A New Matrix-Energy Method for Analyzing All Stresses in Rigidly Connected Highway Bridge Trusses

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An effort is made to advance a unified logical method for analyzing the true stress states in rigidly connected bridge trusses. The report is divided into two parts: (a) basic research and development, and (b) integrated synthetic application with automatic logical checks.

The theoretical development of the proposed matrix-energy method and examples for the orientation of its exact and simplified versions are set forth. The method of panel-load superposition is advanced as a powerful substitute for influence lines and the methods of substitution and transformation as efficient ways for reducing the number of unknowns and the unavoidable solution of large sets of simultaneous equations is clarified. Also included is an integrated synthetic application of those points developed to a complication analysis. To illustrate a comprehensive scheme with only a small computer available, a three-span continuous highway bridge truss has been selected for the objective analysis. The close correspondence of all maximum design axial stresses to these determined by the conventional method demonstrates the validity of the proposed method.

The proposed method yields most expediently two maximum stress states: (a) maximum axial stresses and simultaneous end moments and transverse shears; and (b) maximum end moments and simultaneous axial stresses and transverse shears. The larger requirement of the two constitutes the absolute maximum stress state that should govern the design. Automatic logical checks for programmed computation are presented. An independent proof for the symmetric coefficient matrix and a validity demonstration for the method of transformation are included.

•FOR MORE THAN eight decades, methods originally developed for analyzing pin-connected trusses have inappropriately continued to be used in determining stresses in modern rigidly connected trusses. Pinned joints were being replaced by riveted joints in the 1870's and became almost completely obsolete about half a century ago. Since World War II, additional versions of rigidly jointed trusses—welded and bolted—have gained increasing importance. All these modern rigidly connected trusses, with or without internal or external redundancy, are inherently highly statically indeterminate rigid frames. The rigidity of the joints constitutes the main cause for end moments, transverse shear, and axial stress in each member.

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THEORETICAL DEVELOPMENT

Including Manderla's (1) first enunciation of a method 85 years ago, at least nine independent methods have been developed for the solution of the so-called "secondary stresses," stresses caused by conditions ignored in the conventional analysis of "primary stresses." The problem of secondary stress has actually arisen from improper solution of rigidly connected trusses, rather than from its being truly secondary in nature. By analyzing a rigidly connected truss under a given loading as an assemblage or chain of rigid frames, only one true set of perfectly normal genuine stresses will be found, thus dispelling the misnomer of secondary stresses.

To achieve the ideal of solving all genuine stresses including secondary stresses in each member of a rigidly connected truss of any configuration with any redundancy under any externally applied loading, a matrix-energy formulation is proposed. The method enables the determination of all genuine stresses in a unified setup; it adapts to programmed electronic computation, provides both exact and simplified solutions, and applies to both determinate and indeterminate rigidly connected bridge trusses.

BASIC CONCEPTS

A rigidly connected truss under a given loading is structurally much more complicated than an otherwise ideal pin-connected version identically loaded. There exist, as the truss deflects, couples acting on the bar ends (equal to the internal resisting moments at those points) plus transverse shears. Any determinate truss thus becomes indeterminate in its logical correct solution.

In the most general case, a rigidly connected indeterminate truss of any redundancy would be completely determined by statics, if all of the following were known: (a) the internal resisting moments at the ends of the members, (b) the axial stresses in the redundant members, and (c) the redundant reactions at the supports. These three types of quantities are treated as unknowns in the proposed method. To insure that all unknowns are statically independent, equations of static equilibrium must be fully applied to eliminate dependent unknowns. Consequently, the number of statically independent unknowns is just equal to the degree of statical indeterminateness of the truss viewed as an assemblage of rigid frames.

In general, for an asymmetrical rigidly connected truss of m members under asymmetrical loading, there will be 2m unknown end moments. In a symmetrical rigidly connected truss and under symmetrical loading, if n is the number of joints, the number N of statically independent unknown end moments is given by

$$N = \frac{1}{2}(2m - n) = m - \frac{n}{2} \tag{1}$$

All internal axial stresses, bending moments, and shears in the members, and hence, the total strain energy of the truss can be expressed in terms of the externally applied panel loads and the independent unknowns. ("Axial stress" denotes "total axial stress" or "total internal axial force" as distinguished from "unit axial stress.") By appropriate partial differentiations, all the necessary simultaneous equations will be evolved. On these basic concepts is founded the development of the problem solution in its operative sequence.

Fundamental Notations and Sign Convention

The exaggerated elastic curve of any truss member I-J in the plane of the truss is represented in Figure 1. Symbols applying to this member are as follows:

M_{ij}, M_{ji} = unknown internal resisting end moment at I- and J-end, respectively (kip-in.);

N_{ij} = axial stress (kips);

Q_{ij} = transverse shear (kips);

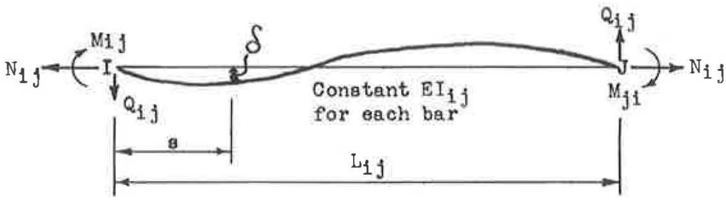


Figure 1.

A_{ij} = cross-sectional area (sq in.);

I_{ij} = moment of inertia (in.^{4});

L_{ij} = length (in.);

s = distance from I-end (in.);

δ = displacement at s, normal to line I-J (in.);

E = modulus of elasticity of material (ksi);

G = modulus of rigidity of material (ksi);

μ = Poisson's ratio of material (may be taken as 0.03 for structural steel); and

U_{ij}, V_{ij}, W_{ij} = strain energy due to bending moment, transverse shear, and axial stress, respectively (in.-kip).

The sign convention is defined such that (a) positive end moments produce clockwise rotation of the member ends; (b) positive axial stresses are in tension; and (c) a positive pair of shears forms a counterclockwise couple.

Constituent Strain-Energy Matrix

The matrix of constituent strain-energy expressions may now be formulated. In Figure 1, recognizing that the moment due to axial stress and deviation from the line I-J is usually negligibly small, the true moment about any point at a distance s from the I-end,

$$M_S = M_{ij} - Q_{ij}\, s - N_{ij}\, \delta \tag{2}$$

may take the simplified form of

$$M_S = M_{ij} - Q_{ij}\, s \tag{3}$$

where

$$Q_{ij} = \frac{M_{ij} + M_{ji}}{L_{ij}} \tag{4}$$

Following the original suggestion of Ménabréa (2) (containing the earliest suggestion in the use of the expression for the strain energy of the truss), but in the present-day complete form, we may write the matrix of the constituent strain-energy expressions of any member I-J as:

$$\begin{bmatrix} W_{ij} \\ \\ U_{ij} \\ \\ V_{ij} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \int_0^{L_{ij}} \frac{N_{ij}^2}{EA_{ij}} ds \\ \\ \int_0^{L_{ij}} \frac{M_s^2}{EI_{ij}} ds \\ \\ \int_0^{L_{ij}} \frac{Q_{ij}^2}{GA_{ij}} ds \end{bmatrix} = \frac{1}{E} \begin{bmatrix} \frac{1}{2} \frac{L_{ij}}{A_{ij}} N_{ij}^2 \\ \\ \frac{L_{ij}}{6I_{ij}} \left(M_{ij}^2 - M_{ij}M_{ji} + M_{ji}^2\right) \\ \\ \frac{1+\mu}{A_{ij}L_{ij}} \left(M_{ij} + M_{ji}\right)^2 \end{bmatrix} \tag{5}$$

where $G = E/2(1 + \mu)$ in the last equation.

Summing up $\{W\ U\ V\}_{ij}$ for all members of the truss, the total strain energy U of any truss is then

$$U = \sum_{1}^{m} \lfloor 1\ 1\ 1 \rfloor \left\{W_{ij}\ U_{ij}\ V_{ij}\right\} \tag{6}$$

where m is the number of members in the truss.

Matrix Equation of Unknowns and Their Solution

With all joints enormously rigid, all components ideally fit, and all supports unyielding, the application of Castigliano's second theorem (3), or the theorem of least work, to the problem of trusses with any degree of redundancy, will yield the following relations:

$$\left\{\frac{\partial U}{\partial M}\ \frac{\partial U}{\partial N}\ \frac{\partial U}{\partial R}\right\} = \{0\ 0\ 0\} \tag{7}$$

where M is any statically independent unknown end moment, N is the unknown axial stress in any redundant member, and R is any unknown redundant reaction.

Whereas Eqs. 7 represent minimization of strain energy or zero "relative" displacements, the last also denotes the condition of zero settlement of support. In the case of non-zero settlement, according to Castigliano's first theorem (3), $\frac{\partial U}{\partial M}$ would be equal to the rotation, $\frac{\partial U}{\partial N}$ to an over- or underrun, and $\frac{\partial U}{\partial R}$ to the support settlement.

The unknown M's, N's, and R's of any loaded plane truss of any configuration may be generalized as the unknown column vector $\{X_i\}$, where $i = 1, 2, \ldots, n$. Repeated application of $\frac{\partial U}{\partial X_i} = 0$ yields a set of n nonhomogeneous simultaneous algebraic linear equations:

$$\left[a_{ij}\right] \left\{X_i\right\} = \left\{C_i\right\} \tag{8}$$

in which both i and $j = 1, 2, \ldots,$ n and the constant vector $\{C_i\}$ has been transposed to the righthand side.

It follows analogously from Maxwell's theorem of reciprocity (4) that the coefficient a_{ji} of X_i in the j th equation is identical both in sign and magnitude as the coefficient a_{ij} of X_j in the i th equation, and, consequently,

$$a_{ij} = a_{ji} \tag{9}$$

where $i \neq j$, giving a symmetric coefficient matrix. An independent proof for the symmetry of the coefficient matrix is given in Appendix A.

The system of Eqs. 8 will always have a general solution by inverting $[a_{ij}]$ unless it is singular; i.e., if $|a_{ij}| \neq 0$, the solution will be

$$\{X_i\} = [a_{ij}]^{-1}\{C_i\} \tag{10}$$

Because the premultiplication of a matrix by its inverse is uniquely equal to a unit matrix, the vector of solutions given by the right side of Eqs. 10 constitutes the only solutions.

By virtue of a symmetric matrix in Eqs. 8, only $\frac{1}{2}$ n(n + 1) coefficients must be evaluated and, consequently, the computer time for inverting the matrix will be correspondingly reduced. In inverting large matrices, an efficient and fast method such as Li's algorisms (5) is recommended.

ANALYSIS OF A BRIDGE TRUSS BY MATRIX-ENERGY METHOD

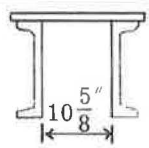
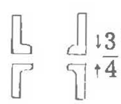
An Example for Orientation of Exact Method

To exemplify the numerical process and compare the results with those obtained by recognized conventional methods, the simple bridge truss given by Sutherland and Bowman (6) is first solved by the proposed exact matrix-energy method.

It is desired to find all genuine stresses at the ends of each member of the rigidly connected truss shown in Figure 2 due to vertical loads of 166 kips at each lower panel point except supports. The makeup of members is given in Table 1. For simplicity,

TABLE 1

MAKEUP OF MEMBERS AND SECTION PROPERTIES

Bar	Section	A(in.^{2})	I(in.^{4})	L(in.)	I/c(in.^{3})	Sketch
1-3	2-[15 × 33.9 1-Pl 18 × 7/16	27.68	961.0	450.44	167.5 99.1	
3-5	2-[15 × 33.9 1-Pl 18 × 3/8	26.55	922.8	300.00	156.0 97.6	
1-2 2-4	4-∟s 6 × $3\frac{1}{2}$ × $\frac{1}{2}$	18.00	175.3	300.00	27.5	
2-3	4-∟s 6 × $3\frac{1}{2}$ × 7/16	15.88	153.8	336.00	24.1	
3-4	4-∟s 6 × $3\frac{1}{2}$ × 3/8	13.68	131.8	450.44	20.7	
4-5	4-∟s 5 3 × 3/8	11.44	79.1	336.00	14.7	

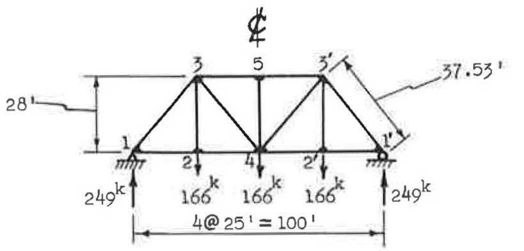


Figure 2.

centroidal axes of members are taken as intersecting exactly at theoretical panel points, thus eliminating eccentric moments.

Independent and Dependent Unknowns.—In the present case, referring to Eq. 1, m = 13 and n = 8; therefore, N = 13 - 4 = 9. That is, the present truss is determinate when pin connected, but becomes indeterminate to the 9th degree when rigidly connected. The nine statically independent unknown end moments may be represented, element for element, by the matrix:

$$\begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{bmatrix} = \begin{bmatrix} M_{13} & M_{21} & M_{24} \\ M_{31} & M_{32} & M_{35} \\ M_{42} & M_{43} & M_{53} \end{bmatrix} = -\begin{bmatrix} M_{1'3'} & M_{2'1'} & M_{2'4'} \\ M_{3'1'} & M_{3'2'} & M_{3'5'} \\ M_{4'2'} & M_{4'3'} & M_{5'3'} \end{bmatrix} \qquad (11)$$

Then, by $\Sigma M = 0$ at joints 1, 2, 3 and 1', 2', 3', six of the remaining dependent unknown end moments can be expressed as:

$$\begin{bmatrix} M_{12} \\ M_{23} \\ M_{34} \end{bmatrix} = -\begin{bmatrix} M_{1'2'} \\ M_{2'3'} \\ M_{3'4'} \end{bmatrix} = -\begin{bmatrix} X_1 \\ X_2 + X_3 \\ X_4 + X_5 + X_6 \end{bmatrix} \qquad (12)$$

By symmetry, this yields:

$$\{M_{45} \; M_{54} \; Q_{45}\} = \{0 \; 0 \; 0\} \qquad (13)$$

Extended Methods of Moments, Shears, and Joints.—The axial stress in each member is readily determined by the extended methods of moments, shears, or joints, which are illustrated for members 1-2, 1-3, and 2-3 in the following paragraphs.

In the extended method of moments, by passing a section just to the left of member 2-3 and considering the equilibrium of the free body to the left, as shown in Figure 3, we have by $\Sigma M = 0$ about joint 3,

$$\lfloor -N_{12} \; X_2 \; 249 \; X_4 \rfloor \{336 \; 1 \; 300 \; 1\} = 0 \text{ and}$$

$$N_{12} = \lfloor X_2 \; X_4 \; 1 \rfloor \{0.002976 \; 0.002976 \; 222.321\}$$

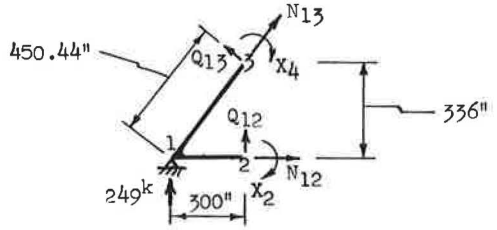


Figure 3.

The extended method of shears, taking the same free body as shown in Figure 3, yields:

$$\{Q_{12} \; Q_{13}\} = \left\{ \frac{-X_1 + X_2}{300.00} \; \frac{X_1 + X_4}{450.44} \right\}$$

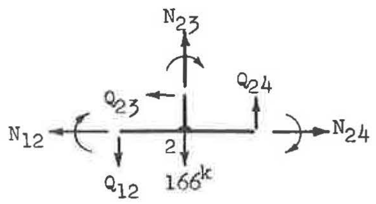


Figure 4.

and by $\Sigma Y = 0$,

$$\lfloor N_{13} \quad X_1 + X_4 \quad -X_1 + X_2 \quad 249 \rfloor \begin{Bmatrix} \frac{28}{37.53} & \frac{25}{450.44(27.53)} & \frac{1}{300} & 1 \end{Bmatrix} = 0$$

Therefore,

$$N_{13} = \lfloor X_1 \quad -X_2 \quad -X_4 \quad -1 \rfloor \{0.002486 \quad 0.004469 \quad 0.001982 \quad 333.808\}$$

In the extended method of joints, by passing a horseshoe section around joint 2, as shown in Figure 4, we have by $\Sigma Y = 0$,

$$\lfloor N_{23} \quad Q_{24} \quad Q_{12} \quad 166 \rfloor \{1 \quad 1 \quad -1 \quad -1\} = 0$$

Substitution of the values of Q_{24} and Q_{12} yields:

$$\lfloor N_{23} \quad X_3 + X_7 \quad -X_1 + X_2 \quad 166 \rfloor \begin{Bmatrix} 1 & \frac{1}{300} & -\frac{1}{300} & -1 \end{Bmatrix} = 0$$

or

$$N_{23} = \lfloor -X_1 \quad X_2 \quad -X_3 \quad -X_7 \quad 1 \rfloor \{0.003333 \quad 0.003333 \quad 0.003333 \quad 0.003333 \quad 166.000\}$$

Axial-Stress Expressions and Strain-Energy Constants. —By applying the preceding methods, axial stresses in all members of the truss may be found as given in Table 2. The constant term in each N expression is exactly equal to the "primary stress" in the same bar if it is pin jointed. Constants in the strain-energy expressions of Eqs. 5 are given in Table 3, taking Poisson's ratio μ as 0.3. With the aid of Tables 2 and 3, $(10)^6$ E times the strain energy in each truss member as given by Eqs. 5 is recorded in Table 4.

