the evaluation of the inner resistance of a macromeritic liquid, the dependence of $V$ on $\sigma$, instead of Eq. 62b, must be considered, according to an empirical equation proposed long ago by Tait (108). This relation states that there is a strain-hardening during the process of compression. For a given amount of compression, greater all-round stresses must be applied if there is already a stress of this type acting. This means that the bulk modulus of compressibility is not a constant, but a function of the all-round pressure itself. If

$$
\begin{equation*}
C=\frac{d p}{d \epsilon}=\frac{p+b}{a b} \tag{68}
\end{equation*}
$$

the volume change is

$$
\begin{equation*}
\mathrm{V}-\mathrm{V}_{\mathrm{o}}=\mathrm{a} \log \left(\frac{\mathrm{p}+\mathrm{b}}{\mathrm{~b}}\right) \tag{69}
\end{equation*}
$$

This equation is of the same form as that for the compression of soils proposed by Terzaghi (112).

The $\operatorname{term}\left(\mathrm{V}-\mathrm{V}_{0}\right)$ is given as a trigonometric function of $\beta$; it may be replaced without loss of accuracy by a parabola of the second degree. The deviation from the exact value can be made smaller than 1 percent. Then

$$
\begin{equation*}
V-V_{0}=C_{1} B^{2} \tag{70}
\end{equation*}
$$

and from Eq. 69

$$
\begin{equation*}
p=b\left[\left(\exp \frac{V-V_{0}}{a}\right)-1\right]=b\left[\left(\exp \frac{C_{1} \beta_{1}^{2}}{a}\right)-1\right] \tag{7ii}
\end{equation*}
$$

Eq. 59, giving a measure for the available potential energy, may be replaced also by a parabola of the second degree. Then

$$
\begin{equation*}
\mathrm{F}=1-\mathrm{C}_{2} \beta^{2} \tag{72}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau=\partial \mathrm{F} / \partial \beta=2 \mathrm{C}_{2} \beta \tag{73}
\end{equation*}
$$

The shearing resistance is given by

$$
\begin{equation*}
\tan \phi=\frac{\tau}{\rho}=\frac{2 C_{2} \beta}{b\left[\left(\exp \frac{C_{1} \beta^{2}}{a}\right)-1\right]} \tag{74}
\end{equation*}
$$

Because the relation between the porosity and the angle of orientation ( $\alpha=90^{\circ}-\beta$ ) is known, Eq. 74 furnishes the solution to the problem. However, the physical meaning of the laws expressed by this equation will be better understood, if instead of $\beta$, the void ratio $\epsilon$ is used (notation used to replace the usual e, in order to avoid confusion with the base of the natural logarithm). Then, Eq. 70 yields

$$
\begin{equation*}
\beta=\sqrt{\frac{V-V_{0}}{C_{1}}}=\sqrt{\frac{\epsilon-\epsilon \min }{C_{1}}} \tag{75}
\end{equation*}
$$

and

$$
\begin{equation*}
\tan \phi=\frac{2 C_{2} \sqrt{\left(\epsilon-\epsilon_{\min }\right) / C_{1}}}{\mathrm{~b}\left[\left(\exp \frac{\epsilon-\epsilon_{\min }}{\mathrm{a}}\right)-1\right]}=\frac{C \sqrt{\epsilon-\epsilon_{\min }}}{\left(\exp \frac{\epsilon-\epsilon \min }{\mathrm{a}}\right)-1} \tag{76}
\end{equation*}
$$

Eq. 76 should be checked against available shear test data obtained by dependable research workers. In Figure 35 data from direct shear tests on Ottawa standard sand (88) have been plotted as $\tan \phi=\mathrm{f}(\epsilon)$; the curve according to Eq. 76 is also shown using the constants $\epsilon_{\min }=0.2 \mathrm{a}=0.5$, and $\mathrm{C}=1.26$. The deviations are small (Table 19).

It is likely that a more precise determination of the constants would result in a still better agreement; this sample calculation has been presented to show the generaltrend of Eq. 76. It may be assumed that the constants $\epsilon_{\min }$ and a vary within close limits.

A third approximation is also available to evaluate the function $\tan \phi=f(\epsilon)$. The variation of the volume of the unit cell with the variation of the imaginary coordination

number is a linear relationship. The shearing stress, necessary to produce movement by overcoming the friction on the surface of the spheres, may be assumed to be directly proportional to ( $\mathrm{N}-6$ ). In the case of the cubic arrangement of the spheres ( $\mathrm{N}=6$ ), the shearing resistance is zero (melting point). Then Eq. 61 yields

$$
\begin{equation*}
N-6=6(2+\sqrt{2})(1-V) \tag{77}
\end{equation*}
$$

TABLE 19
EXPERIMENTAL VS CALCULATED DATA ON SHEARING STRENGTH

| Void <br> Ratio $¢$ | Tan $\varnothing$ |  |  | Dev. | Dev. <br> $(g)$ |
| :--- | :--- | ---: | :--- | :--- | :--- |
|  | Calc. | Exp. |  |  |  |
| 0.58 | 0.684 | 0.682 |  | -0.002 | -0.29 |
| 0.60 | 0.651 | 0.648 |  | -0.003 | -0.46 |
| 0.62 | 0.620 | 0.614 |  | -0.006 | -0.98 |
| 0.64 | 0.593 | 0.589 | -0.004 | -0.67 |  |
| 0.66 | 0.570 | 0.579 |  | +0.009 | +1.38 |

and

$$
\begin{equation*}
\tau=C(N-6)=C_{1}\left[a-b\left(V-V_{0}\right)\right] \tag{78}
\end{equation*}
$$

Assuming, as in the first approximation,

$$
\begin{equation*}
\sigma=C_{2}\left(V-V_{0}\right) \tag{79}
\end{equation*}
$$

the result is

$$
\begin{equation*}
\tan \phi=\frac{C}{V-V_{0}}-B \tag{80}
\end{equation*}
$$

This equation differs from Winterkorn's Eq. 34a only in the term (-B). This may account for the activation energy neglected in Batschinski's equation. It would be


Figure 37. Cubic packing as limiting state.