TABLE 2

AXIAL-STRESS EXPRESSIONS

Member	$(10)^3$ Times Axial-Stress Expressions
1-2	$2.976X_2 + 2.976X_4 + 222321$
1-3	$2.486X_1 - 4.469X_2 - 1.982X_4 - 333808$
2-3	$-3.333X_1 + 3.333X_2 - 3.333X_3 - 3.333X_7 + 166000$
2-4	$-2.976X_3 + 2.976X_4 + 2.976X_5 + 222321$
3-4	$4.469X_3 - 1.982X_4 - 1.982X_5 + 2.486X_6 + 4.469X_7 + 1.982X_8 +$ $4.469X_9 + 111269$
3-5	$-2.976X_7 - 2.976X_8 - 2.976X_9 - 296429$
4-5	$-6.667X_6 - 6.667X_9$

TABLE 3

CONSTANTS IN STRAIN-ENERGY EXPRESSIONS

Member	L/A	L/6I	$2(1 + \mu)/AL$
1-2	16.66667	0.285225	0.000481
1-3	16.27311	0.078120	0.000209
2-3	21.15869	0.364109	0.000487
2-4	16.66667	0.285253	0.000481
3-4	32.92689	0.569600	0.000422
3-5	11.29944	0.054183	0.000326
4-5	29.37063	0.707965	0.000676

TABLE 4

STRAIN ENERGY OF TRUSS MEMBERS

Member	$(10)^6$ E Times Strain Energy in Member I-J = $10^6\,E(W_{ij} + U_{ij} + V_{ij})$
1-2	$\frac{1}{2}(16.66667)\,(2.976X_2 + 2.976X_4 + 222321)^2 + 285225(X_1^2 + X_1X_2 + X_2^2) + \frac{1}{2}(481)\,(-X_1 + X_2)^2$
1-3	$\frac{1}{2}(16.27311)\,(2.486X_1 - 4.469X_2 - 1.982X_4 - 333808)^2 + 78120(X_1^2 - X_1X_4 + X_4^2) + \frac{1}{2}(209)\,(X_1 + X_4)^2$
2-3	$\frac{1}{2}(21.15869)\,(-3.333X_1 + 3.333X_2 - 3.333X_3 - 3.333X_7 + 166000)^2 + 364109\left[(-X_2 - X_3)^2 + (X_2 + X_3)X_5 + X_5^2\right] + \frac{1}{2}(487)\,(-X_2 - X_3 + X_5)^2$
2-4	$\frac{1}{2}(16.66667)\,(-2.976X_3 + 2.976X_4 + 2.976X_5 + 222321)^2 + 285253(X_3^2 - X_3X_7 + X_7^2) + \frac{1}{2}(481)\,(X_3 + X_7)^2$
3-4	$\frac{1}{2}(32.92689)\,(4.469X_3 - 1.982X_4 - 1.982X_5 + 2.486X_6 + 4.469X_7 + 1.982X_8 + 4.469X_9 + 111269)^2 + 569600\left[(-X_4 - X_5 - X_6)^2 + (X_4 + X_5 + X_6)X_8 + X_8^2\right] + \frac{1}{2}(422)\,(-X_4 - X_5 - X_6 + X_8)^2$
3-5	$\frac{1}{2}(11.29944)\,(-2.976X_7 - 2.976X_8 - 2.976X_9 - 296429)^2 + 54183(X_6^2 - X_6X_9 + X_9^2) + \frac{1}{2}(326)\,(X_6 + X_9)^2$
4-5 center vertical	$\frac{1}{2}(29.37063)\,(-6.667X_6 - 6.667X_9)^2$ (use one-half in computing $\frac{1}{2}$ EU)

Simultaneous Equations and Their Solution. —Because of symmetry in both structure and loading in this particular example, it is necessary to write only one-half of the total strain energy. After eliminating 1/2 and E, repeated application of $\frac{\partial U}{\partial X_i} = 0$ yields Eqs. 10 where

$$\left\{X_i\right\} = \left\{X_1\ X_2\ X_3\ X_4\ X_5\ X_6\ X_7\ X_8\ X_9\right\},$$

$$\left[a_{ij}\right]^{-1} = \begin{bmatrix} 727716 & 284328 & 235 & -77992 & 0 & 0 & 235 & 0 & 0 \\ & 1300346 & 728471 & 292 & 363622 & 0 & -235 & 0 & 0 \\ & & 1300734 & -439 & 363183 & 366 & -283879 & 292 & 658 \\ & & & 1296559 & 1139899 & 1139460 & -292 & 569049 & -292 \\ & & & & 1868605 & 1139460 & -292 & 569049 & -292 \\ & & & & & 1249171 & 366 & 569340 & -52838 \\ & & & & & & 571981 & 392 & 758 \\ & & & & & & & 1139852 & 392 \\ & & & & & & & & 110103 \end{bmatrix}^{-1}$$

a_{ij} below main diagonal $= a_{ji}$ above it

and

$$C_i = (10)^6\{25.2144\ -47.0097\ 6.3636\ -25.5609\ -3.7656\ -9.1098\ -14.6329\ -17.2309\ -26.3407\}.$$

The solution of $\{X_i\}$ in kips-inches by electronic digital computer or otherwise is recorded, element for element, as

$$\begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{bmatrix} = \begin{bmatrix} 66.20 & -84.47 & 39.19 \\ -13.41 & 42.50 & -40.54 \\ -5.803 & -9.309 & -258.8 \end{bmatrix} = \begin{bmatrix} M_{13} & M_{21} & M_{24} \\ M_{31} & M_{32} & M_{35} \\ M_{42} & M_{43} & M_{53} \end{bmatrix} \qquad (14)$$

TABLE 5

BENDING STRESSES AT MEMBER ENDS

Member	End	End Moment (kips-in.) Cross' Method	End Moment (kips-in.) Proposed Method	$\frac{I}{c}$ (in.^{3})	Unit Bending Stress (ksi)
1	2	-67	-66.20	27.5	2.262
	3	67	66.20	167.5	0.395[a]
				99.1	0.668[b]
2	1	-85	-84.47	27.5	3.072
	3	45	45.28	24.1	1.879
	4	40	39.19	27.5	1.425
3	1	-11	-13.41	167.5	0.080[a]
				99.1	0.135[b]
	2	43	42.50	24.1	1.763
	4	12	11.45	20.7	0.553
	5	-44	-40.54	156.0	0.260[a]
				97.6	0.415[b]
4	2	-5	-5.803	27.5	0.211
	3	-9	-9.309	20.7	0.450
	5	0	0	14.7	0
5	3	-263	-258.7	156.0	1.658[a]
				97.6	2.651[b]
	4	0	0	14.7	0

[a]Top. [b]Bottom.

TABLE 6

AXIAL STRESSES AND TRANSVERSE SHEARS

Member	N_{ij} (kips)	N_{ij}/A_{ij} (ksi)	Q_{ij} (kips)
1-2	222.030	12.335	-0.502
1-3	-333.239	-12.039	0.118
2-3	165.387	10.415	0.261
2-4	222.291	12.350	0.111
3-4	110.085	8.047	0.005
3-5	-295.614	-11.134	-0.998
4-5	1.996	0.174	0

Bending Stresses. —Dividing end moments of members thus found by their respective section moduli (I/c) given in Table 1 yields the unit bending stresses at member ends recorded in Table 5. These correspond to the so-called secondary stresses. Values of end moments for the same truss members as found by Sutherland and Bowman (6) by the Cross method are also given. The closeness of end-moment values by both methods testifies to the validity of the proposed method. But the results of the proposed method are "truer" because more accurate axial-strain energy has been used and shearing-strain energy has been taken into consideration.

Axial Stresses and Transverse Shears.—Axial stresses and transverse shears, simultaneously obtained by substituting the values of X_i into Table 2 and Eq. 4, are recorded in Table 6. Unit axial stresses are also calculated.

Streamlining and Simplification. —By treating the rigidly connected truss as an assemblage of rigid frames, the exact matrix-energy method proposed herein, as demonstrated by the former example, has yielded the solution of axial, bending, and shearing stresses in all members of the truss in one unified single setup. With widespread use of electronic computers, the entire process can be programmed from given data to end results. It is shorter and more straightforward than the conventional methods when secondary stresses are considered.

Although the exact method should be used for special investigations and particular designs requiring a high degree of accuracy, for ordinary design purposes a simplified method should be used.

An Example for Orientation of Simplified Method

A study of the equations obtained from $\frac{\partial U}{\partial X_i} = 0$ suggests a simplified method which saves much time in writing the energy expressions and in evaluating the elements of the coefficient matrix.

The process for obtaining the first equation of Eqs. 10, after dividing $(10)^6$ EU by the planted $(10)^6$, from the true value of

$$\frac{1}{2} E \frac{\partial U}{\partial X_1} = 0$$

yields, on rearrangement, the relation:

$$\begin{aligned}
0 = \ & \left.\begin{array}{l} 0.285225(2X_1 + X_2) \\ +0.078120(2X_1 - X_4) \end{array}\right\} \text{contribution by moments} \\
& \left.\begin{array}{l} -16.27311(0.002486)(333.808) \\ -21.15869(0.003333)(166.000) \end{array}\right\} \text{axial stress (corresponding to primary stress)} \\
& \left.\begin{array}{l} +16.27311(0.002486)(0.002486X_1 \\ \qquad - 0.004469X_2 \\ \qquad - 0.001982X_4) \\ +21.15869(0.003333)^2(X_1 - X_2 + X_3 + X_7) \end{array}\right\} \text{axial stress (affected by end moments and transverse shears)} \\
& \left.\begin{array}{l} +0.000481(-1)(-X_1 + X_2) \\ +0.000209(X_1 + X_4) \end{array}\right\} \text{transverse shears}
\end{aligned}$$

It is evident that the coefficients of the unknowns X_1, X_2, and X_4 above the dashed line are about 1,000 times greater than those below the dashed line.

An approximate but much simplified solution sufficiently accurate for usual engineering purposes can, therefore, be most expediently obtained by deleting all strain-energy terms contributed by transverse shears in writing the energy expressions and all terms affecting axial stress contributed by end moments and transverse shears after partial differention. All terms contributed by moments and the term corresponding to the primary stress should be retained.

The simplified form of the first equation thus becomes

$$\lfloor 7262 \quad 2852 \quad -781 \rfloor \; \{X_1 \; X_2 \; X_4\} = 25.214\,(10)^4$$

and the symmetric matrix equation reduces to:

$$\begin{bmatrix} 7262 & 2852 & 0 & -781 & 0 & 0 & 0 & 0 & 0 \\ & 12987 & 7282 & 0 & 3641 & 0 & 0 & 0 & 0 \\ & & 12988 & 0 & 3641 & 0 & -2853 & 0 & 0 \\ & & & 12954 & 11392 & 11392 & 0 & 5696 & 0 \\ & & & & 18674 & 11392 & 0 & 5696 & 0 \\ & & & & & 12476 & 0 & 5696 & -542 \\ \multicolumn{5}{l}{a_{ij} \text{ below main diagonal}} & & 5705 & 0 & 0 \\ \multicolumn{5}{l}{\quad = a_{ji} \text{ above it.}} & & & 11392 & 0 \\ & & & & & & & & 1084 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \end{bmatrix} = (10)^4 \begin{bmatrix} 25.214 \\ -47.010 \\ 6.364 \\ -25.561 \\ -3.766 \\ -9.110 \\ -14.633 \\ -17.231 \\ -26.341 \end{bmatrix}$$

whose solution by electronic digital computer yields:

$$\begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{bmatrix} = \begin{bmatrix} 66.9 & -84.9 & 39.0 \\ -10.7 & 43.4 & -44.5 \\ -6.15 & -9.25 & -265 \end{bmatrix} = \begin{bmatrix} M_{13} & M_{21} & M_{24} \\ M_{31} & M_{32} & M_{35} \\ M_{42} & M_{43} & M_{53} \end{bmatrix} \qquad (15)$$

after which all axial, bending, and shearing stresses in each member of the truss can be determined by statics. The accuracy of the simplified method can be seen by comparing Eqs. 15 with Eqs. 14. The closeness of the results testifies to the validity of the simplified method.

SPECIALLY DEVELOPED TECHNIQUES FOR CONTINUOUS HIGHWAY BRIDGE TRUSSES

Method of Panel-Load Superposition for Continuous Trusses

In determining the maximum tensile or compressive stress, or maximum and minimum stresses in case of reversal, in all members of a determinate truss due to moving live loads, there are available two methods of approach—the influence-line method and the maximum-stress load-position criterion method—both giving the live-load positions producing maximum tensile or compressive stresses.

Due to its inherently complicated nature in an indeterminate system, however, the maximum-stress load-position criterion method has never been heretofore applied to a continuous bridge truss. Theoretically, such a criterion can be deduced for any continuous truss, but the resulting expression would be unwieldy. This explains why the influence-line method has remained the only means by which live-load positions are determined for computing the maximum tensile and compressive stresses in any member of a continuous truss.

Nevertheless, it must be recognized that continuous bridge trusses are usually built only for comparatively long-span crossings. Even in a moderate three-span continuous deck truss, such as the Hawk Falls Bridge (on the Northeast Extension of the Pennsylvania Turnpike) which measures 616 ft horizontally between end bearings and is built on a 1.54 percent grade, there are altogether 113 members if taken unsymmetrically and 57 members when considered as symmetrical about the centerline of the bridge. The formulation of influence-line equations, computation of influence ordinates, and plotting of influence broken lines for so many different members in the composition of the said continuous truss are all very time-consuming tasks.

It must be further recognized that influence lines constituted a visual aid in determining live-load positions in the days of manual computation. With modern electronic computation, typing out influence ordinates, plotting them into influence broken lines, and then retyping in positioned loads for maximum tensile and compressive stresses form the slowest links in automatically programmed continuous computation.

Moreover, it is evident that there are far less live-load panel points than stress-carrying members. In the case of the Hawk Falls Bridge, if considered symmetrical, there are only 15 (14 in a pin-connected truss) live-load panel points vs 57 members (55 if L_0U_0 and U_0U_1 are excepted which would be true in a pin-connected truss) in one-half of the continuous truss. It is, therefore, much more expedient to compute the stresses in all members under each of the 14 or 15 live panel loads than to compute 29 influence ordinates for each of the 55 or 57 members of the truss.

For the reasons just stated, it is proposed to abandon the classical influence-line method and, in its place, use the panel-load superposition method. The procedure is as follows:

1. Convert the lane loadings and concentrated loads for moment and shear, for a given number of lanes and specified reduction, and for a given roadway width to panel loads and concentrations when the lane loadings are placed nearest to the truss;
2. Load the bridge truss with one stress-producing live panel load at a time;
3. Compute the axial, bending, and shearing stresses in each member according to the proposed matrix-energy method;
4. Repeat the process until all stress-producing live-load panel points are covered from one end of the truss to and including the center panel point, if there is one, and if the truss is symmetrical about its centerline;
5. Tabulate the stresses thus found, labeling members symmetrically on the other side of the bridge centerline as primed members;
6. Add all plus-sign tensile stresses and minus-sign compressive stresses for each of the umprimed and primed members;
7. Obtain the concentrated load factors for moment and shear by dividing the respective converted concentrated load by the converted lane-loading panel load;
8. Multiply the appropriate concentrated load factor with the maximum stress among the plus-sign tensile stresses and among the minus-sign compressive stresses caused by single live panel loads, using the concentrated load factor for shear or for moment, respectively, as the stress in the member is dictated by shear or by moment, applying one or two concentrations for moment according to the specifications in use; and
9. Determine the maximum live-load stress of plus sign and minus sign in any member by summing up the plus-sign or minus-sign stresses obtained in Step 6 for the unprimed member and for the corresponding primed member, and in Step 8 for the unprimed member.

The maximum live-load stress obtained by this method will be almost identical with that obtained with the true lane-loading length as deduced from influence broken lines. We shall prove by a random example that the difference is generally less than one percent which is well within engineering accuracy because neither modulus of elasticity, moments of inertia, most probable load estimation, nor allowable stresses are probably more accurate than within one percent.

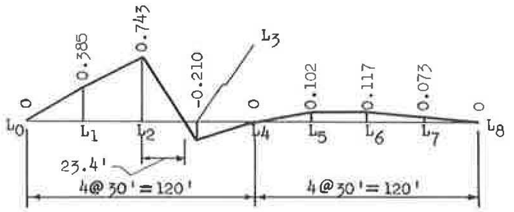


Figure 5.

Panel-Load Superposition Method vs Classical Influence-Line Method

A random proof of the closeness of the results obtained by these two methods is based on the influence line of the diagonal L_2U_3 of a two-span parallel-chord Warren-type continuous truss with verticals consisting of 4 panels at 30 ft in each span. This influence line was constructed to a close order of approximation by the three-moment theorem which can be applied to the case with expediency. An excerpt (7) of the correct influence line is shown in Figure 5.

For unit panel load, the corresponding fractional load over 23.4 ft of the stringer from L_2 to L_3 will be 23.4/30 = 0.780 and, therefore, the right stringer reaction = 0.78(23.4)/2(30) = 0.304 and the left stringer reaction = 0.780 - 0.304 = 0.476, making the panel load at L_2 = 0.476 + 0.500 = 0.976 instead of unity.

If the trusses were spaced at 38 ft c. to c., carrying two roadways of 26 ft each with an additional 4-ft divider, as in the Hawk Falls Bridge, applying the maximum lane live loading (converted from H20-S16-44) that may act on one truss, 1.26 kips/ft plus one concentration of 51.1 kips for shear, to this influence line will give the maximum tension in L_2U_3 as T_i and T_s, respectively, for the influence-line method and the panel-load superposition method. Thus,

$$\begin{bmatrix} T_i \\ T_s \end{bmatrix} = 1.26(30) \begin{bmatrix} 1 & 0.976 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.385 \\ 0.743 \\ 0.102 \\ 0.117 \\ 0.073 \end{bmatrix} + 51.1 \begin{bmatrix} 0.743 \\ 0.743 \end{bmatrix} = \begin{bmatrix} 90.97 \\ 91.65 \end{bmatrix} \text{kips}$$

which shows that the proposed panel-load superposition method results in a stress error of only (91.65 - 90.97)/90.97 = 0.00747, on the order of 7/10 of one percent. This is sufficiently accurate for all designing purposes. Similar verification may be shown for any member in any truss.

The panel-load superposition method is recommended for use in all analyses of indeterminate highway bridge trusses, especially when the maximum axial stresses are governed by lane loading plus concentrations.

Methods for Reducing the Number of Unknowns

Large-capacity computers, if available, can usually solve large systems of non-homogeneous algebraic linear equations. The number of unknowns is generally immaterial. A process of solution which is easier to formulate and program will prove more expedient. However, when a moderately large set of equations has to be solved with a small-size computer, the capacity may not be enough to handle the necessary numerical operations. In addition, larger rounding-off errors are as a rule associated with a larger number of unknowns. Working with the smallest possible number of unknowns has the advantages that the solution is more easily accommodated by most computers, and the rounding-off errors are kept to a minimum.

Method of Substitution. —This consists of substituting an unsymmetrical loading by symmetrical and antisymmetrical loadings. This method has been developed to reduce to only one-half the number of statically independent unknowns in symmetrical longer span continuous trusses under unsymmetrical loading.

Figure 6a shows arbitrarily a symmetrical three-span continuous bridge truss carrying an unsymmetrical load of 2P applied at a certain panel point (e.g., the first panel point), with redundant reactions R_1 and R_2 indicated at interior supports. By the prin-

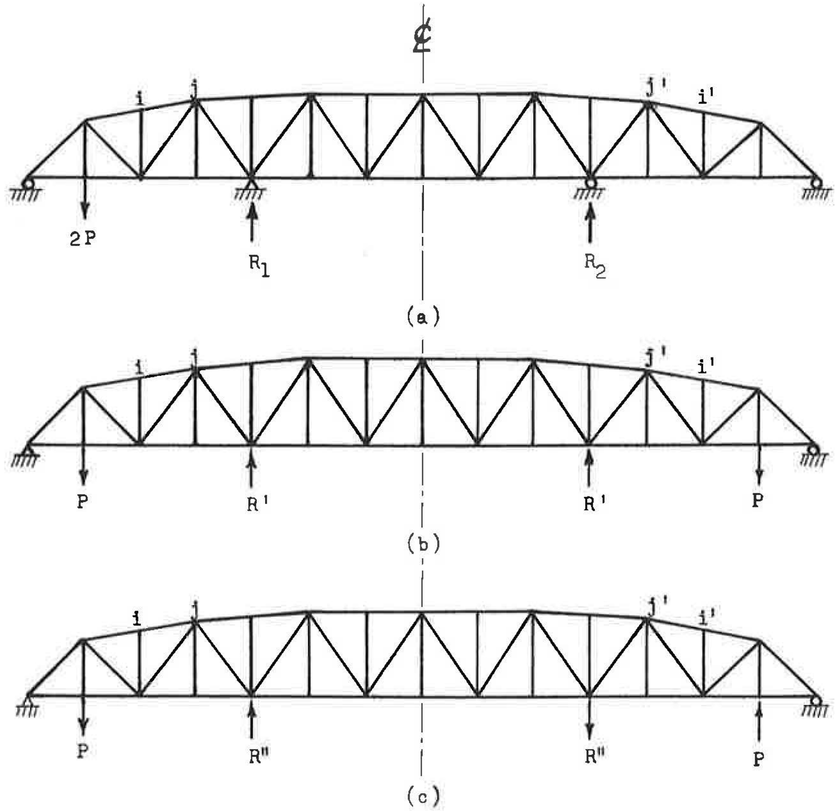


Figure 6.

ciple of superposition, R_1, R_2 and 2P, viewed as a loading system acting on the symmetrical bridge structure, can be represented as the sum of a set of symmetrical loading and a set of antisymmetrical loading acting separately on the same structure as indicated in Figures 6b and 6c. The magnitudes of R' and R'' are related to R_1 and R_2 by

$$\begin{bmatrix} R' + R'' \\ R' - R'' \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \tag{16}$$

Solving for R' and R'' yields

$$\begin{bmatrix} R' \\ R'' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} R_1 + R_2 \\ R_1 - R_2 \end{bmatrix} \tag{17}$$

In the case of symmetrical trusses, the number of statically independent unknowns in the matrix-energy method for symmetrical loading will be just equal to one-half of that for unsymmetrical loading. By using unprimed subscripts for joints on the left half of the structure and primed subscripts for those on the right half, we have

$$\begin{bmatrix} N_{ij} \\ M_{ij} \\ Q_{ij} \end{bmatrix} = \begin{bmatrix} N_{i'j'} \\ -M_{i'j'} \\ -Q_{i'j'} \end{bmatrix} \tag{18}$$

In even-panel symmetrical trusses with a center-vertical member, the internal bending moments and shearing stress in this member under symmetrical loading will be equal to zero. But the axial stress in the center-vertical member will not be equal to zero due to axial stresses and shears of inclined chord members meeting at the center joints, or due to shears alone if those chord members are parallel. In computing the strain energy in one-half of the truss, of course, only one-half the strain energy in the center vertical should be counted. The same applies if a center top chord exists instead of a center vertical.

In the same symmetrical truss under the substituted antisymmetrical loading, the number of statically independent unknowns will also be equal to one-half of that under unsymmetrical loading; that is, using the unprimed and primed subscripts as before, we have

$$\begin{bmatrix} N_{ij} \\ M_{ij} \\ Q_{ij} \end{bmatrix} = \begin{bmatrix} -N_{i'j'} \\ M_{i'j'} \\ Q_{i'j'} \end{bmatrix} \tag{19}$$

where the absolute values of axial stresses, bending moments, and shearing stresses are the same in corresponding unprimed and primed members. As the strain energy is a scalar quantity, it may again be computed for only one-half of the truss.

Thus, in symmetrical bridge trusses, any unsymmetrical loading may be substituted by a set of symmetrical loading and a conjugate set of antisymmetrical loading, whereby not only are the unknowns reduced to one-half of the original number but also one-half of the total strain energy of the truss will be needed in later formulation. The algebraic sum of the solutions under the substituted symmetrical and antisymmetrical loadings will give the desired solution under the original unsymmetrical loading.

Method of Transformation. —If, after reducing the number of unknowns by the method of substitution, the size of the system of equations is still greater than the capacity of the computer available, further reduction of unknowns can be effected by the method of transformation, i. e., transformation of unknown end moments into unknown reference tangential deflection angles. In the matrix-energy method for analyzing rigidly connected trusses, the recognized unknowns, except redundants, are the statically independent end resisting moments of members. There are three such unknowns in each triangular closed figure. But there are fewer joints than independent unknown end moments. At each joint there is only one unknown rotation, which in rigid frames is the principal advantage of the slope-deflection method.

In a truss, however, it is more expedient to choose Manderla's (1) approach which was the origin of the modern slope-deflection method. To adapt his approach to the problem under consideration, the number of unknowns may be reduced by applying the relationship

$$M_{ij} = \frac{2EI_{ij}}{L_{ij}} \left(2\tau_{ij} + \tau_{ji}\right) \tag{20}$$

between the unknown end moment and the tangential deflection angle τ, which is the angle between the tangent at the end of the deformed member and the straight line joining the ends of that deformed member.

There are as many τ's as twice the number of members in a truss. However, all the τ's around a joint can be expressed in terms of the angle changes ($\Delta\alpha$'s) between the straight lines joining the ends of the neighboring members which experience deformations under a given loading, and a selected reference τ. Thus, all end moments can be expressed in terms of reference τ's which are the new intermediate unknowns. As there is only one reference τ at each joint, the number of τ's will be exactly equal to the number of truss joints, which is far less than the number of unknown end moments in the set of simultaneous equations in the energy method.

Hence, when all end moments are substituted by their corresponding equivalent expressions represented by Eq. 20, the set of n simultaneous equations will result in n equations with j unknowns, where n is the number of original equations or statically independent unknowns in the energy method, j is the number of unknown reference τ's plus the number of redundants, and $n > j$.

As both the original n equations with n unknowns and the present n equations with j unknowns are all genuinely correct and exact equations, there is no need to normalize (8) the n equations into a new set of j equations for solving the j new unknowns. The n equations are to be distinguished from "conditional equations" of observation. Instead of normalizing, any j equations out of the n equations that contain the j unknowns will give identically correct solutions. With the j unknowns (reference τ's and any redundant reactions and/or axial stress) solved, back substitution into M-τ relations represented by Eq. 20 will give all end moments.

To utilize a given computer capacity, partial reduction of unknown end moments may also be permissible with the result of mixed unknowns consisting of all redundants, some reference τ's, and some end moments. A numerical demonstration of the method of transformation for reducing the number of unknowns is given in Appendix B, where identical results are obtained as by the simplified energy method.

SOLUTION OF SIMULTANEOUS ALGEBRAIC LINEAR EQUATIONS

Scores of direct and indirect (or iterative) methods have been developed for solving simultaneous algebraic linear equations. Proper choice of method to suit the problem, to adapt to the computer capacity, and to attain the desired accuracy and efficiency lies in the skill of the programmer.

The well-known direct methods include (a) determinants of matrices (slowest); (b) lower triangular matrices; (c) upper triangular matrices including unit upper triangular matrices; (d) post multipliers; (e) elimination; (f) row operation; (g) row operator with and without augmentation; (h) decomposition; (i) submatrices, escalator, or block decomposition; (j) symmetrical matrices; (k) Cayley-Hamilton theorem; (l) Gauss-Doolittle method and Crout method of LDU decomposition; (m) orthogonalization; (n) inverting modified matrices; and (o) Li's algorisms for mono- and polyset constant terms with and without inversion for both asymmetrical and symmetrical matrices (5, 9, 10, 11).

Among the indirect or iterative methods, the following may be cited from the geometrical approach: (a) Wittmeyer process, (b) special Wittmeyer processes, (c) Seidel, (d) back-and-forth Seidel process, (e) optimum or steepest gradients, (f) conjugate gradient, (g) relaxation, (h) hyperplane interpretation, and (i) residual vector. From the analytic approach are (a) Cesari's method, (b) method of von Mises and Geiringer, and (c) method of Hotelling and Bodewig. To these, must be added the Monte Carlo method, a nondeterministic or statistical method. No attempt is made to exhaust the list (5, 9, 10, 11).

To choose the best method for a given set of equations requires a clear comprehension of the underlying theory, the synthesis of the procedures, and the formulation of the algorism of the preceding methods. Whenever a large-capacity high-speed computer is available, because computing time is generally insignificant, the most easily programmed method (except the slowest) should be chosen. With low-speed small-capacity computers the fastest method requiring the least storage capacity should be used. When the capacity is too limited, a flexible method that can be adapted to the computer should be chosen, such as "Simultaneous Equations A La King" for the IBM 1620 or the relaxation method.

Large Sets of Equations and Computing Time

Two- or three-span continuous highway bridge trusses of moderate length may easily run into the inversion of a matrix of the order of more than 100 × 100. The largest set of simultaneous linear equations solved by a computer in the United States up to June 1963 consisted of a matrix of 700 × 700.

Using STRETCH at the highest speed yet built (500,000 multiplications per second) and Li's algorism for a symmetrical coefficient matrix of a set of 199 equations, the computing time will be about 4 sec. With STRETCH and using an already inverted symmetric matrix according to Li's algorisms, the time of solution of 199 equations for each set of constant terms will be further reduced to only 0.115 sec.

To solve the same n = 199 equations by the conventional determinental method would require $(n+1)!$ or approximately $789(10)^{372}$ multiplications, plus $(n + 1)(n)(n - 1) = n(n^2-1) = 7,880,400$ additions or 3,940,200 equivalent multiplications, taking each addition time as approximately equal to one-half of each multiplication time. Even using the fastest computer, STRETCH, it would still require $1,578(10)^{366}$ sec or slightly more than $5(10)^{361}$ yr. Before the advent of the electronic digital computer, the task would have been impossible.

Structural analyses generally involve symmetrical matrices. Using Li's algorism for symmetrical matrices with one set of constant terms, the solution of n unknowns requires an equivalent number of multiplications on the order of about $\frac{n^3}{2}$.

Simultaneous Equations A La King

Usually, computers available to bridge engineers are of moderate or smaller size than STRETCH and have limited storage capacity. For instance, in the use of the basic IBM 1620 computer, having only a storage capacity of 20,000, the solution of a moderately large set of linear equations will need special programming. "Simultaneous Equations A La King," developed by D. N. Leeson and designated Program Number 5.0.008 in the 1620 General Program Library, can solve a set of 58 linear equations requiring a core storage of 55,510 and 58,937, respectively, for the recommended mantissa length of 12 or a longer length of 13 (the latter for more accuracy). To facilitate the use of this program, the Source Program Deck (Cards) and the SPS II Processor Deck (or Assembly Deck) should be prepared or secured in advance.

Relaxation Method

This method of successive approximations has the inherent advantage of easy programming, may be broken into as many segments as any small computer can hold—especially if the coefficient matrix is band-like—and can attain any desired accuracy. The mathematical technique and the physical facility of the computer can be used in an infinite variety of ways to accelerate the convergence of the process of solution. Besides use as a mathematical tool for solving simultaneous linear equations, the method may be directly applied to stress calculation in frameworks (12).

A system ready for relaxation with the main diagonal elements equal to -1 and the constant K vector shifted to the left of the equations may be written:

$$\begin{bmatrix} -1 & a_{12} & \cdots & a_{1n} \\ a_{21} & -1 & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ \cdot \\ k_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad (21)$$

The difference of the left-hand side of the ith equation may be denoted by r_i (residuals) for any reasonably assumed set of starting values $x_j^{(o)}$; thus,

$$\begin{bmatrix} -1 & a_{12} & \cdots & a_{1n} \\ a_{21} & -1 & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & -1 \end{bmatrix} \begin{bmatrix} x_1^{(o)} \\ x_2^{(o)} \\ \cdot \\ \cdot \\ \cdot \\ x_n^{(o)} \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ \cdot \\ k_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ r_n \end{bmatrix} \tag{22}$$

The relaxation procedure simply consists of altering the starting values of $x_j^{(o)}$, one or more at a time, until all the r_i become zero or negligibly small for the desired accuracy. If a given $x_j^{(o)}$, e.g., $x_k^{(o)}$, is altered by an amount Δx_k, then r_k alters by $-\Delta x_k$ and the other r_i alter by $a_{ik}\,\Delta x_k$. Consequently, to reduce a given r_i, e.g., r_k, to zero, we alter $x_k^{(o)}$ by $x_k = r_k$. Simultaneously, the other r_i will also alter and must be reduced to zero, one by one, by suitable alterations, Δx_j. It is expedient to eliminate the largest residual appearing in the system at any stage in the process.

The entire procedure may be most conveniently carried out in tabular form by entering the starting value of each $x_j^{(o)}$ and its successive alterations Δx_j in a left column (or computer locations), and the residuals in the right column (on another location of the computer). Thus, the relaxation table has two columns (or sequential locations) for each x_j. As soon as the residuals have gradually vanished to the degree of accuracy desired, the sum of $x_j^{(o)}$ and of all the Δx_j gives the final value of x_j. With this outline of the procedure, the relaxation method may be programmed for any particular version of the computer.

CONTINUOUS TRUSS, LOADINGS AND UNKNOWNS

Continuous Highway Bridge Truss and Loadings

Applicability of the proposed matrix-energy method to an indeterminate highway structure is demonstrated using as an example the three-span continuous bridge over the Missouri River near Wolf Creek, Montana, on Federal Aid Project 172D Unit 2. The bridge is an economical structure and has a pleasing appearance, mainly due to its excellent proportions and simplicity in details. The main reason for choosing this bridge truss to exemplify the indeterminate analysis lies in its having only two redundant reactions symmetrical in arrangement, a moderate number of members and, hence, comparatively few unknowns, which can be handled by the smaller digital computers possessed by most engineering organizations.

As shown in Figure 7, the skeleton truss of the bridge is of the Warren type with verticals and slightly inclined upper chords. It has spans of 135 ft: 180 ft: 135 ft, carrying a roadway of 20 ft. It was designed in 1932 for Standard H15 loading according to Montana State Highway Commission Standard Design Specifications for Highway Structures as revised in February 1932, which are the same as the AASHO Standard Specifications for Highway Bridges and Incidental Structures, 1931.

Makeup of Members. —Section components of each member, its gross sectional area, gross moment of inertia, theoretical length, and section modulus are as given in Table 7 for use in later computations. Centroidal axes are taken as intersecting at theoretical panel points, thus eliminating eccentric moments.

Dead Panel Loads. —These have been duly distributed to lower and upper panel points as given in Table 8.

Live Loads and Impact Formula. —The design live load for the bridge as used in 1932 was Standard H15 loading. The equivalent loading by which the design was then governed, as it is today, consists of a uniform load of 480 lb/lin ft of loaded lane plus a concentrated load of 13,500 lb for moment or of 19,500 lb for shear. That is, the

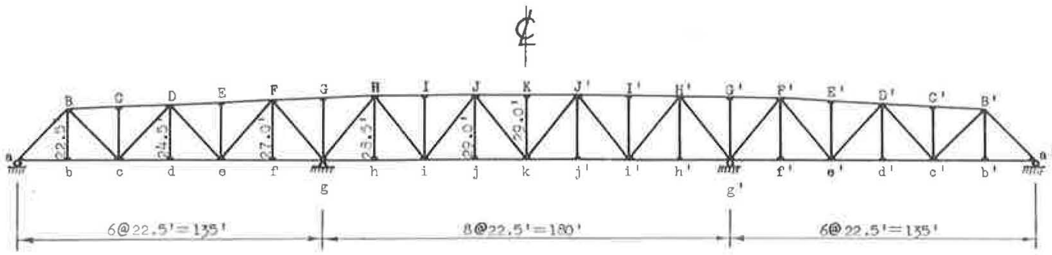


Figure 7.

TABLE 7

MAKEUP OF MEMBERS AND SECTION PROPERTIES

Member	Section	A(in.^{2})	I(in.^{4})	L(in.)	$\frac{I}{c}$(in.^{3})
ab to jk	2 - 12[20.7	12.06	256.2	270	42.8
BC CD DE EF FG GH HI IJ JK	2 - 12[20.7 1 - Pl 15 × $\frac{5}{16}$	16.75	384.1	270.26654 270.41635 270.14996 270.01667 270	83.7[a] 49.7[b]
Bb Dd Ff Hh Jj	$8\frac{1}{16}$ CB 31	9.12	37.0	270 294 324 342 348	9.2
Cc	4 - Ls 4 × 3 × $\frac{5}{16}$	8.36	30.3	282	7.3
Ee Gg Ii Kk	4 - Ls 5 × 3 × $\frac{5}{16}$	9.60	57.4	309 333 345 348	11.1
aB	2 - 12[20.7 1 - Pl 15 × $\frac{5}{16}$	16.75	384.1	381.83766	83.7[a] 49.7[b]
Bc cD	2 - 9 [15	8.78	101.4	381.83766 399.16914	22.6
De	2 - 12[20.7	12.06	256.2	399.16914	42.8
eF	2 - 12[25	14.64	287.0	421.75349	47.8
Fg	2 - 12[30 1 - Pl 16 × $\frac{3}{8}$	23.58	493.7	421.75349	102.8[a] 65.2[b]
gH	2 - 12[35 1 - Pl 16 × $\frac{3}{8}$	26.44	618.4	435.73387	124.4[a] 83.5[b]
Hi iJ	2 - 12[25	14.64	287.0 287	435.73387 440.45886	47.8
Jk	2 - 12[20.7	12.06	256.2	440.45886	42.8

[a] Top. [b] Bottom.

TABLE 8

DEAD PANEL LOADS

Lower Panel Pt.	D. L. (kips)	Upper Panel Pt.	D. L. (kips)
a	19.93	—	—
b	33.11	B	5.22
c	33.62	C	4.25
d	33.59	D	4.42
e	34.17	E	4.28
f	33.22	F	6.03
g	38.69	G	3.21
h	33.96	H	6.58
i	34.95	I	4.71
j	33.27	J	4.98
k	35.04	K	4.50

lighter concentrated load is used in computing the stresses in members in which the greater part of the stress is produced by bending moments; the heavier concentrated load is used when the greater part of the stress in a member is produced by shearing forces or when it is to be in equilibrium with that in a member such as at the end joint. There seems no stipulation, at the time when the bridge was designed, that two concentrations be placed in adjacent spans for the maximum stresses of chord members near the intermediate supports.

To conform further with the provisions for obtaining the greater maximum stress in a member at the time when the bridge was designed, the roadway is considered loaded over its entire width of 20 ft with both uniform and concentrated loads per foot of width equal to one-ninth (the lane width then being 9 ft) of the load of one traffic lane; but the load intensity is reduced by 20 (roadway in feet) - 18 (two-lane width in feet) = 2 percent. As the result of this method of applying live loads, the lane loading will be increased by a factor of $\frac{10}{9}(1 - 0.02) = 1.08889$.

For the bridge under consideration with a typical panel length of 22.5 ft, the typical live panel load is P = 480(22.5) = 10,800 lb, or 10.8 kips. The transversely modified live panel load P_m for producing maximum stresses is, therefore, $P_m = 10.8(1.08889) = 11.76$ kips.

The dynamic vibratory and impact effects will be accounted for as a fraction of the live-load stress by the formula:

$$I = \frac{50}{L + 125} \tag{23}$$

in which I is impact fraction and L is length, in feet, of the portion (or portions) of the span (or spans) which is loaded to produce the maximum stress in the member considered. There was no 30 percent impact ceiling when the bridge was designed.

By tracing the most possible former loading conditions as summarized previously one insures the closest check of maximum axial stresses determined by the proposed method with those obtained by the Montana State Highway Commission when the bridge was designed. This check will testify to the validity of the proposed method.

Reduction of Statically Independent and Dependent Unknowns

Statically Independent Unknowns. —There are 38 closed triangular figures (f) and two redundant supports (r) in the truss. If the structure is viewed as a chain of rigidframes, it is statically indeterminate to the $(3f + r) = 3(38) + 2 = 116$th degree under unsymmetrical loading. By using the method of substitution for an unsymmetrical loading by a set of symmetrical loading and a set of antisymmetrical loading, the number of statically independent unknowns are reduced to only one-half of 116, or 58.

With the letter designations for joints and numeral designations for independent unknown-end-moment subscripts as indicated in Figure 8, the 58 statically independent unknowns under any unsymmetrical loading are as defined in the following:

1. For the set of symmetrical loading,

$$\begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_{57} \\ X_{58} \end{bmatrix}_s = \begin{bmatrix} M_{ab} \\ M_{Ba} \\ \cdot \\ \cdot \\ \cdot \\ M_{kJ} \\ R_g \end{bmatrix}_s = - \begin{bmatrix} M_{a'b'} \\ M_{B'a'} \\ \cdot \\ \cdot \\ \cdot \\ M_{k'J'} \\ -R_{g'} \end{bmatrix}_s \tag{24}$$

2. For the set of antisymmetrical loading,

$$\begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_{57} \\ X_{58} \end{bmatrix}_a = \begin{bmatrix} M_{ab} \\ M_{Ba} \\ \cdot \\ \cdot \\ \cdot \\ M_{kJ} \\ R_g \end{bmatrix}_a = \begin{bmatrix} M_{a'b'} \\ M_{B'a'} \\ \cdot \\ \cdot \\ \cdot \\ M_{k'J'} \\ -R_{g'} \end{bmatrix}_a \tag{25}$$

Statically Dependent Unknowns. —The dependent unknown end moments are given by joint equilibrium as follows:

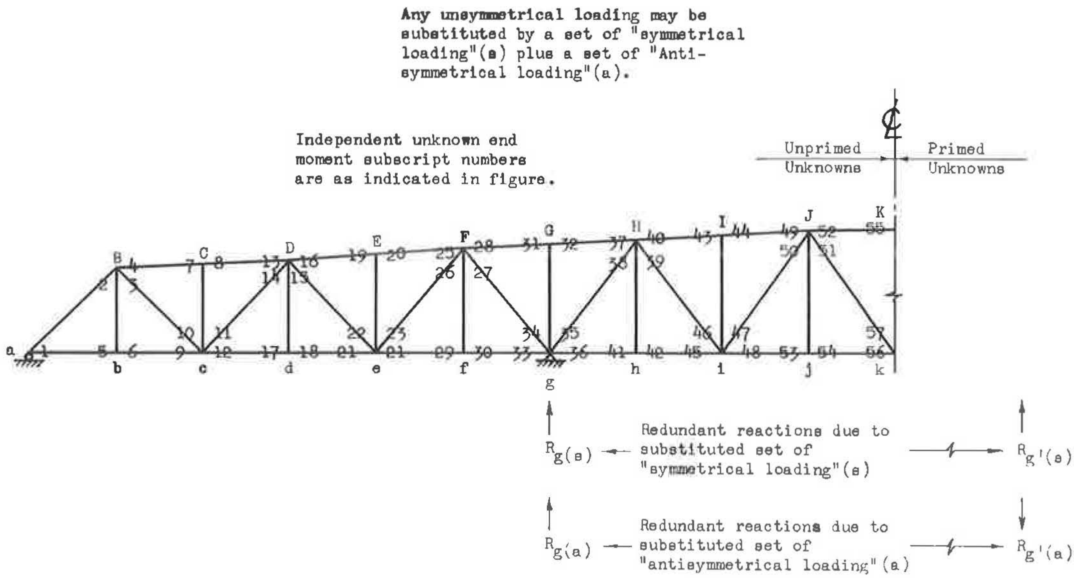


Figure 8.