Figure 38. Substitution of isolated forces.
worthwhile to check this equation against experimental data; it is not impossible that cases yielding negative values for ${ }^{\epsilon}$ min , which is physically difficult to visualize, would fit Eq. 80.

## LATERAL PRESSURES OF PACKINGS

The study of systematic packings can be extended to the investigation of lateral pressures exerted by them. This investigation furnishes some interesting results that may be of value for the better understanding of earth pressure phenomena (110).

If in cubic packing, there is, according to Winterkorn's conception, a melting point of the grain assembly, the coefficient of lateralpressures, as interpreted in the theory of earth pressure, has to be unity (Fig. 20). This packing, however, gives at the first glance zero value (Fig. 36); there are no horizontal forces between the spheres. The slightest tilting of the wall AB will produce, however, lateral forces; if there is no friction between the spheres, this horizontal force, $H$, if $\Delta \alpha \rightarrow 0$, in the first row is $W$ (Fig. 37). In the second row, H is 2 W , and so on, until in the n-th row $H$ is $n W$. The coefficient of earth pressure is

$$
\begin{equation*}
\mathbf{K}_{0}=\frac{\mathrm{E}_{0}}{\mathrm{~h}^{2} \gamma}=\frac{\sigma_{\mathrm{X}}}{\sigma_{\mathrm{Z}}} \tag{81}
\end{equation*}
$$

This definition assumes a continuous distribution of the forces transmitted through the particles; that is, the number of contacts becomes infinity. To calculate, the packing must be substituted by a continuous mass. This can be accomplished in the following manner (Fig. 38):

$$
\begin{gather*}
\mathrm{DE}_{0}=W+2 W+\ldots+n W=\frac{1}{2} n(n+1) W  \tag{82}\\
W=\frac{\pi}{6} D^{3} \gamma_{S}  \tag{83}\\
E_{0}=\frac{1}{2} n(n+1) \frac{\pi}{6} D^{2} \gamma_{s} \tag{84}
\end{gather*}
$$

in which D is the width of the back of wall. This gives a triangular distribution for the horizontal stresses, with an intensity

$$
\begin{equation*}
\sigma_{\mathrm{x}}=\frac{2 \sum E}{\mathrm{~h}}=(\mathrm{n}+1) \frac{\pi}{6} \mathrm{D} \gamma_{\mathrm{S}} \tag{85}
\end{equation*}
$$

because $h$ is $n D$.
The vertical force exerted by a vertical row of spheres is

$$
\begin{equation*}
N=n W=\frac{\pi}{6} D \gamma_{s} \tag{86}
\end{equation*}
$$



Figure 39. Forces in orthorhombic system.
giving a uniform pressure

$$
\begin{equation*}
\frac{N}{D^{2}}=n D \frac{\pi}{6} \gamma_{s} \tag{87}
\end{equation*}
$$

and a coefficient of the earth pressure at rest

$$
\begin{equation*}
\mathrm{K}_{0}=\frac{\sigma_{\mathrm{z}}}{\sigma_{\mathrm{Z}}}=1+\frac{1}{\mathrm{n}} \tag{88}
\end{equation*}
$$

which, if $n \rightarrow \infty$, is 1 .
Although a simpler expression for this coefficient is

$$
\begin{equation*}
-K_{0}=\frac{H}{N}=\frac{n W}{n W}=1 \tag{89}
\end{equation*}
$$

the more complicated way shown here is useful for the treatment of other cases.
The density of the orthorhombic system is midway between that of the densest and the loosest ( $[3,6,3]$ and $[1,4,1]$ ) systems. It is, in fact, the densest of the plane systems. The determination of forces between the spheres and those acting on a plane with the angle of $\alpha=60^{\circ}$, respectively, is shown in Figure 39.

With the same procedure as before, the resultant force on $\overline{\mathrm{AB}}$ is

$$
\begin{equation*}
D E_{0}=\frac{1}{2} n(n+1) \frac{\pi}{6} D^{3} \gamma_{S} \tag{90}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{E}_{0}=\mathrm{K}_{0} \frac{\mathrm{~s}^{2} \gamma_{\mathrm{S}}}{2} \tag{91}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\mathrm{K}_{0}=\left(1+\frac{1}{\mathrm{n}}\right) \frac{\pi}{6}=0.523(\mathrm{n} \rightarrow \infty) \tag{92}
\end{equation*}
$$

and, for the vertical plane

$$
\begin{gather*}
D E_{o h}=\frac{1}{2} n(n+1) \frac{1}{0.866^{2}} \times H  \tag{33}\\
H=W \tan 30^{\circ}  \tag{94}\\
K_{o h}=\left(1+\frac{1}{n}\right) \frac{1}{0.866^{2}} \times \frac{\pi}{6} \times 0.577=0.404 \tag{95}
\end{gather*}
$$

These results are in very good agreement with the measured values for sand of middle density.

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[^1]:    ${ }^{a_{\text {Primary }}}$ radius $=1$.

[^2]:    ${ }^{a}$ On samples of rounded gravel.