1. For the set of symmetrical loading,

$$\begin{bmatrix} M_{aB} \\ M_{Bb} \\ M_{bB} \\ \cdot \\ \cdot \\ \cdot \\ M_{jJ} \\ M_{Kk} \\ M_{kK} \end{bmatrix}_s = - \begin{bmatrix} M_{a'B'} \\ M_{B'b'} \\ M_{b'B'} \\ \cdot \\ \cdot \\ \cdot \\ M_{j'J'} \\ M_{K'k'} \\ M_{k'K'} \end{bmatrix}_s = - \begin{bmatrix} X_1 \\ X_2 + X_3 + X_4 \\ X_5 + X_6 \\ \cdot \\ \cdot \\ \cdot \\ X_{53} + X_{54} \\ 0 \\ 0 \end{bmatrix}_s \tag{26}$$

2. For the set of antisymmetrical loading,

$$\begin{bmatrix} M_{aB} \\ M_{Bb} \\ M_{bB} \\ \cdot \\ \cdot \\ \cdot \\ M_{jJ} \\ M_{Kk} \\ M_{kK} \end{bmatrix}_a = \begin{bmatrix} M_{a'B'} \\ M_{B'b'} \\ M_{b'B'} \\ \cdot \\ \cdot \\ \cdot \\ M_{j'J'} \\ M_{K'k'} \\ M_{k'K'} \end{bmatrix}_a = - \begin{bmatrix} X_1 \\ X_2 + X_3 + X_4 \\ X_5 + X_6 \\ \cdot \\ \cdot \\ \cdot \\ X_{53} + X_{54} \\ 2X_{55} \\ 2(X_{56} + X_{57}) \end{bmatrix}_a \tag{27}$$

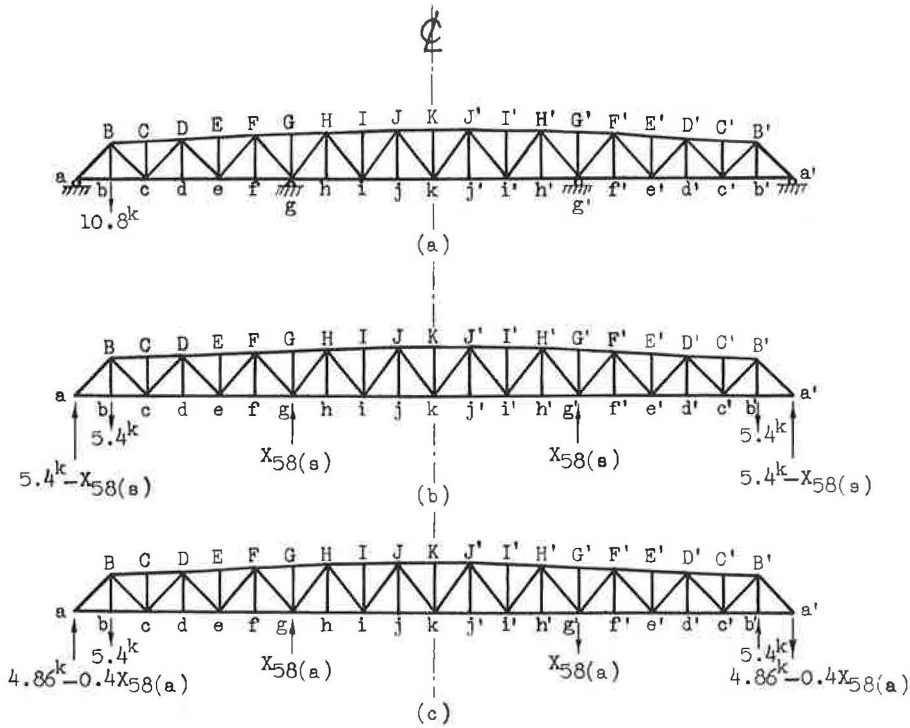


Figure 9.

Application of Method of Substitution

When the computer available is of small capacity, to analyze the truss under unsymmetrical loading with the reduced number of 58 statically independent unknowns, solutions of stresses in truss members for the substituted sets of symmetrical and antisymmetrical loadings must first be carried out. The solutions for the original unsymmetrical case may then be obtained at once by the principle of superposition.

As shown in Figure 9a, any unsymmetrical typical live panel load of 10. 8 kips applied, e. g., at b may be substituted by two sets of loadings shown in Figures 9b and 9c. The reactions in Figure 9b are obvious; R_a in Figure 9c follows directly from the equation of couple equilibrium:

$$20R_a + 8X_{58} = 18(5.4)$$

If a sufficiently large computer is at hand for solving all the unknowns under any unsymmetrical loading, the process will be faster without resort to this method of substitution.

FORMULATION OF AXIAL-STRESS EXPRESSIONS AND CHARACTERISTIC SIMPLIFICATIONS

The basic techniques for formulating axial-stress expressions have been presented under "Extended Methods of Moments, Shears, and Joints." As a general rule, the axial stress in each of the top and bottom chord members and each of the diagonal members in the panels having inclined upper chords may be determined by the extended method of moments; that of the end posts and of the diagonal members in panels having parallel chords, by the extended method of shears; and that of the verticals, by the method of joints.

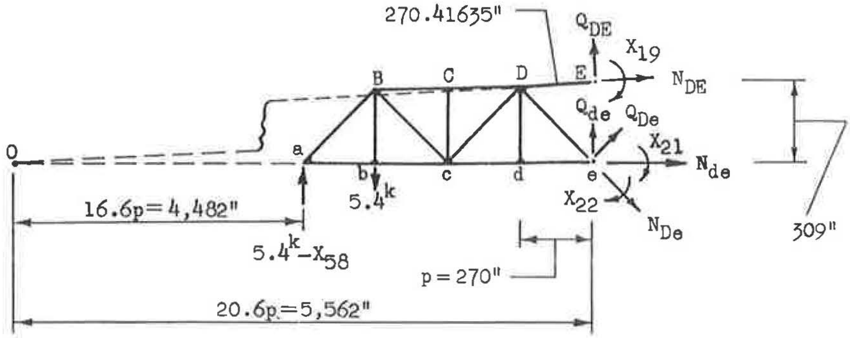


Figure 10.

As typical examples of application of this general rule, we may formulate the axial-stress expressions for members De, DE, aB, Jk, Bb, and Cc in the truss shown in Figure 9b under the substituted set of symmetrical half-panel loads.

Diagonal De and Upper Chord DE

These two axial stresses may be determined most expediently by the extended method of moments. Taking the free body diagram just to the left of member Ee, as shown in Figure 10, and noting that

$$\left\{ Q_{de} \quad Q_{De} \quad Q_{DE} \right\} = \left\{ \frac{X_{18} + X_{21}}{p} \quad \frac{X_{15} + X_{22}}{De} \quad \frac{X_{16} + X_{19}}{DE} \right\} \tag{28}$$

we have, by $\Sigma M_O = 0$,

$$\lfloor \text{Row vector of stresses} \rfloor \ \{\text{Column vector of lever arms}\} = 0$$

or

$$\lfloor N_{De} \ \ -Q_{De} \ \ -Q_{de} \ \ -Q_{DE} \ \ (X_{19} + X_{21} + X_{22}) \ \ 5.4 \ \ -(5.4 - X_{58}) \rfloor$$

$$\left\{ \frac{Dd(oe)}{De} \quad \frac{de(oe)}{De} \quad oe \quad 20.6(DE) \quad 1 \quad 17.6p \quad oa \right\} = 0 \tag{29}$$

Substituting Q's from Eq. 28 and all known distances into Eq. 29, and transposing, yields:

$$N_{De} = \frac{\lfloor X_{15} \ \ (X_{16} + X_{18}) \ \ (X_{19} + X_{21}) \ \ X_{22} \rfloor}{(10)^{10}} \begin{bmatrix} 23006973 \\ 50285858 \\ 47844797 \\ 20565912 \end{bmatrix} - \lfloor X_{58} \ \ 1 \rfloor \begin{bmatrix} 1.0940836 \\ 0.35590670 \end{bmatrix} \tag{30}$$

where the last product of two vectors corresponds to the conventional primary stress in member De.

Taking the same free body diagram shown in Figure 10, by $\Sigma M_e = 0$, we have again the product of a row vector and a column vector equal to zero,

$$\lfloor N_{DE} \quad -Q_{DE} \quad (X_{19} + X_{21} + X_{22}) \quad -5.4 \quad (5.4 - X_{58}) \rfloor$$

$$\left\{ Ee \quad \frac{p}{DE} \quad \frac{Ee(Ee - Dd)}{DE} \quad 1 \quad 3p \quad 4p \right\} = 0 \qquad (31)$$

Substituting Q's from Eq. 28 and all known distances into Eq. 31 and transposing, yields:

$$N_{DE} = \frac{\lfloor X_{18} \quad -X_{19} \quad -(X_{21} + X_{22}) \rfloor}{(10)^{10}} \begin{bmatrix} 2054445.1 \\ 30357919 \\ 32412364 \end{bmatrix} + \lfloor X_{58} \quad -1 \rfloor \begin{bmatrix} 3.5005353 \\ 4.7257226 \end{bmatrix} \qquad (32)$$

where the last product of two vectors also corresponds to the conventional primary stress in member DE.

End Post aB and Center Diagonal Jk

These two axial stresses may most expediently be determined by the extended method of shears. Taking the free body diagram just to the left of member Bb (Fig. 11), and noting that

$$\left\{ Q_{ab} \quad Q_{aB} \right\} = \left\{ \frac{X_1 + X_5}{p} \quad \frac{-X_1 + X_2}{aB} \right\} \qquad (33)$$

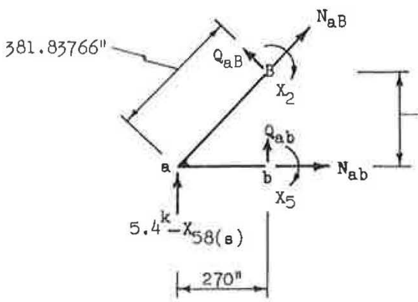


Figure 11.

we have, by $\Sigma Y = 0$,

$$N_{aB}\frac{1}{\sqrt{2}} + \frac{Q_{aB}}{\sqrt{2}} + Q_{ab} + 5.4 - X_{58} = 0 \qquad (34)$$

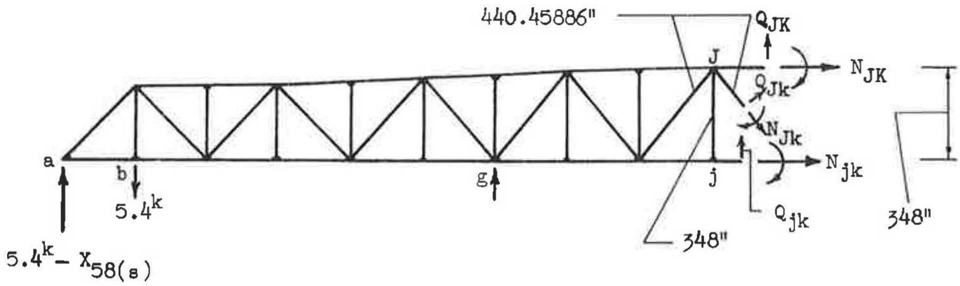


Figure 12.

Substituting Q's from Eq. 33 and known distances into Eq. 34, and transposing, yields:

$$N_{aB} = -\frac{\lfloor (X_1 + X_2) \quad X_5 \rfloor}{(10)^{10}} \begin{bmatrix} 26189140 \\ 52378280 \end{bmatrix} + \lfloor X_{58} \quad -1 \rfloor \begin{bmatrix} 1.4142136 \\ 7.6367532 \end{bmatrix} \tag{35}$$

in which the last product of two vectors is again the conventional primary stress in member aB.

To determine N_{Jk}, taking the free body diagram to the left of panel jk, as shown in Figure 12, and noting that

$$\left\{ Q_{jk} \quad Q_{Jk} \quad Q_{JK} \right\} = \left\{ \frac{X_{54} + X_{56}}{p} \quad \frac{X_{51} + X_{57}}{Jk} \quad \frac{X_{52} + X_{55}}{p} \right\} \tag{36}$$

we have, by $\Sigma Y = 0$,

$$-N_{Jk} \frac{Jj}{Jk} + Q_{Jk} \frac{p}{Jk} + Q_{JK} + Q_{jk} + X_{58} - 5.4 + (5.4 - X_{58}) = 0 \tag{37}$$

Substituting Q's from Eq. 36 and all known distances into Eq. 37, and transposing, yields:

$$N_{Jk} = \frac{\lfloor X_{51} \quad (X_{52} \; X_{54} \; X_{55} \; X_{56}) \quad X_{57} \rfloor}{(10)^{10}} \begin{bmatrix} 17614859 \\ 46877273 \\ 17614859 \end{bmatrix} + 0 \tag{38}$$

where the last term means the conventional primary stress in member Jk is zero.

Vertical Members Bb and Cc

The extended method of joints may most advantageously be applied to determine the axial stresses in vertical members Bb and Cc. Taking the free body diagram around joint b (Fig. 13a), and noting that

$$\left\{ Q_{ab} \quad Q_{bc} \right\} = \left\{ \frac{X_1 + X_5}{p} \quad \frac{X_6 + X_9}{p} \right\} \tag{39}$$

we have, by $\Sigma Y = 0$,

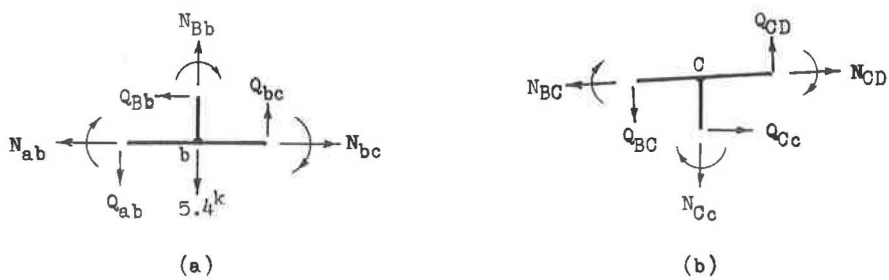


Figure 13.

$$N_{Bb} - Q_{ab} - 5.4 + Q_{bc} = 0 \tag{40}$$

Substituting Q's from Eq. 39 and the known panel length into Eq. 40, and transposing, yields:

$$N_{Bb} = \frac{37037037}{(10)^{10}} (X_1 + X_5 - X_6 - X_9) + 5.4 \tag{41}$$

where the last term represents the conventional primary stress in member Bb.

In the free body diagram shown in Figure 13b, noting that

$$\begin{bmatrix} Q_{Cc} \\ \\ Q_{CD} \\ \\ Q_{BC} \end{bmatrix} = \begin{bmatrix} -\dfrac{\sum_{i=7}^{12} X_i}{Cc} \\ \\ \dfrac{X_8 + X_{13}}{CD} \\ \\ \dfrac{X_4 + X_7}{BC} \end{bmatrix} \tag{42}$$

we have, by summing up the stress components along the axis perpendicular to member BC (or CD),

$$-N_{Cc} \left(\frac{p}{BC}\right) + Q_{Cc} \left(\frac{12}{BC}\right) + Q_{CD} - Q_{BC} = 0 \tag{43}$$

Substituting Q's from Eqs. 42 and all known distances into Eq. 43, and transposing, yields:

$$N_{Cc} = \frac{\begin{bmatrix} -X_4 & -X_7 & X_8 & \sum_{i=9}^{12} X_i & X_{13} \end{bmatrix}}{(10)^{10}} \begin{bmatrix} 37037037 \\ 35460993 \\ 38613081 \\ 15760441 \\ 37037037 \end{bmatrix} + 0 \tag{44}$$

where the last term denotes that the conventional primary stress in member Cc is zero.

Summary of Axial-Stress Expressions

The axial stresses in all other members of the left half of the truss shown in Figure 9b under typical symmetrical half live panel loads at b and b′, have been similarly determined. They are summarized in Eqs. 45:

Least Significant Part of Axial Stress — Most Significant Part of Axial Stress

Lower Chords

$$\begin{bmatrix} N_{ab} \\ N_{bc} \end{bmatrix} = \frac{37037037}{(10)^{10}} \begin{bmatrix} X_2 + X_5 \\ -(X_3 + X_4 + X_6) \end{bmatrix} + \begin{bmatrix} -X_{58} & 1 \\ -X_{58} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5.4 \end{bmatrix}$$

$$\begin{bmatrix} N_{cd} \\ N_{de} \end{bmatrix} = \frac{34013605}{(10)^{10}} \begin{bmatrix} X_{13} + X_{14} + X_{17} \\ -(X_{15} + X_{16} + X_{18}) \end{bmatrix} + \begin{bmatrix} -X_{58} & 1 \\ -X_{58} & 1 \end{bmatrix} \begin{bmatrix} 2.7551020 \\ 4.9591837 \end{bmatrix}$$

$$\begin{bmatrix} N_{ef} \\ N_{fg} \end{bmatrix} = \frac{30864198}{(10)^{10}} \begin{bmatrix} X_{25} + X_{26} + X_{29} \\ -(X_{27} + X_{28} + X_{30}) \end{bmatrix} + \begin{bmatrix} -X_{58} & 1 \\ -X_{58} & 1 \end{bmatrix} \begin{bmatrix} 4.1666667 \\ 4.5000000 \end{bmatrix}$$

$$\begin{bmatrix} N_{gh} \\ N_{hi} \end{bmatrix} = \frac{29239766}{(10)^{10}} \begin{bmatrix} X_{37} + X_{38} + X_{41} \\ -(X_{39} + X_{40} + X_{42}) \end{bmatrix} + \begin{bmatrix} -X_{58} & 1 \\ -X_{58} & 1 \end{bmatrix} \begin{bmatrix} 4.7368422 \\ 4.2631579 \end{bmatrix}$$

$$\begin{bmatrix} N_{ij} \\ N_{jk} \end{bmatrix} = \frac{28735632}{(10)^{10}} \begin{bmatrix} X_{49} + X_{50} + X_{53} \\ -(X_{51} + X_{52} + X_{54}) \end{bmatrix} + \begin{bmatrix} -X_{58} & 1 \\ -X_{58} & 1 \end{bmatrix} \begin{bmatrix} 4.6551724 \\ 4.1896552 \end{bmatrix}$$

Upper Chords

$$N_{BC} = \frac{\lfloor X_4 \quad -X_7 \quad -(X_9 + X_{10}) \rfloor}{(10)^{10}} \begin{bmatrix} 1644467.1 \\ 33851532 \\ 35495999 \end{bmatrix} + \lfloor X_{58} \quad -1 \rfloor \begin{bmatrix} 1.9167840 \\ 5.1753167 \end{bmatrix}$$

$$N_{CD} = \frac{\lfloor X_8 \quad (X_{11} + X_{12}) \quad X_{13} \rfloor}{(10)^{10}} \begin{bmatrix} 37140466 \\ 35495999 \\ 1644467.1 \end{bmatrix} + \lfloor X_{58} \quad -1 \rfloor \begin{bmatrix} 1.9167840 \\ 5.1753167 \end{bmatrix}$$

$$N_{DE} = \frac{\lfloor X_{16} \quad -X_{19} \quad -(X_{21} + X_{22}) \rfloor}{(10)^{10}} \begin{bmatrix} 2054445.1 \\ 30357919 \\ 32412364 \end{bmatrix} + \lfloor X_{58} \quad -1 \rfloor \begin{bmatrix} 3.5005352 \\ 4.7257226 \end{bmatrix}$$

$$N_{EF} = \frac{\lfloor X_{20} \quad (X_{23} + X_{24}) \quad X_{25} \rfloor}{(10)^{10}} \begin{bmatrix} 34466809 \\ 32412364 \\ 2054445.1 \end{bmatrix} + \lfloor X_{58} \quad -1 \rfloor \begin{bmatrix} 3.5005352 \\ 4.7257226 \end{bmatrix}$$

$$N_{FG} = \frac{\lfloor X_{28} \quad -X_{31} \quad -(X_{33} + X_{34}) \rfloor}{(10)^{10}} \begin{bmatrix} 1233882.6 \\ 28812826 \\ 30046709 \end{bmatrix} + \lfloor X_{58} \quad -1 \rfloor \begin{bmatrix} 4.8675668 \\ 4.3808102 \end{bmatrix}$$

Least Significant Part of Axial Stress	Most Significant Part of Axial Stress

Upper Chords (Cont'd)

$$N_{GH} = \frac{\lfloor X_{32} \quad (X_{35} + X_{36}) \quad X_{37} \rfloor}{(10)^{10}} \begin{bmatrix} 31280592 \\ 30046709 \\ 1233882.6 \end{bmatrix} + \lfloor X_{58} \quad -1 \rfloor \begin{bmatrix} 4.8675668 \\ 4.3808102 \end{bmatrix}$$

$$N_{HI} = \frac{\lfloor X_{40} \quad -X_{43} \quad (X_{45} + X_{46}) \rfloor}{(10)^{10}} \begin{bmatrix} 411497.23 \\ 28575800 \\ 28987297 \end{bmatrix} + \lfloor X_{58} \quad -1 \rfloor \begin{bmatrix} 4.6959420 \\ 4.2263451 \end{bmatrix}$$

$$N_{IJ} = \frac{\lfloor X_{44} \quad (X_{47} + X_{48}) \quad X_{49} \rfloor}{(10)^{10}} \begin{bmatrix} 29398794 \\ 28987297 \\ 411497.23 \end{bmatrix} + \lfloor X_{58} \quad -1 \rfloor \begin{bmatrix} 4.6959420 \\ 4.2263451 \end{bmatrix}$$

$$N_{JK} = -0.0028735632(X_{55} + X_{56} + X_{57}) + \lfloor X_{58} \quad -1 \rfloor \begin{bmatrix} 4.6551724 \\ 4.1896552 \end{bmatrix}$$

Verticals

$$N_{Bb} = 0.0037037037(X_1 + X_5 - X_6 - X_9) + 5.4000000$$

$$N_{Cc} = \frac{\left\lfloor X_4 \quad -X_7 \quad X_8 \quad \sum_{i=9}^{12} X_i \quad X_{13} \right\rfloor}{(10)^{10}} \begin{bmatrix} 37037037 \\ 35460993 \\ 38613081 \\ 1576044.1 \\ 37037037 \end{bmatrix} + 0$$

$$N_{Dd} = 0.0037037037(X_{12} + X_{17} - X_{18} - X_{21}) + 0$$

$$N_{Ee} = \frac{\left\lfloor X_{16} \quad -X_{19} \quad X_{20} \quad \sum_{i=21}^{24} X_i \quad X_{25} \right\rfloor}{(10)^{10}} \begin{bmatrix} 37037037 \\ 35239123 \\ 38834951 \\ 1797914.4 \\ 37037037 \end{bmatrix} + 0$$

$$N_{Ff} = 0.0037037037(X_{24} + X_{29} - X_{30} - X_{33}) + 0$$

$$N_{Gg} = \frac{\left\lfloor -X_{28} \quad -X_{31} \quad X_{32} \quad \sum_{i=33}^{36} X_i \quad X_{37} \right\rfloor}{(10)^{10}} \begin{bmatrix} 37037037 \\ 36036036 \\ 38038038 \\ 1001001.0 \\ 37037037 \end{bmatrix} + 0$$

$$N_{Hh} = 0.0037037037(X_{36} + X_{41} - X_{42} - X_{45}) + 0$$

Least Significant Part of Axial Stress	Most Significant Part of Axial Stress

Verticals (Cont'd)

$$N_{Ii} = \frac{\left\lfloor -X_{40} \quad -X_{43} \quad X_{44} \quad \sum_{i=45}^{48} X_i \quad X_{49} \right\rfloor}{(10)^{10}} \begin{bmatrix} 37037037 \\ 36714976 \\ 37359098 \\ 322061.19 \\ 37037037 \end{bmatrix} + 0$$

$$N_{Jj} = 0.0037037037(X_{48} + X_{53} - X_{54} - X_{56}) \qquad + 0$$

$$N_{Kk} = 0.0074074074(X_{52} + X_{55}) \qquad + 0$$

Diagonals

$$N_{aB} = \frac{\left\lfloor -(X_1 + X_2) \quad -X_5 \right\rfloor}{(10)^{10}} \begin{bmatrix} 26189140 \\ 52378280 \end{bmatrix} + \left\lfloor X_{58} \quad -1 \right\rfloor \begin{bmatrix} 1.4142135 \\ 7.6367532 \end{bmatrix}$$

$$N_{Bc} = \frac{\left\lfloor X_3 \quad (X_4 + X_6) \quad (X_7 + X_9) \quad X_{10} \right\rfloor}{(10)^{10}} \begin{bmatrix} 26189140 \\ 52378280 \\ 50149417 \\ 23960277 \end{bmatrix} - \left\lfloor X_{58} \quad 1 \right\rfloor \begin{bmatrix} 1.2938550 \\ 0.32496822 \end{bmatrix}$$

$$N_{cD} = \frac{-\left\lfloor X_8 \quad X_{11} \quad X_{12} \quad X_{13} \quad X_{14} \quad X_{17} \right\rfloor}{(10)^{10}} \begin{bmatrix} 52425682 \\ 25146797 \\ 52425682 \\ 50285858 \\ 23006973 \\ 50285858 \end{bmatrix} + \left\lfloor X_{58} \quad 1 \right\rfloor \begin{bmatrix} 1.2421677 \\ 0.31198630 \end{bmatrix}$$

$$N_{De} = \frac{\left\lfloor X_{15} \quad (X_{16} + X_{18}) \quad (X_{19} + X_{21}) \quad X_{22} \right\rfloor}{(10)^{10}} \begin{bmatrix} 23006973 \\ 50285858 \\ 47844797 \\ 20565912 \end{bmatrix} - \left\lfloor X_{58} \quad 1 \right\rfloor \begin{bmatrix} 1.0940836 \\ 0.35590670 \end{bmatrix}$$

$$N_{eF} = \frac{-\left\lfloor X_{20} \quad X_{23} \quad X_{24} \quad X_{25} \quad X_{26} \quad X_{29} \right\rfloor}{(10)^{10}} \begin{bmatrix} 50551779 \\ 22099137 \\ 50551779 \\ 48211419 \\ 19758777 \\ 48211419 \end{bmatrix} + \left\lfloor X_{58} \quad 1 \right\rfloor \begin{bmatrix} 1.0489494 \\ 0.34122451 \end{bmatrix}$$

$$N_{Fg} = \frac{\left\lfloor X_{27} \quad (X_{28} + X_{30}) \quad (X_{31} + X_{33}) \quad X_{34} \right\rfloor}{(10)^{10}} \begin{bmatrix} 19758777 \\ 48211419 \\ 46908408 \\ 18455766 \end{bmatrix} - \left\lfloor X_{58} \quad 1 \right\rfloor \begin{bmatrix} 1.0906205 \\ 0.18997905 \end{bmatrix}$$

Least Significant Part of Axial Stress | Most Significant Part of Axial Stress

Diagonals (Cont'd)

$$N_{gH} = \frac{-\lfloor X_{32} \;\; X_{35} \;\; X_{36} \;\; X_{37} \;\; X_{38} \;\; X_{41} \rfloor}{(10)^{10}} \begin{bmatrix} 48463338 \\ 19393605 \\ 48463338 \\ 47187987 \\ 18118254 \\ 47187987 \end{bmatrix} + \lfloor -X_{58} \;\; 1 \rfloor \begin{bmatrix} 0.20660686 \\ 0.18594617 \end{bmatrix}$$

$$N_{Hi} = \frac{\lfloor X_{39} \;\; (X_{40} + X_{42}) \;\; (X_{43} + X_{45}) \;\; X_{46} \rfloor}{(10)^{10}} \begin{bmatrix} 18118254 \\ 47187987 \\ 46777657 \\ 17707924 \end{bmatrix} + \lfloor X_{58} \;\; -1 \rfloor \begin{bmatrix} 0.066473512 \\ 0.059826161 \end{bmatrix}$$

$$N_{iJ} = \frac{-\lfloor X_{44} \;\; X_{47} \;\; X_{48} \;\; X_{49} \;\; X_{50} \;\; X_{53} \rfloor}{(10)^{10}} \begin{bmatrix} 47284901 \\ 18022487 \\ 47284901 \\ 46877273 \\ 17614859 \\ 46877273 \end{bmatrix} + \lfloor -X_{58} \;\; 1 \rfloor \begin{bmatrix} 0.066035812 \\ 0.059432230 \end{bmatrix}$$

$$N_{Jk} = \frac{\lfloor X_{51} \;\; (X_{52} + X_{54} + X_{55} + X_{56}) \;\; X_{57} \rfloor}{(10)^{10}} \begin{bmatrix} 17614859 \\ 46877273 \\ 17614859 \end{bmatrix} + \quad 0 \qquad (45)$$

Those axial stresses in members on the right half of the truss shown in Figure 9b under typical symmetrical half live panel loads at b and b′ may be obtained by using these axial-stress expressions and Eqs. 18.

Most and Least Significant Parts of Axial Stresses

As indicated in Eqs. 45, the most significant part of each axial stress represented by the last product of two vectors (or otherwise zero) consists of a constant term and another containing the redundant reaction X_{58}, and corresponds to the conventional primary stress in each member. The least significant part of each axial stress is due to end moments and transverse shears by virtue of rigid-frame action.

The entirety of an axial-stress expression should be used in writing the strain-energy expression due to each axial stress before partial differentiation, but thereafter the least significant part may be neglected in formulating the equations and in computing axial stresses.

It should be noted that the coefficients of the unknown end moments in the axial-stress expressions are dependent only on the properties of the composition of the truss. They are independent of any external loading. The coefficients of the unknown end moments in the axial-stress expressions are, therefore, always the same regardless of loading conditions. Only the most significant or primary-stress terms are subject to change for different loadings. This important fact necessitates determination of only a single set of coefficients of unknown end moments in the axial-stress expressions for all members under various loading conditions. For each different loading condition, only those coefficients in the most significant (or primary-stress) terms need further calculation, thus saving much time and labor.

Since the reactions at the end supports due to the redundant reactions (treated as loads) at the interior supports always have -1 as the coefficient of $X_{58(s)}$ for any substituted set of symmetrical loading and -0.4 as the coefficient of $X_{58(a)}$ for any substituted set of antisymmetrical loading, as shown in Figure 9, the coefficient of $X_{58(s)}$, in the axial-stress expressions will always be the same as will that of $X_{58(a)}$. Therefore, axial-stress expressions under any different symmetrical or antisymmetrical loading can be readily determined by revising only the constant terms in the most significant part of axial stresses, or primary stresses.

As has been pointed out, the least significant terms in the axial-stress expressions for antisymmetrical loadings are the same as for those for symmetrical loadings given previously (this applies even to unsymmetrical loading). To formulate the axial-stress expressions for the antisymmetrical set of loadings at b and b′, as shown in Figure 9c, we need only write explicitly the most significant part of axial stresses (corresponding to primary stresses) as grouped in Eqs. 46:

Lower Chords

$$N_{ab} = N_{bc} = \frac{1}{(10)^8} \lfloor -X_{58} \quad 1 \rfloor \{40,000,000 \quad 486,000,000\}$$

$$N_{cd} = N_{de} = \frac{1}{(10)^8} \lfloor -X_{58} \quad 1 \rfloor \{110,204,080 \quad 347,142,860\}$$

$$N_{ef} = N_{fg} = \frac{1}{(10)^8} \lfloor -X_{58} \quad 1 \rfloor \{166,666,670 \quad 225,000,000\}$$

$$N_{gh} = N_{hi} = \frac{1}{(10)^8} \lfloor -X_{58} \quad 1 \rfloor \{142,105,260 \quad 127,894,740\}$$

$$N_{ij} = N_{jk} = \frac{1}{(10)^8} \lfloor -X_{58} \quad 1 \rfloor \{46,551,724 \quad 41,896,552\}$$

Upper Chords

$$N_{BC} = N_{CD} = \frac{1}{(10)^8} \lfloor X_{58} \quad -1 \rfloor \{76,671,359 \quad 414,025,340\}$$

$$N_{DE} = N_{EF} = \frac{1}{(10)^8} \lfloor X_{58} \quad -1 \rfloor \{140,021,410 \quad 283,543,360\}$$

$$N_{FG} = N_{GH} = \frac{1}{(10)^8} \lfloor X_{58} \quad -1 \rfloor \{194,702,670 \quad 175,232,410\}$$

$$N_{HI} = N_{IJ} = \frac{1}{(10)^8} \lfloor X_{58} \quad -1 \rfloor \{93,918,842 \quad 84,526,957\}$$

$$N_{JK} = 0$$

Verticals

$$N_{Bb} = +5.4$$

$$N_{Cc} = N_{Dd} = N_{Ee} = N_{Ff} = N_{Gg} = N_{Hh} = N_{Ii} = N_{Jj} = N_{Kk} = 0$$

Diagonals

$$N_{aB} = \frac{1}{(10)^8} \lfloor X_{58} \quad -1 \rfloor \{56,568,542 \quad 687,307,790\}$$

$$N_{Bc} = \frac{-1}{(10)^8} \lfloor X_{58} \quad 1 \rfloor \{51,754,198 \quad 102,364,990\}$$

$$N_{cD} = \frac{1}{(10)^8} \lfloor X_{58} \quad 1 \rfloor \{49,686,707 \quad 98,275,685\}$$

$$N_{De} = \frac{-1}{(10)^8} \lfloor X_{58} \quad 1 \rfloor \{43,763,343 \quad 94,671,183\}$$

$$N_{eF} = \frac{1}{(10)^8} \lfloor X_{58} \quad 1 \rfloor \{41,957,976 \quad 90,765,719\}$$

$$N_{Fg} = \frac{1}{(10)^8} \lfloor -X_{58} \quad -1 \rfloor \{43,624,819 \quad 77,891,410\}$$

$$N_{gH} = \frac{1}{(10)^8} \lfloor -X_{58} \quad 1 \rfloor \{84,708,813 \quad 76,237,932\}$$

$$N_{Hi} = \frac{1}{(10)^8} \lfloor X_{58} \quad -1 \rfloor \{77,774,009 \quad 69,996,608\}$$

$$N_{iJ} = \frac{1}{(10)^8} \lfloor -X_{58} \quad 1 \rfloor \{77,261,899 \quad 69,535,709\}$$

$$N_{Jk} = \frac{1}{(10)^8} \lfloor X_{58} \quad -1 \rfloor \{75,941,183 \quad 68,347,064\} \tag{46}$$

The most significant part of axial stresses in members on the right half of the truss may be obtained by using these products of vectors and Eqs. 19.

STRAIN-ENERGY EXPRESSIONS, REDUNDANT REACTIONS, AND SETS OF SIMULTANEOUS EQUATIONS

To apply the simplified matrix-energy method for analyzing the three-span continuous truss under consideration, the constants in the strain-energy expressions of truss members have been computed and the results are given in Table 9. With axial-stress expressions formulated as shown previously and strain-energy constants of truss members as given in Table 9, the strain-energy expressions can be readily formulated in practically the same manner as given in Table 4, except that to conform with the simplified matrix-energy method used herein the strain energy due to transverse shears will be neglected in the present analysis; the error so introduced will be negligible.

With the axial-stress expression for member ab represented by N_{ab}, that for member bc by N_{bc}, etc., the general strain-energy expressions may be written, according to Eqs. 5, using U to denote total internal strain energy in the member represented by the subscripts:

TABLE 9

CONSTANTS IN STRAIN-ENERGY EXPRESSIONS OF TRUSS MEMBERS

Member	$\frac{L}{A}$	$\frac{L}{6I}$	Member	$\frac{L}{A}$	$\frac{L}{6I}$
ab	22.388060	0.17564403	Cc	33.732057	1.5511551
bc	22.388060	0.17564403	Dd	32.236842	1.3243243
cd	22.388060	0.17564403	Ee	32.187500	0.89721254
de	22.388060	0.17564403	Ff	35.526316	1.4594595
ef	22.388060	0.17564403	Gg	34.687500	0.96689895
fg	22.388060	0.17564403	Hh	37.500000	1.5405405
gh	22.388060	0.17564403	Ii	35.937500	1.0017422
hi	22.388060	0.17564403	Ji	38.157895	1.5675676
ij	22.388060	0.17564403	Kk	36.250000	1.0104530
jk	22.388060	0.17564403	aB	22.796278	0.16568500
BC	16.135316	0.11727265	Bc	43.489483	0.62760957
CD	16.135316	0.11727265	cD	45.463456	0.65609655
DE	16.144260	0.11733765	De	33.098602	0.25967287
EF	16.144260	0.11733765	eF	28.808298	0.24492073
FG	16.128356	0.11722206	Fg	17.886068	0.14237847
GH	16.128356	0.11722206	gH	16.480101	0.11743582
HI	16.120398	0.11716422	Hi	29.763242	0.25303941
IJ	16.120398	0.11716422	iJ	30.085988	0.25578331
JK	16.119403	0.11715699	Jk	36.522294	0.28653322
Bb	29.605263	1.21621620	—	—	—

$$E\begin{bmatrix} U_{ab} \\ U_{bc} \\ U_{cd} \\ U_{de} \\ U_{ef} \\ U_{fg} \\ U_{gh} \\ U_{hi} \\ U_{ij} \\ U_{jk} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}N^2_{ab} & (X_1^2 - X_1X_5 + X_5^2) \\ \frac{1}{2}N^2_{bc} & (X_6^2 - X_6X_9 + X_9^2) \\ \frac{1}{2}N^2_{cd} & (X_{12}^2 - X_{12}X_{17} + X_{17}^2) \\ \frac{1}{2}N^2_{de} & (X_{18}^2 - X_{18}X_{21} + X_{21}^2) \\ \frac{1}{2}N^2_{ef} & (X_{24}^2 - X_{24}X_{29} + X_{29}^2) \\ \frac{1}{2}N^2_{fg} & (X_{30}^2 - X_{30}X_{33} + X_{33}^2) \\ \frac{1}{2}N^2_{gh} & (X_{36}^2 - X_{36}X_{41} + X_{41}^2) \\ \frac{1}{2}N^2_{hi} & (X_{42}^2 - X_{42}X_{45} + X_{45}^2) \\ \frac{1}{2}N^2_{ij} & (X_{48}^2 - X_{48}X_{53} + X_{53}^2) \\ \frac{1}{2}N^2_{jk} & (X_{54}^2 - X_{54}X_{56} + X_{56}^2) \end{bmatrix} \begin{bmatrix} 22.388060 \\ \\ 0.17564403 \end{bmatrix}$$

$$E\begin{bmatrix} U_{BC} \\ U_{CD} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} N^2_{BC} & (X_4^2 - X_4X_7 + X_7^2) \\ \frac{1}{2} N^2_{CD} & (X_8^2 - X_8X_{13} + X_{13}^2) \end{bmatrix} \begin{bmatrix} 16.135316 \\ 0.11727265 \end{bmatrix}$$

$$E\begin{bmatrix} U_{DE} \\ U_{EF} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} N^2_{DE} & (X_{16}^2 - X_{16}X_{19} + X_{19}^2) \\ \frac{1}{2} N^2_{EF} & (X_{20}^2 - X_{20}X_{25} + X_{25}^2) \end{bmatrix} \begin{bmatrix} 16.144260 \\ 0.11733765 \end{bmatrix}$$

$$E\begin{bmatrix} U_{FG} \\ U_{GH} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} N^2_{FG} & (X_{28}^2 - X_{28}X_{31} + X_{31}^2) \\ \frac{1}{2} N^2_{GH} & (X_{32}^2 - X_{32}X_{37} + X_{37}^2) \end{bmatrix} \begin{bmatrix} 16.128356 \\ 0.1172220 \end{bmatrix}$$

$$E\begin{bmatrix} U_{HI} \\ U_{IJ} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} N^2_{HI} & (X_{40}^2 - X_{40}X_{43} + X_{43}^2) \\ \frac{1}{2} N^2_{IJ} & (X_{44}^2 - X_{44}X_{49} + X_{49}^2) \end{bmatrix} \begin{bmatrix} 16.120398 \\ 0.11716422 \end{bmatrix}$$

$$E\ U_{JK} = \begin{bmatrix} \frac{1}{2} N^2_{JK} & (X_{52}^2 - X_{52}X_{55} + X_{55}^2) \end{bmatrix} \begin{bmatrix} 16.119403 \\ 0.11715699 \end{bmatrix}$$

$$E\ U_{Bb} = \begin{bmatrix} \frac{1}{2} N^2_{Bb} \begin{bmatrix} \sum_{i=2}^{4} X_i & -\sum_{i=2}^{4} X_i & \sum_{i=5}^{6} X_i \end{bmatrix} \begin{bmatrix} \sum_{i=2}^{4} X_i \\ \sum_{i=5}^{6} X_i \\ \sum_{i=5}^{6} X_i \end{bmatrix} \end{bmatrix} \begin{bmatrix} 29.605263 \\ 1.2162162 \end{bmatrix}$$

$$E\ U_{Cc} = \left[\frac{1}{2} N^2_{Cc} \begin{bmatrix} \sum\limits_{i=7}^{8} X_i & -\sum\limits_{i=7}^{8} X_i & \sum\limits_{i=9}^{12} X_i \end{bmatrix} \begin{bmatrix} \sum\limits_{i=7}^{8} X_i \\ \sum\limits_{i=9}^{12} X_i \\ \sum\limits_{i=9}^{12} X_i \end{bmatrix} \right] \begin{bmatrix} 33.732057 \\ 1.5511551 \end{bmatrix}$$

$$E\ U_{Dd} = \left[\frac{1}{2} N^2_{Dd} \begin{bmatrix} \sum\limits_{i=13}^{16} X_i & -\sum\limits_{i=13}^{16} X_i & \sum\limits_{i=17}^{18} X_i \end{bmatrix} \begin{bmatrix} \sum\limits_{i=13}^{16} X_i \\ \sum\limits_{i=17}^{18} X_i \\ \sum\limits_{i=17}^{18} X_i \end{bmatrix} \right] \begin{bmatrix} 32.236842 \\ 1.3243243 \end{bmatrix}$$

$$E\ U_{Ee} = \left[\frac{1}{2} N^2_{Ee} \begin{bmatrix} \sum\limits_{i=19}^{20} X_i & -\sum\limits_{i=19}^{20} X_i & \sum\limits_{i=21}^{24} X_i \end{bmatrix} \begin{bmatrix} \sum\limits_{i=19}^{20} X_i \\ \sum\limits_{i=21}^{24} X_i \\ \sum\limits_{i=21}^{24} X_i \end{bmatrix} \right] \begin{bmatrix} 32.187500 \\ 0.89721254 \end{bmatrix}$$

$$E\ U_{Ff} = \left[\frac{1}{2} N^2_{Ff} \left[\sum_{i=25}^{28} X_i \quad -\sum_{i=25}^{28} X_i \quad \sum_{i=29}^{30} X_i \right] \begin{bmatrix} \sum\limits_{i=25}^{28} X_i \\ \sum\limits_{i=29}^{30} X_i \\ \sum\limits_{i=29}^{30} X_i \end{bmatrix} \right] \begin{bmatrix} 35.526316 \\ 1.4594595 \end{bmatrix}$$

$$E\ U_{Gg} = \left[\frac{1}{2} N^2_{Gg} \left[\sum_{i=31}^{32} X_i \quad -\sum_{i=31}^{32} X_i \quad \sum_{i=33}^{36} X_i \right] \begin{bmatrix} \sum\limits_{i=31}^{32} X_i \\ \sum\limits_{i=33}^{36} X_i \\ \sum\limits_{i=33}^{36} X_i \end{bmatrix} \right] \begin{bmatrix} 34.687500 \\ 0.96689895 \end{bmatrix}$$

$$E\ U_{Hh} = \left[\frac{1}{2} N^2_{Hh} \left[\sum_{i=37}^{40} X_i \quad -\sum_{i=37}^{40} X_i \quad \sum_{i=41}^{42} X_i \right] \begin{bmatrix} \sum\limits_{i=31}^{40} X_i \\ \sum\limits_{i=41}^{42} X_i \\ \sum\limits_{i=41}^{42} X_i \end{bmatrix} \right] \begin{bmatrix} 37.500000 \\ 1.5405405 \end{bmatrix}$$

$$E\,U_{Ii} = \left[\tfrac{1}{2} N_{Ii}^2 \left\lfloor \sum_{i=43}^{44} X_i \quad -\sum_{i=43}^{44} X_i \quad \sum_{i=45}^{48} X_i \right\rfloor \begin{bmatrix} \sum_{i=43}^{44} X_i \\ \sum_{i=45}^{48} X_i \\ \sum_{i=45}^{48} X_i \end{bmatrix}\right] \begin{bmatrix} 35.937500 \\ 1.0017422 \end{bmatrix}$$

$$E\,U_{Jj} = \left[\tfrac{1}{2} N_{Jj}^2 \left\lfloor \sum_{i=49}^{52} X_i \quad -\sum_{i=49}^{52} X_i \quad \sum_{i=53}^{54} X_i \right\rfloor \begin{bmatrix} \sum_{i=49}^{52} X_i \\ \sum_{i=53}^{54} X_i \\ \sum_{i=53}^{54} X_i \end{bmatrix}\right] \begin{bmatrix} 38.157895 \\ 1.5675676 \end{bmatrix}$$

$$E \begin{bmatrix} U_{Kk(s)} \\ \\ U_{Kk(a)} \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2} N_{Kk}^2 & & & 0 \\ \tfrac{1}{2} N_{Kk}^2 & 4 \left\lfloor X_{55} \quad -X_{55} \quad \sum_{i=56}^{57} X_i \right\rfloor & & \begin{bmatrix} X_{55} \\ \sum_{i=56}^{57} X_i \\ \sum_{i=56}^{57} X_i \end{bmatrix} \end{bmatrix} \begin{bmatrix} 36.250000 \\ 1.0104530 \end{bmatrix}$$

where (s) denotes symmetrical set of loading, and (a) denotes antisymmetrical set of loading. Only one-half of EU_{Kk} should be used in computing $\frac{1}{2}EU$.

$$E\begin{bmatrix}U_{aB}\\ U_{Bc}\\ U_{cD}\\ U_{De}\\ U_{eF}\end{bmatrix} = \begin{bmatrix} \begin{bmatrix}\frac{1}{2}N_{aB}^2 & (X_1^2 + X_1X_2 + X_2^2)\end{bmatrix}\begin{bmatrix}22.796278\\ 0.16568500\end{bmatrix} \\ \begin{bmatrix}\frac{1}{2}N_{Bc}^2 & (X_3^2 - X_3X_{10} + X_{10}^2)\end{bmatrix}\begin{bmatrix}43.489483\\ 0.62760957\end{bmatrix} \\ \begin{bmatrix}\frac{1}{2}N_{cD}^2 & (X_{11}^2 - X_{11}X_{14} + X_{14}^2)\end{bmatrix}\begin{bmatrix}45.463456\\ 0.65609655\end{bmatrix} \\ \begin{bmatrix}\frac{1}{2}N_{De}^2 & (X_{15}^2 - X_{15}X_{22} + X_{22}^2)\end{bmatrix}\begin{bmatrix}33.098602\\ 0.25967287\end{bmatrix} \\ \begin{bmatrix}\frac{1}{2}N_{eF}^2 & (X_{23} - X_{23}^2X_{26} + X_{26}^2)\end{bmatrix}\begin{bmatrix}28.808298\\ 0.24492073\end{bmatrix}\end{bmatrix}$$

$$E\begin{bmatrix}U_{Fg}\\ U_{gH}\\ U_{Hi}\\ U_{iJ}\\ U_{Jk}\end{bmatrix} = \begin{bmatrix} \begin{bmatrix}\frac{1}{2}N_{Fg}^2 & (X_{27}^2 - X_{27}X_{34} + X_{34}^2)\end{bmatrix}\begin{bmatrix}17.886068\\ 0.14237847\end{bmatrix} \\ \begin{bmatrix}\frac{1}{2}N_{gH}^2 & (X_{35}^2 - X_{35}X_{38} + X_{38}^2)\end{bmatrix}\begin{bmatrix}16.480101\\ 0.11743582\end{bmatrix} \\ \begin{bmatrix}\frac{1}{2}N_{Hi}^2 & (X_{39}^2 - X_{39}X_{46} + X_{46}^2)\end{bmatrix}\begin{bmatrix}29.763242\\ 0.25303941\end{bmatrix} \\ \begin{bmatrix}\frac{1}{2}N_{iJ}^2 & (X_{47}^2 - X_{47}X_{50} + X_{50}^2)\end{bmatrix}\begin{bmatrix}30.085988\\ 0.25578331\end{bmatrix} \\ \begin{bmatrix}\frac{1}{2}N_{Jk}^2 & (X_{51}^2 - X_{51}X_{57} + X_{57}^2)\end{bmatrix}\begin{bmatrix}36.522294\\ 0.28653322\end{bmatrix}\end{bmatrix} \tag{47}$$

Solution of Redundant Reactions

In the simplified matrix-energy method which we are now applying, not only is the strain energy due to transverse shears neglected but also the part other than the conventional primary stress (the most significant part) in the axial-stress expression is deleted after partial differentiation. Thus, the 58th equation obtained from

$$\frac{1}{2}E\,\frac{\partial U}{\partial X_{58}} = 0 \tag{48}$$

assumes the form of an algebraic linear equation with X_{58} as the "unique unknown." This redundant reaction may now be solved. The size of the set of simultaneous equations is consequently reduced to 57 equations with 57 unknowns.

The pair of symmetrical half-panel loads applied at b and b′, and c and c′, etc., may be represented by b++, c++, etc., respectively; the pair of antisymmetrical half-panel loads applied at b and b′, c and c′, etc., may be represented by b+-, c+-, etc., respectively; the panel load symmetrically applied at k may be represented by k; and the symmetrically located dead loads may be represented by DL. The redundant reactions at the interior supports are found as in Table 10. The left redundant reaction is upward positive, and the right redundant reaction is upward positive under symmetrical loadings and downward positive under antisymmetrical loadings.

It must be noted that the redundant reactions can first be solved independently only in the simplified matrix-energy method. In the exact matrix-energy method, Eq. 48 will contain some of the unknown end moments X_i; it must be solved simultaneous with the other 57 equations.

TABLE 10

REDUNDANT REACTIONS AT INTERIOR SUPPORTS

Due to	X_{58} (kips)	Due to	X_{58} (kips)
a+-	0	DL	269.39998
b++	1.0696012	a+-	0
c++	2.1265065	b+-	1.5125273
d++	3.1326161	c+-	2.9914410
e++	4.0066760	d+-	4.2743738
f++	4.7827808	e+-	5.0799756
g++	5.4000000	f+-	5.5290950
h++	5.8704096	g+-	5.4000000
i++	6.1854863	h+-	4.5959812
j++	6.3893981	i+-	3.2634630
k++	6.4439991	j+-	1.7249734

Formulation of Sets of Simultaneous Equations

For the present problem, the complete systems of simultaneous equations in poly-set unknown and constant vectors, after partially differentiating the total strain energy given by Eq. 6 with respect to each unknown, take the following general matrix forms:

1. For symmetrical loadings,

$$\left[a_{ij}\right]_{\overline{57}\times 57}\left[X_{ik}\right]_{\overline{57}\times 10}=\left[c_{ik}\right]_{\overline{57}\times 10} \tag{49}$$

2. For antisymmetrical loadings,

$$\left[a_{ij}\right]_{\overline{57}\times 57}\left[X_{ik}\right]_{\overline{57}\times 8}=\left[c_{ik}\right]_{\overline{57}\times 8} \tag{50}$$

The solutions of these equations by matrix inversion are, respectively,

$$\left[X_{ik}\right]_{\overline{57}\times 10}=\left[a_{ij}\right]^{-1}_{\overline{57}\times 57}\left[c_{ik}\right]_{\overline{57}\times 10} \tag{51}$$

and

$$\left[X_{ik}\right]_{\overline{57}\times 8}=\left[a_{ij}\right]^{-1}_{\overline{57}\times 57}\left[c_{ik}\right]_{\overline{57}\times 8} \tag{52}$$

where the barred subscripts denote number of rows, the unbarred subscripts denote number of columns, and $[a_{ij}]$'s of Eqs. 51 and 52 differ by a 3 × 3 trailing sub-matrix as will be explained later by Eq. 53. There are as many unknown column vectors and known constant column vectors as there are loading conditions.

To explain the formulation of these matrices, Table 10 is used to express all primary axial stresses in truss members under the 18 different sets of loading conditions in their numerical values.

After substituting each of the 18 sets of axial stresses for N_{ij}'s in Eqs. 47, repeated operations of $\frac{1}{2}E\,\frac{\partial U}{\partial X_i}=0$, where i = 1, 2, . . . , 57, will yield two sets of 57 × 57 coefficient matrices with 57 × (10 + 8) known constant matrix to solve the 57 × (10 + 8) unknown end-moment matrix in two inversions.

It is especially noteworthy that, in the exact matrix-energy method, the coefficient matrix has to be determined from coefficients of all unknowns in the strain-energy expressions. But in the simplified matrix-energy method, the coefficient matrix is dependent only on coefficients of unknown bending moments.

Because the center vertical member has no bending moment under any symmetrical loading and experiences certain bending moment under each antisymmetrical loading, the coefficient matrix for symmetrical loadings will be different from that for antisymmetrical loadings but limited only to a 3 × 3 trailing sub-matrix:

TABLE 11

SOLUTION-VECTOR MATRIX

X_i	b++	c++	d++	e++	f++	h++	i++	j++	k++	DL	b+-	c+-	d+-	e+-	f+-	h+-	i+-	j+-
1	-2.4055712	-0.65158492	-0.44561134	-0.23444840	-0.10281838	0.07771276	0.12956796	0.16319313	0.17216851	-22.601690	-2.3931343	-0.62683546	-0.41309940	-0.20388586	-0.08156693	0.03615346	0.03720177	0.02475908
2	-1.7866888	1.2635521	-0.00829977	0.12806769	0.02666502	-0.03225614	-0.05127491	-0.06454358	-0.06799405	-3.1196718	-1.7915954	1.2537955	-0.02112347	0.11601286	0.01828284	-0.01589822	-0.01490431	-0.00996878
3	-0.12838390	0.30899713	0.50109965	0.16925690	0.08274663	-0.05746442	-0.09636843	-0.12137484	-0.12808581	4.6758872	-0.13763611	0.29056382	0.47691270	0.14652010	0.06693673	-0.02654108	-0.02764315	-0.01838373
4	2.0213862	-2.3228607	-0.77416475	-0.52292620	-0.20159021	0.16349613	0.27021036	0.34029667	0.35895171	-7.5058655	2.0473085	-2.2712827	-0.70640325	-0.45922739	-0.15729772	0.07691076	0.07775873	0.05179775
5	-5.5822360	0.23300047	-0.64982916	-0.12064142	-0.07776867	0.04548907	0.07763424	0.09778671	0.10324706	-39.382299	-5.5747760	0.24785096	-0.63032579	-0.10230734	-0.06502020	0.02053776	0.02218832	0.01472642
6	5.5762074	-1.0276941	0.33612084	-0.13566824	-0.02840189	0.03913274	0.06309882	0.07947646	0.08377560	32.352004	5.5822554	-1.0156644	0.35192910	-0.12102762	-0.01806878	0.01894826	0.01822875	0.01218260
7	-0.14146649	-6.1964330	-0.13404790	-0.60869145	-0.12603469	0.15040469	0.23852585	0.30022241	0.31624145	-48.248830	-0.11864673	-6.1510599	-0.07440935	-0.55262857	-0.08705185	0.07434184	0.06940060	0.04642552
8	0.51965699	6.1409189	-0.52303822	0.33346171	-0.01619998	-0.05099498	-0.07087894	-0.08903395	-0.09332900	47.292198	0.51314131	6.1275989	-0.54057605	0.31697538	-0.02766347	-0.02877603	-0.02141083	-0.01452296
9	2.7011943	-3.9179179	1.0103041	-0.58201852	-0.14454435	0.17205030	0.27924865	0.35172813	0.37082656	-2.3316837	2.7279681	-3.8646588	1.0802870	-0.51623142	-0.09879972	0.08267367	0.08057331	0.05380428
10	-0.71528609	-0.41386032	0.75245478	0.10433609	0.07735723	-0.04128430	-0.07107946	-0.08952959	-0.09455194	-2.9366250	-0.72212068	-0.42746347	0.73459116	0.08754351	0.06568061	-0.01842425	-0.02028336	-0.01344771
11	-0.57948913	0.45681994	0.67337035	0.28241392	0.09992315	-0.08028630	-0.13130510	-0.16530588	-0.17429783	3.1734162	-0.59207308	0.43178650	0.64047785	0.25149354	0.07842287	-0.03828237	-0.03793236	-0.02528706
12	-0.95407966	3.8190626	-3.1465206	-0.06017790	-0.16707785	0.04139364	0.07815523	0.09838119	0.10413855	1.1380347	-0.94654636	3.8340780	-3.1268192	-0.04165787	-0.15420002	0.01612041	0.02202245	0.01442974
13	0.13521703	1.7095313	-5.5289161	-1.0823744	-1.0644630	0.58575011	1.0290680	1.2970930	1.3708844	-10.988757	0.23432973	1.9069313	-5.2697549	-0.83875134	-0.89506024	0.25371412	0.29143947	0.19296750
14	-0.47341413	-0.33616160	-0.34742034	0.07611524	-0.06878610	0.01570982	0.03544835	0.04482901	0.04770800	-7.2444291	0.46995109	-0.32924072	-0.33835554	0.08463658	-0.06286068	0.00398718	0.00945325	0.00611846
15	-0.13595642	-0.81489770	0.73211959	0.65764024	0.74431732	-0.24785093	-0.43876494	-0.55210111	-0.58335930	-2.6551421	-0.17811526	-0.89885374	0.62189436	0.55402363	0.67226772	-0.10677054	-0.12528398	-0.08265829
16	0.66309564	0.05100597	5.2407294	0.04150114	0.34812161	-0.27882113	-0.50847040	-0.64217696	-0.67975665	27.716671	0.61389651	-0.04705766	5.1120449	-0.07946800	0.26400567	-0.11352240	-0.14144507	-0.09353500
17	-0.50572645	0.66396883	-6.9620071	-0.72363399	-0.79625230	0.32868463	0.58074216	0.73140022	0.77295665	-41.065560	-0.44985226	0.77524823	-6.8159131	-0.58649915	-0.70075685	0.14158637	0.16506028	0.10907590
18	0.68775553	-0.02889219	7.0588445	0.40253191	0.75530715	-0.24444154	-0.44750584	-0.56360808	-0.59619890	48.330125	0.64463767	-0.11480698	6.9460900	0.29653762	0.68160404	-0.09987727	-0.12639183	-0.08317717
19	0.03822173	-0.30927867	1.0461165	-3.6899920	0.03037325	0.33733584	0.36983475	0.48610521	0.50485224	-14.147154	0.07433210	-0.23600011	1.1402660	-3.6014875	0.09191095	0.21979247	0.12736597	0.08763112
20	0.22310959	0.75279677	-0.00712064	4.0755531	-0.33992840	-0.29167934	-0.27224320	-0.33730822	-0.34606176	29.827371	0.19854237	0.70462057	-0.07104321	4.0150633	-0.38170732	-0.21335799	-0.09669358	-0.06802215
21	0.37725826	0.28439378	4.2117706	-1.9441958	1.1035305	0.29177691	0.35375152	0.45586580	0.46543850	34.753101	0.41069719	0.35057497	4.2990603	-1.8621398	1.1605854	0.18159577	0.10828211	0.07523870
22	-0.31072201	-0.73354224	-1.1953132	-0.85735860	0.98054577	0.06662861	0.02183810	0.02795683	0.02620353	-14.063428	-0.30894417	-0.73024294	-1.1907454	-0.85306459	0.98353002	0.06168564	0.01047638	0.00916413
23	-0.16793446	-0.12371195	-1.5571038	0.38604819	0.82993214	-0.20611611	-0.25297200	-0.31311734	-0.32529684	-11.413357	-0.19117679	-0.16959811	-1.6176845	0.32909981	0.79033548	-0.13069681	-0.08445275	-0.05701338
24	0.33158936	0.94571633	-0.41108908	2.7515356	-3.3334884	-0.11706900	-0.02085407	-0.03223272	-0.02905207	-4.2307470	0.32959863	0.94211539	-0.41622042	2.7467119	-3.3368399	-0.11131401	-0.00777019	-0.01066295
25	0.40111082	0.86978292	0.98779296	2.2548192	-2.4273704	0.01402609	0.75701785	0.97291025	1.0603415	34.942066	0.47917216	1.0275663	1.1928871	2.4476172	-2.2932952	-0.25981274	0.15350067	0.06965335
26	-0.19673197	-0.32652911	-1.0465026	-0.66768542	-0.57281358	-0.09788775	0.18214831	0.24013352	0.26973795	-16.923849	-0.17651921	-0.28512399	-0.99315271	-0.61753402	-0.53793424	-0.17186928	0.02025696	0.00796719
27	-0.26184031	-0.57667665	-0.50045100	-1.3279476	0.13174628	0.15053370	-0.71563464	-0.85158727	-0.92116705	-31.692732	-0.32858927	-0.71116089	-0.67507342	-1.4921005	0.01759149	0.37510673	-0.21669832	-0.10979957
28	0.16441224	0.25829069	0.82620075	0.23137432	3.0702096	0.05084900	-0.13369718	-0.24931959	-0.29496071	25.607025	0.14094568	0.20930721	0.76332559	0.17226839	3.0290996	0.14867415	0.07569294	0.03621399
29	0.17437499	0.42170487	0.04990060	0.89715714	-5.4328576	-0.02608133	0.53327507	0.65929161	0.71417252	-13.112572	0.22648363	0.52674697	0.18648019	1.0255481	-5.3435729	-0.20469918	0.13790239	0.07536310
30	-0.05333637	-0.16780892	0.24734709	-0.36025800	5.6518827	0.15081458	-0.43710928	-0.53753457	-0.58981060	26.207579	-0.09660475	-0.25555640	0.13377853	-0.46701761	5.5776374	0.30112742	-0.10510967	-0.05077649
31	0.47559890	0.91400610	1.4405461	1.4840139	2.4092233	2.2919404	2.1362276	2.4957959	2.5120882	104.11260	0.65021333	1.2522162	1.8921983	1.9085864	2.7044008	1.7693453	0.95003622	0.62525704
32	-0.42480843	-0.81627582	-1.2741754	-1.3178571	-1.9888553	-2.7252284	-2.3603637	-2.7577936	-2.7635036	-105.29828	-0.61641789	-1.1879933	-1.7694389	-1.7834260	-2.3125289	-2.1561345	-1.0667521	-0.70973806
33	0.33310457	0.63521013	1.0824476	1.0937395	5.1828471	1.9172024	0.84442922	1.1086160	1.0925378	87.719904	0.40975158	0.78213605	1.2812547	1.2806270	5.3127621	1.6805194	0.31136616	0.27287611
34	0.07048108	0.11640711	0.22011957	-0.00923143	-0.62373044	2.2556948	0.80042410	1.0805594	1.0475374	32.275207	0.14365220	0.25529315	0.40969313	0.16897617	-0.49985916	2.0324040	0.29666898	0.28381705
35	-0.21118262	-0.38880163	-0.67893268	-0.55074967	-2.4503809	0.79538343	-0.27388066	-0.57630781	-0.60789832	-33.123103	-0.25501463	-0.47392634	-0.79353310	-0.65847944	-2.5252397	0.94262050	0.05112506	-0.08633756
36	-0.15282103	-0.28764806	-0.48888944	-0.40992221	-1.6572565	-5.4398604	-1.5431580	-1.8265748	-1.7331765	-87.959510	-0.27239143	-0.51515579	-0.79753106	-0.70005871	-1.8589602	-5.0898924	-0.74546971	-0.54138506
37	0.04095351	0.08964540	0.05200698	-0.00689930	-0.13473939	-2.9692443	-0.55428409	-1.1753670	-0.99495951	-34.139866	-0.02295553	-0.04322767	-0.10968101	-0.15889282	-0.24022918	-2.8220371	-0.20585081	-0.52472739
38	0.16144737	0.32970206	0.38444753	0.40937812	-0.34348754	0.23653954	1.9566442	1.4008345	1.5725211	38.714136	0.27316557	0.53804340	0.67485743	0.68237523	-0.15346573	-0.11364589	1.1608078	0.20908886
39	-0.06340884	-0.11629529	-0.13799928	-0.12652243	0.13110234	0.47355231	0.77658286	1.3225113	0.88871784	20.025351	-0.00567651	-0.01339034	0.00858039	0.01126941	0.22678453	0.33410540	0.44885581	0.66488646
40	-0.14601067	-0.31748473	-0.32227210	-0.29798550	0.26862753	2.0390582	-2.7216111	-1.9807853	-1.9303057	-36.078333	-0.28320848	-0.55597775	-0.67895167	-0.63327943	0.03539365	2.4693764	-1.7422989	-0.47806857
41	0.09425310	0.18344889	0.21151241	0.21324714	-0.25157926	-5.6121169	0.50893052	-0.01362089	0.33328523	-29.075157	0.12038331	0.23843637	0.21801333	0.27858021	-0.20600480	-5.7126972	0.28526575	-0.27175321
42	-0.10103609	-0.19682162	-0.23412354	-0.23313583	0.17362356	5.3690614	-1.1155331	-0.47517065	-0.86670422	16.358942	-0.16357697	-0.32176649	-0.39724215	-0.38647320	0.06682804	5.5709852	-0.65667928	0.15471104
43	-0.07948148	0.06015082	0.21736944	0.20575054	0.28212552	0.30350409	-3.9015765	-0.51858638	-0.50152356	-30.405164	0.04511832	0.08627907	0.13017144	0.12378291	0.22421059	0.37490326	-3.6895819	-0.21107886
44	-0.07965307	-0.07090397	-0.21549129	-0.20396396	-0.25739744	-0.10974162	3.4007691	-0.69422558	-0.58212544	15.816520	-0.11871576	-0.22968882	-0.31705721	-0.29944278	-0.32268874	0.02571563	3.6583517	-0.21078768
45	0.04893288	0.09213600	0.13075044	0.11762796	0.18076167	3.3537017	-2.8669826	-0.11241000	-1.3051976	-2.2794002	-0.04029740	-0.07943475	-0.09777128	-0.09718722	0.03079375	3.5834232	-2.3025323	0.59673903
46	0.03925561	0.07782856	0.10793250	0.10392450	0.13397896	-0.78559190	-0.04992132	1.9151619	1.0068163	14.621883	0.10740699	0.20826127	0.28363052	0.26909121	0.24840713	-0.98763141	-0.49312500	1.0417684
47	-0.03401738	-0.06979503	-0.09139617	-0.08476517	-0.11452342	-0.96656059	0.96805857	1.8392039	0.98414658	13.912573	0.03295626	0.06469364	0.08022139	0.07656355	-0.00183479	-1.1418466	0.53950895	1.0379950
48	-0.05390201	-0.10470336	-0.14441411	-0.13429427	-0.17466042	-1.2927190	1.5188031	-4.8334508	-1.6321349	-38.815250	-0.16382687	-0.31780032	-0.42780849	-0.40070669	-0.35922198	-0.96442703	2.2351068	-3.1572898
49	0.17121155	0.35944683	0.43300239	0.40638420	0.24935031	-0.39709212	-0.09373528	-5.2460866	-4.8009543	-41.926645	-0.14346427	-0.27873920	-0.37202312	-0.35037916	-0.27311483	0.47068516	1.7741729	-1.8466660
50	0.08261970	0.15168972	0.20954801	0.19757789	0.12084320	-0.64753623	-0.35549312	-0.16234413	-0.41069584	-4.5931378	0.06014301	0.11714411	0.15357134	0.14495579	0.08597193	-0.58986080	-0.28365599	0.05790758
51	0.07384836	-0.14837182	-0.18704346	-0.17580545	-0.10885632	0.29191092	0.15183665	1.5031540	1.4287052	14.274203	0.01460782	0.02828850	0.03870577	0.03644096	0.03700170	0.03348163	-0.35935557	0.58936284
52	-0.18061244	-0.36287597	-0.45745599	-0.42997095	-0.26623211	0.71393246	0.37134997	3.6762942	3.4942132	29.753810	0.04732155	0.09174071	0.12438311	0.11695914	0.10982178	0.09951566	-0.91453444	1.0872526
53	0.11720172	0.22557943	0.29684052	0.27947709	0.16956973	-0.72751213	-0.33315265	-7.5161612	-3.3279276	-61.517262	-0.09884623	-0.19202850	-0.25592636	-0.24017268	-0.18937191	-0.12209875	0.96133051	-3.9217534
54	-0.11797132	-0.22655631	-0.29912591	-0.28158345	-0.17462995	0.67745227	0.39865353	7.2611762	3.0016079	58.549610	0.07460945	0.14492849	0.19327295	0.18132147	0.14438861	0.13160722	-0.73629605	3.7353447
55	0.06576978	0.11903060	0.16936816	0.15925937	0.12711374	-0.03836554	-0.74793431	0.26396040	-0.60506109	-6.7231919	0.00306067	0.00593757	0.00800552	0.00752193	0.00670405	0.00608558	-0.04936143	0.03808340
56	0.04511919	0.08713653	0.11598601	0.10883475	0.08626189	0.07503440	-0.42340351	4.7530400	-0.06812434	29.971847	0.01196322	0.02326183	0.03075854	0.02882204	0.02080152	0.01903548	-0.06039943	-0.28763932
57	0.02689179	0.04866894	0.06925901	0.06511758	0.05197395	-0.01568661	-0.30581352	0.10792745	-0.24739589	-0.64042248	-0.00823036	-0.01691669	-0.02103010	-0.01968646	-0.01298277	-0.01192791	0.00897120	0.30519093

$$\text{sub - }\begin{bmatrix}a_{ij}\end{bmatrix} = \begin{bmatrix}a_{55,55} & a_{55,56} & a_{55,57}\\ a_{56,55} & a_{56,56} & a_{56,57}\\ a_{57,55} & a_{57,56} & a_{57,57}\end{bmatrix} \qquad (53)$$

which is a diagonal matrix for symmetrical loadings or a symmetrical matrix for antisymmetrical loadings. This difference is evident by inspecting the strain-energy expressions of the center-vertical member Kk in Eqs. 47.

Coefficient Matrices and Constant-Vector Matrices

The two sets of 57 × 57 coefficient matrices for symmetrical and antisymmetrical loadings and the 57 × 10 and 57 × 8 constant-vector matrices for the 18 basic loading conditions are regimented arrays of numbers. To save space, they are not shown herein.

Solution by Inversion Using Electronic Digital Computer

Eqs. 51 and 52 were solved by matrix inversion with the program "Simultaneous Equations A La King" furnished by IBM for use in the IBM 1620 computer (capacity, 60,000 core storage) available at the South Dakota Department of Highways.

Solutions of end moments of all members for substituted symmetrical and antisymmetrical sets of loadings are given in Table 11, from which full panel-load end moments may be obtained.

Appropriate combinations of panel-load end moments will yield all end moments simultaneous with maximum design stresses and maximum end moments of all members.

COMPOSITION OF MAXIMUM DESIGN AXIAL STRESSES

Dead-Load Stresses

Dead-load stresses are computed by the same general rule and in a similar manner as described previously by applying all upper and lower dead panel loads given in Table 8 simultaneously.

Maximum and Minimum Live-Load Stresses

Maximum live-load stresses, and minimum live-load stresses or maximum live-load stresses of the opposite sign, of all members of the truss are the most important part of live-load-stress analysis for later stress combination to arrive at maximum design stresses and maximum range of stress reversals.

In a programmed computation by the basic scheme of the proposed method, if the computer available is of sufficient capacity, neither method of substitution nor method of transformation would be needed. The procedure would be to load the truss with live panel loads, one at a time, from b to f, then from h to k, as shown in Figure 9a. By the panel-load superposition method, summation of all-plus-sign stresses and of all-minus-sign stresses in each member will give, respectively, the maximum tensile and compressive stresses.

From Figures 9b and 9c, with the method of substitution introduced in the present case because only a smaller computer is available, half live panel loads will be placed for both symmetrical and antisymmetrical sets, one pair at a time, at b and b′ to f and f′, then at h and h′ to j and j′, but one full panel load will be placed at k only once. Then from Eqs. 18 and 19, the axial stress, N, in member IJ is given by

$$\underset{\substack{\text{(due to any P}\\ \text{at any panel}\\ \text{point x)}}}{N_{ij}} = \underset{\substack{\text{(due to symm.}\\ \text{set of } \frac{1}{2}\text{P at}\\ \text{x and x}')}}{N_{ij(s)}} + \underset{\substack{\text{(due to antisymm.}\\ \text{set of } \frac{1}{2}\text{P at x}\\ \text{and x}')}}{N_{ij(a)}} \qquad (54)$$

and

$$\underset{\substack{\text{(due to any P}\\ \text{at panel}\\ \text{point x}')}}{N_{ij}} = \underset{\substack{\text{(due to any P}\\ \text{at panel}\\ \text{point x)}}}{N_{i'j'}} = \underset{\substack{\text{(due to symm.}\\ \text{set of } \frac{1}{2}\text{P at}\\ \text{x and x}')}}{N_{ij(s)}} - \underset{\substack{\text{(due to antisymm.}\\ \text{set of } \frac{1}{2}\text{P at x}\\ \text{and x}')}}{N_{ij(a)}} \tag{55}$$

It follows, therefore, that

$$\begin{bmatrix} \text{Max. Total Pos. } N_{ij} \\ \\ \text{Max. Total Neg. } N_{ij} \end{bmatrix} = \begin{bmatrix} \sum_{b}^{b'} (\text{Pos. } N_{ij}) + (\text{Max. Pos. } N_{ij})\, F_c \\ \\ \sum_{b}^{b'} (\text{Neg. } N_{ij}) + (\text{Max. Neg. } N_{ij})\, F_c \end{bmatrix} \tag{56}$$

where the concentrated-load factors for moment and for shear are given by

$$\begin{bmatrix} F_c \text{ for moment} \\ F_c \text{ for shear} \end{bmatrix} = \frac{1}{10.8} \begin{bmatrix} 13.5 \\ 19.5 \end{bmatrix} \tag{57}$$

and in Eqs. 56 b to b′ attached to the Σ sign means summation of all positive or negative axial stresses when panel points from b to b′ are loaded such that like-sign stresses are produced.

Impact Stresses

The loaded length L in the impact formula of Eq. 23 will be obtained for either the maximum plus-sign or maximum negative-sign live-load stress by summing up the corresponding panel lengths. Impact stresses are then obtained from

$$\text{Impact stress} = \text{Max. L. L. stress} \left(\frac{50}{L + 125} \right) \tag{58}$$

Maximum and Minimum Design Stresses

These will be determined by summing up dead-load, maximum or minimum live-load (or maximum of opposite sign), and impact stresses in the usual manner.

Provision for Overload Stresses

These stresses are differently stipulated in different design specifications according to the judgment of those who have jurisdiction over formulating the specifications. The 100 percent increase of maximum live-load and impact fraction was stipulated in the 1932 design specifications. Because under this provision there is usually reversal of stresses, both the algebraic sum of one sign and that of the opposite sign are increased by 50 percent of the smaller value. When so increased for overload provision, the results of the analysis of this study check almost identically with those of a study conducted by the Montana State Highway Commission.

Summary of Axial Stresses

Dead-load, maximum live-load, impact, and maximum design stresses together with certain overload and reversal stresses are summarized in Tables 12 and 13. Results of the proposed method are given as well as those of the Montana study for comparison.

As shown in these tables, all dead-load, maximum and minimum (or of opposite sign) live-load, impact, and overload stresses check almost identically with the corresponding results of the Montana State Highway Commission. This further testifies to the validity and soundness of the proposed method.

All numerical values, except the minimum DL stresses for overload provision, were independently computed. To conform with the original design provisions, Montana values were used under "energy method."

TABLE 12

SUMMARY OF MAXIMUM AXIAL STRESSES, NO OVERLOAD

Member	Montana (kips)				Energy Method (kips)			
	D. L.	L. L.	I.	Total	D. L.	L. L.	I.	Total
ab	63.9	42.8	5.4	112.1	63.9	42.8	5.4	112.1
bc	63.9	42.8	5.4	112.1	63.9	42.8	5.4	112.1
cd	70.9	55.1	7.0	133.0	71.0	55.2	7.0	133.2
de	70.9	55.1	7.0	133.0	71.0	55.2	7.0	133.2
ef	-51.6	-32.2	-4.6	-88.4	-51.4	-32.2	-4.6	-88.2
fg	-51.6	-32.2	-4.6	-88.4	-51.4	-32.2	-4.6	-88.2
gh	-40.8	-26.3	-3.8	-70.9	-40.6	-26.3	-3.8	-70.7
hi	-40.8	-26.3	-3.8	-70.9	-40.6	-26.3	-3.8	-70.7
ij	80.7	55.3	9.1	145.1	81.0	55.3	9.1	145.4
jk	80.7	55.3	9.1	145.1	81.0	55.3	9.1	145.4
BC	-85.7	-55.0	-7.0	-147.7	-85.8	-55.0	-7.0	-147.8
CD	-85.7	-55.0	-7.0	-147.7	-85.8	-55.0	-7.0	-147.8
FG	154.0	58.0	6.6	218.6	153.9	57.9	6.6	218.4
GH	154.0	58.0	6.6	218.6	153.9	57.9	6.6	218.4
JK	-96.0	-61.1	-10.	-167.1	-96.2	-61.1	-10.	-167.2
Bb	33.1	33.0	9.9	76.0	33.1	33.0	9.7	75.8
Cc	-4.2	—	—	-4.2	-4.3	—	—	-4.3
Dd	33.6	33.0	9.9	76.5	33.6	33.0	9.7	76.3
Ee	-4.3	—	—	-4.3	-4.3	—	—	-4.3
Ff	33.2	33.0	9.9	76.1	33.2	33.0	9.7	75.9
Gg	-3.2	—	—	-3.2	-3.2	—	—	-3.2
Hh	34.0	33.0	9.9	76.9	34.0	33.0	9.7	76.7
Ii	-4.7	—	—	-4.7	-4.7	—	—	-4.7
Jj	33.3	33.0	9.9	76.2	33.3	33.0	9.7	76.0
Kk	-4.5	—	—	-4.5	-4.5	—	—	-4.5
aB	-90.4	-60.4	-7.7	-158.5	-90.4	-60.5	-7.7	-158.6
De	-70.1	-44.9	-6.0	-121.0	-70.0	-44.9	-6.0	-120.9
eF	117.3	60.4	7.7	185.4	117.2	60.4	7.6	185.2
Fg	-160.0	-74.2	-8.4	-242.6	-160.0	-74.2	-8.4	-242.6
gH	-182.6	-84.2	-9.6	-276.4	-182.6	-84.3	-9.6	-276.5
Hi	123.9	64.5	8.2	196.6	123.9	64.6	8.2	196.7
iJ	-72.9	-49.1	-6.6	-128.6	-72.9	-49.1	-6.6	-128.6

TABLE 13

SUMMARY OF MAXIMUM AXIAL STRESSES, OVERLOAD

Member	Loads	Montana (kips)		Energy Method (kips)	
Bc	D. L.	30.8	21.6	30.8	21.6
	L. L.	75.4	-37.2	75.4	-37.2
	I.	10.8	-5.6	10.8	-5.6
	Reversal	10.6	-10.6	10.6	-10.6
	Total	127.6	-31.8	127.6	-31.8
cD	D. L.	21.9	15.3	21.8	15.3
	L. L.	63.4	-45.0	63.4	-45.0
	I.	9.0	-6.8	9.0	-6.8
	Reversal	18.3	-18.3	18.3	-18.3
	Total	112.6	-54.8	112.5	-54.8
DE	D. L.	-23.5	-16.5	-23.6	-16.5
EF	L. L.	-82.4	53.2	-82.4	53.2
	I.	-10.4	8.8	-10.4	8.8
	Reversal	-22.8	22.8	-22.8	22.8
	Total	-139.1	68.3	-139.2	68.3
HI	D. L.	-36.0	-25.2	-36.2	-25.2
IJ	L. L.	-76.8	42.6	-76.8	42.6
	I.	-12.6	5.4	-12.6	5.4
	Reversal	-11.4	11.4	-11.4	11.4
	Total	-136.8	34.2	-137.0	34.2
Jk	D. L.	25.0	17.5	25.0	17.5
	L. L.	71.4	-48.6	71.4	-48.6
	I.	10.2	-7.4	10.2	-7.4
	Reversal	19.3	-19.3	19.3	-19.3
	Total	125.9	-57.8	125.9	-57.8

Further Merits of Panel-Load Method

Eqs. 54 and 55 give live-load stress in any truss member due to any typical panel load. For the investigation under consideration, a typical full live panel load was used so that the panel-load stresses may be later used to redesign the same bridge under current lane width of 10 ft, although the orginal design was made for the 1932 specifications of 9-ft lane-loading width. The transverse effect of shifting the 9-ft lane-loading width on the 10-ft lane width could have been directly obtained by multiplying the typical live panel load by the previously determined factor of 1.08889.

Influence-line ordinates for all members, although not needed in the proposed method of analysis, can be easily obtained from Eqs. 54 and 55 by simply using unit panel load instead of typical full live panel load. Moreover, the panel-load method makes it extremely expedient to obtain bending stresses (secondary stresses) simultaneous with maximum design axial stresses, and maximum bending stresses (maximum secondary stresses) together with simultaneous axial stresses.

END MOMENTS IN TRUSS MEMBERS

The essence of this study is to develop a unified, streamlined matrix-energy method so that engineers can analyze any rigidly connected truss, determinate or indeterminate, under any combination of loadings, and ascertain in each member the two possible governing states of required internal resistances: (a) under loadings of maximum axial stress—maximum axial stress, the larger of the bending stresses simultaneous with maximum axial stress, and transverse shear simultaneous with maximum axial stress; (b) under loadings of maximum bending stress—axial stress simultaneous with the larger of the maximum bending stresses, the larger of the maximum bending stresses, and transverse shear simultaneous with the larger of the maximum bending stresses. The term "bending stress," as used herein, corresponds to conventional secondary stress. It is evident that whichever of these two states requires the larger section should govern the design.

The problem involved is no longer academic. It deserves more serious practical considerations today than ever before: as longer spans of bridges are built, more brittle high-strength steels are used, welded connections are introduced, steel prestressing is applied, more dynamic effects are experienced from high-speed heavy wheel loads, more economical designs are stressed, and thinner sections, plates, and sheets are called for on plans. To best meet all these exacting demands and to insure public safety at minimum cost, the closest analysis of complicated bridge structures must be made.

Since the earliest introduction of a method for analyzing secondary stresses (1), although at least nine independent methods have been developed, the complicated and tedious analysis of secondary stresses has remained mainly of academic interest and even as such has been only rarely performed. Not many already constructed rigidly connected trusses have been given a secondary-stress analysis. None of particular importance have ever been given as thorough an analysis as required to investigate thoroughly the two governing states.

Even with today's technological development coupled with the availability of high-speed electronic digital computers, generally only a conventional maximum axial-stress analysis is made for rigidly connected trusses. By the time the analyst has achieved this conventional task using Müller-Breslau's (13) principle of influence lines, he is reluctant to undertake a secondary-stress analysis. The chief obstacle has been lack of a straight-forward, unified, streamlined method whereby axial stresses, end moments, and transverse shears will be obtained in one single setup. In this manner, all desired results will be yielded once the problem has been formulated and fed into the computer.

With the approach used in the present study, the formidable task of performing and iterating all necessary calculations for all members of a truss under all conceivable loading conditions will become simple. This advantage is inherent to the proposed method of panel-load superposition because at this stage of the analysis, each axial stress in every member due to each individual panel load as well as both end moments in every member due to each same individual panel load has been determined. The remaining computation of simultaneous end moments and shears under loadings of maximum axial stress and of maximum end moments and simultaneous shears and axial stresses under loadings of maximum bending stress is merely a simple arithmetic chore.

Simultaneous End Moments and Transverse Shears

End moments of a member simultaneous with its maximum axial stress may be readily obtained by the method of substitution used in this study, after converting X_i to M_{ij} according to Eqs. 24 to 27, in a manner analogous to Eqs. 54 and 55 for axial stresses. Thus,

$$\underset{\substack{\text{(due to any P}\\ \text{at any panel}\\ \text{point x)}}}{M_{ij}} = \underset{\substack{\text{(due to symm.}\\ \text{set of } \frac{1}{2}\text{P at}\\ \text{x and x}')}}{M_{ij(s)}} + \underset{\substack{\text{(due to antisymm.}\\ \text{set of } \frac{1}{2}\text{P at x}\\ \text{and x}')}}{M_{ij(a)}} \tag{59}$$

$$\underset{\substack{\text{(due to any P}\\ \text{at panel}\\ \text{point x}')}}{M_{ij}} = \underset{\substack{\text{(due to same P}\\ \text{at panel}\\ \text{point x)}}}{-M_{i'j'}} = \underset{\substack{\text{(due to symm.}\\ \text{set of } \frac{1}{2}\text{P at}\\ \text{x and x}')}}{M_{ij(s)}} - \underset{\substack{\text{(due to antisymm.}\\ \text{set of } \frac{1}{2}\text{P at x}\\ \text{and x}')}}{M_{ij(a)}} \tag{60}$$

Then, by the method of superposition,

$$\text{Simul. L. L. } M_{ij} = \sum_{b}^{b'} M_{ij} + F_c\, M_{ij} \text{ (due to the panel load producing max. } N_{ij}) \tag{61}$$

where b to b$'$ and F_c carry the same significances as defined with Eqs. 56 and 57. And by Eq. 23,

$$\text{Simul. Impact } M_{ij} = \text{Simul. L. L. } M_{ij} \left(\frac{50}{L + 125}\right) \tag{62}$$

Hence,

$$\text{Simul. Total } M_{ij} = \text{D. L. } M_{ij} + \text{Simul. L. L. } M_{ij} \left(1 + \frac{50}{L + 125}\right) \tag{63}$$

Simultaneous transverse shears for any member IJ may be obtained from simultaneous end moments, by Eq. 4, as

$$\text{Simul. } Q_{ij} = \frac{\text{Simul. Total } M_{ij} + \text{Simul. Total } M_{ji}}{L_{ij}} \tag{64}$$

Maximum End Moments and Simultaneous Shears

Maximum end moments represent the state in which the resisting moment at either end of a member reaches its possible maximum by loading certain panel points plus a concentration, all producing moments of the same sign. This state has its own simultaneous axial stresses and transverse shears. One of the two maximum end moments of each member will produce maximum secondary stresses.

Maximum live-load end moments M_{ij} and M_{ji} of any member IJ may be obtained from

$$\begin{bmatrix} \text{Max. L. L. } M_{ij} \\ \\ \text{Max. L. L. } M_{ji} \end{bmatrix} = \begin{bmatrix} \sum_{b}^{b'} \begin{matrix}\text{Pos.}\\ \text{or}\\ \text{Neg.}\end{matrix} M_{ij} + \left(\text{Max. } \begin{matrix}\text{Pos.}\\ \text{or}\\ \text{Neg.}\end{matrix} M_{ij}\right) F_c \\ \\ \sum_{b}^{b'} \begin{matrix}\text{Pos.}\\ \text{or}\\ \text{Neg.}\end{matrix} M_{ji} + \left(\text{Max. } \begin{matrix}\text{Pos.}\\ \text{or}\\ \text{Neg.}\end{matrix} M_{ji}\right) F_c \end{bmatrix} \tag{65}$$

in which the summations are to cover all positive or all negative M_{ij}, whichever gives the larger maximum live-load M_{ij}. M_{ji} is treated in the same manner. F_c and b to b′ are as defined before.

Then, maximum total end moments M_{ij} or M_{ji} of any member IJ is given by

$$\text{Max. Total } M_{ij} \text{ or } M_{ji} = \text{D. L. } M_{ij} \text{ or } M_{ji} + \text{Max. L. L. } M_{ij} \text{ or } M_{ji} \left(1 + \frac{50}{L + 125}\right) \tag{66}$$

Simultaneous transverse shear in member IJ under this state is given by

$$Q_{ij} = \frac{\text{Max. Total } M_{ij} \text{ or } M_{ji} + \text{Simul. Total } M_{ji} \text{ or } M_{ij}}{L_{ij}} \tag{67}$$

Governing Maximum Design Stress State

Under usual conditions, especially when the truss members are slender and light, the state under maximum axial-stress loadings will govern the design. But if the members are extremely short and heavy and the joints are enormously rigid, the state under maximum end-moment loadings may require a larger section. To be absolutely sure, both states of stress should be analyzed and compared by the sections required.

Summary of End Moments, Shears, and Unit Fiber Stresses

A summary of end moments, shears, and unit fiber stresses is given in Table 14 for all members of the three-span continuous highway bridge truss. Values are tabu-

TABLE 14

SUMMARY OF MAXIMUM STRESS STATES

Member	M. at End	Maximum Axial-Stress Loading: M. (kips-in.)	Max. Axial Stress (ksi)	Bend. Stress (ksi)	Max. Fiber Stress (ksi)	Shear (kips)	Maximum End-Moment Loading: M. (kips-in.)	Max. Bend. Stress (ksi)	Axial Stress (ksi)	Max. Fiber Stress (ksi)	Shear (kips)
Lower chord:											
ab	a	-37.88					-37.99				
	b	-68.80	9.28	1.61	10.89	-0.394	-69.20	1.62	8.86	10.48	-0.396
bc	b	61.10	9.28	1.43	10.71	0.240	64.70	1.51	7.16	8.67	0.298
	c	3.70					16.10				
cd	c	-9.63					-19.53				
	d	-82.87	11.00	1.94	13.03	-0.343	-85.17	1.99	10.32	12.31	-0.398
de	d	91.60	11.04	2.14	13.18	0.553	92.40	2.16	10.32	12.48	0.553
	e	57.71					56.80				
ef	e	6.83	-7.31	-0.16	-7.47	0.001	-14.91				
	f	-6.47					-44.43	-1.04	-3.16	-4.20	-0.220
fg	f	21.79					52.91				
	g	103.58	-7.31	-2.42	-9.73	0.465	135.00	-3.15	-5.18	-8.33	0.697
gh	g	-103.22	-5.86	-2.41	-8.27	-0.477	-136.20	-3.18	-4.06	-7.24	-0.715
	h	-25.73					-56.90				
hi	h	7.85	-5.86	-0.18	-7.74	0.009	48.41	-1.13	-2.49	-3.62	0.249
	i	-5.47					18.82				
ij	i	-65.26					-67.92				
	j	-105.75	12.10	2.47	14.57	-0.635	-106.09	2.48	11.13	13.61	-0.645
jk	j	100.89	12.06	2.36	14.42	0.551	101.86	2.38	11.39	13.77	0.392
	k	47.86					48.97				
Upper chord:											
BC	B	-23.98					-23.98				
	C	-84.34	-8.82	-1.01	-9.83	-0.401	-84.34	-1.01	-8.82	-9.83	-0.401
CD	C	-12.57	-8.82	-0.25	-9.07	-0.120	83.86	-1.00	-8.10	-9.10	0.296
	D	-19.75					-3.98				
DE	D	43.10	-4.19	-0.51	-4.19	0.032	59.21	-0.71	-3.21	-3.92	0.181
	E	-34.39					-10.27				
EF	E	53.77					54.93				
	F	47.26	-4.19	-0.95	-5.14	0.374	50.49	-1.02	-3.50	-4.52	0.389
FG	F	33.82					44.46				
	G	147.04	13.04	1.76	14.80	0.669	150.06	1.80	12.88	14.68	0.720
GH	G	-149.31	13.04	1.78	14.82	-0.738	-151.45	1.81	12.84	14.65	-0.768
	H	-49.98					-56.17				
HI	H	-52.36	-4.83	-1.05	-5.88	-0.392	-59.96	-1.21	-3.71	-4.92	-0.416
	I	-53.51					-52.46				
IJ	I	32.84					10.66				
	J	-59.90	-4.83	-0.72	-5.55	-0.100	-76.06	-0.91	-3.47	-4.38	-0.242
JK	J	51.79	-9.98	-0.62	-10.60	0.153	54.49	-0.65	-9.26	-9.91	0.175
	K	-10.35					-7.33				
Vert.:											
Bb	B	5.06					12.60				
	b	6.97	8.31	0.76	9.07	0.045	14.16	1.54	3.63	5.17	0.099
Cc	C	0.96					6.61				
	c	0.96	-0.51	-0.13	-0.65	0.007	6.92	-0.95	-0.51	-1.46	0.048
Dd	D	-7.71					12.88				
	d	-8.16	8.37	0.89	9.26	-0.054	13.63	1.48	5.14	6.62	-0.090
Ee	E	-15.68	-0.45	-1.41	-1.86	-0.095	-26.53	-2.39	-0.45	-2.84	-0.163
	e	-13.51					-23.89				
Ff	F	-13.57					-18.12				
	f	-14.89	8.32	1.62	9.94	-0.088	-20.05	2.18	3.79	5.97	-0.118
Gg	G	1.19	-0.33	-0.11	-0.44	0.007	5.72	-0.52	0.33	-0.85	0.034
	g	1.09					5.61				
Hh	H	12.86					17.60				
	h	14.25	8.41	1.55	9.96	0.079	19.53	2.12	5.16	7.28	0.109
Ii	I	14.59	-0.49	-1.31	-1.80	0.079	24.69	-2.22	-0.49	-2.71	0.136
	i	12.56					22.22				
Jj	J	3.82					4.53				
	j	4.70	8.33	0.51	8.84	0.024	5.57	0.61	7.56	7.17	0.029
Kk	K	0.00	-0.47	0.00	-0.47	0.000	0.54	-0.05	-0.47	-0.52	0.003
	k	0.00					0.51				
Diag.:											
aB	a	-42.55	-9.47	-0.86	-10.33	-0.143	-42.55	-0.86	-9.47	-10.33	-0.143
	B	-11.99					-11.99				
Bc	B	8.68	8.43	0.38	8.81	0.014	9.54	0.42	7.73	8.15	0.029
	c	-3.53					1.71				
cD	c	3.87					0.47				
	D	-10.22	6.61	0.45	7.06	-0.016	-12.42	0.55	3.69	4.24	-0.028
De	D	-1.00					-6.83				
	e	-24.26	-10.02	-0.57	-10.59	-0.063	-27.04	-0.63	-9.04	-9.67	-0.085
eF	e	-22.48					-20.34				
	F	-23.93	12.65	0.50	13.15	-0.110	-28.59	0.60	11.66	12.26	-0.116
Fg	F	-42.85					-42.21				
	g	43.03	-10.29	-0.66	-10.95	0.0004	54.52	-0.84	-8.55	-9.39	0.029
gH	g	-41.07					-39.34				
	H	53.18	-10.46	-0.64	-11.10	0.028	60.35	-0.72	-10.28	-11.00	0.046
Hi	H	29.06	13.44	0.61	14.05	0.115	32.60	0.68	12.82	13.50	0.134
	i	21.42					25.80				
iJ	i	26.64	-8.78	-0.56	-9.34	0.051	29.25	-0.61	-8.25	-8.86	0.051
	J	-4.26					-6.76				
Jk	J	20.86	5.46	0.49	5.95	0.043	26.26	0.61	2.19	2.80	0.059
	k	-1.92					-0.10				

TABLE 15

MAXIMUM BENDING STRESSES IN STRUT VERTICALS

Member	End	Max. Axial-Stress Loading[a]	Max. End-Moment Loading[b]
Cc	C	25.5	119
	c		127[c]
Ee	E	316[c]	52.7
Gg	G	33.3	93.2
Ii	I	267	49.7
Kk	K	0	10.6
	k		

[a]Bending stress in percent of maximum axial stress.
[b]Maximum fiber stress (%) over that due to maximum axial-stress loading.
[c]Highest.

lated for two maximum-stress states: (a) under maximum axial-stress loading—maximum axial stress and simultaneous bending stress, and (b) under maximum end-moment loading—maximum bending stress and simultaneous axial stress. Simultaneous shears are given for each stress state. Average and maximum shearing stresses may be obtained by established methods.

Discussion of Results

In Table 14 are underlined the larger of the maximum fiber stresses which should govern the design of the 39 members in one-half of the truss under each of the maximum stress loadings. Although for two members, both maximum stress states produce identical maximum fiber stresses, the maximum axial-stress loading (or state) produces the larger maximum fiber stress in 31 members, and there are as many as six members whose maximum fiber stresses are governed by the maximum end-moment loading (or state) conditions.

The hypothesis previously advanced that the governing state is that of maximum bending stress (maximum secondary stress) and simultaneous axial stress and shearing stress is fully substantiated. In the three-span continuous truss under analysis, the following observations are pertinent:

1. Highest bending stresses (secondary stresses) occur in strut verticals Cc, Ee, Gg, Ii, and Kk (Table 15). In all these strut verticals, maximum end-moment loadings govern. The extraordinarily high bending stresses in these strut verticals would be very serious if these members were not governed by minimum size or slenderness ratio requirements.
2. Bending stresses in hangers increase toward the intermediate supports and reach a maximum of 19.5 percent in Ff of their maximum axial stresses. Maximum axial-stress loadings invariably govern. Bending stresses in these members are generally lower than those in strut verticals.
3. Among compression diagonals, the end posts aB and a'B' have the higher bending stresses which, however, amount to only 9.1 percent of their maximum axial stresses. Both maximum axial-stress and end-moment loadings produce identical results.
4. Among tension diagonals, those nearest the center of the middle span have the higher bending stresses which amount to only 9.0 percent of their maximum axial stresses under loadings for these stresses.
5. Among upper chords, the end upper-chord members BC and B'C' have the higher bending stresses at their inner ends, which amount to 11.5 percent of their maximum axial stresses. Both maximum axial-stress and maximum end-moment loadings produce the same results. The next upper-chord members, CD and C'D', are governed by the maximum end-moment loadings, but their maximum fiber stresses under these loadings are only 0.3 percent over those produced by the maximum axial-stress loadings.
6. Among the lower chords in this continuous truss, those in compression adjacent to the intermediate supports have medium-high bending stresses, i.e., 33.1 percent in fg and f'g', and 41.1 percent in gh and g'h' where g and g' are intermediate supports. Both members are governed by the maximum axial-stress loadings.
7. These stress observations apply only to vertical loadings of the continuous truss under analysis. Bending stresses would be increased when both chord members are analyzed to transmit wind loads (centrifugal forces) resisted by upper and lower lateral systems or when verticals and end posts are analyzed to transmit sway portal action due to lateral loads.

It is thus clear that in the design of any important structure, in any critical design, or in any important investigation, bending (or secondary) stresses in rigidly connected trusses should never be taken as negligible, nor should maximum axial stress plus simultaneous bending stress be assumed as adequate to dictate a safe design.

AUTOMATIC LOGICAL CHECKS

The development up to this point has been in proper logical sequence which allows an automatic computation program to be written from a few basic parameters (span lengths, panel lengths, truss heights, roadway width, median-divider width, sidewalk width, live load, impact-fraction formula, modulus of elasticity, and Poisson's ratio) for maximum design axial stresses and simultaneous end moments, or maximum end moments and simultaneous axial stresses, with simultaneous transverse shears.

Individual programs must be written for each particular electronic digital computer in its most efficient machine language and must be adapted by the individual engineer and programmer to their own machine. To include any specific program would be extraneous to this basic investigation and would detract from its importance.

However, for the sake of helping those who are going to apply the proposed method and associated techniques, some intermediate, sub-final, and final automatic logical checks are given. To provide more general applicability, an indeterminate highway bridge truss of the continuous type will be assumed as the hypothetical analysis to be programmed.

Accuracy Desired

As is inherent in any indeterminate structural analysis, the terms and coefficients involve very small linear and angular displacements and their arithmetic operations. The solution involves rather large sets of simultaneous equations. Unless more significant figures are retained in the initial and intermediate stages, the final answers may not have a three-figure accuracy. For instance, the subtraction of two eight-digit figures differing by the last three digits would become zero if only five digits were kept, and the solution of a large set of simultaneous equations would generate large rounding-off errors. For this reason, it is not only desirable, but even imperative, to use as many digits as the available computer can accommodate.

Logical Checks

Intermediate, sub-final, and final answers should be strategically checked by logical criteria, either mathematical, statical, or according to conservation of energy. At any stage of the machine computation, logical checks can be devised and incorporated into the program; typing out the checking indication may be programmed. In any plane truss, for instance:

1. Computation of lengths of inclined members may be carried out by any process but the results must conform to the Pythagorean rule:

$$a^2 + b^2 = c^2 \tag{68}$$

2. Reactions may be determined by any determinate or indeterminate methods, as the case may be, but they must satisfy statical requirements:

$$\left.\begin{aligned} \Sigma X &= 0 \\ \Sigma Y &= 0 \end{aligned}\right\} \tag{69}$$

where X and Y include, respectively, all horizontal and vertical components of all applied loads and reactions if the entire truss is considered, or of all applied loads and reactions on one side of a section and all stresses on the same side of the same section if such a section is under consideration.

3. Moments may be determined in any way that is expedient, but

$$\Sigma M = 0 \tag{70}$$

must hold when moments are taken about any joint, or about any point through any section.

4. To test for correct internal strain energy, its total must be a minimum, that is

$$\frac{\partial U}{\partial X} = 0 \tag{71}$$

A change from negative to positive implies that nature does not do any foolish work or any more than the minimum or least work.

5. In axial-stress computation with parallel chords at least in center panels, unless the center vertical is a hanger in a pony or through truss or a strut in a deck truss, its live-load axial stress (corresponding to primary stress) must be zero, or

$$N_{Oo} = 0 \tag{72}$$

where O and o are, respectively, upper and lower center joints.

6. For end joint equilibrium, under the present lane loading with shear concentration much larger than moment concentration, the compressive stress in the end-post is governed by shear, and hence, the maximum stress in the end segment of the lower chord in a through truss will not be governed by maximum moment, but by

$$\text{Max. } N_{ab} = -\text{Max. } N_{aB} \frac{L_{ab}}{L_{aB}} \tag{73}$$

where a and B, respectively, denote end and hip joints, and b the lower joint of the first hanger.

7. At all T-shape joints, e.g., j or J, chord stresses (corresponding to primary stresses) are always equal, i.e.,

$$\left.\begin{aligned} N_{ij} &= N_{jk} \\ \text{or} \qquad N_{IJ} &= N_{JK} \end{aligned}\right\} \tag{74}$$

8. With the simplified energy method, all axial stresses, N, should check with the primary stresses, S, by the conventional method, i.e.,

$$N_{ij} = S_{ij} \tag{75}$$

where ij denotes any member.

9. With the exact energy method, the resulting exact N's are in equilibrium under the principle of extended methods of moments, shears, and joints, i.e., when all components of N, M, Q are considered, or

$$F(N, M, Q) = 0 \tag{76}$$

10. With the simplified energy method, all bending stresses, f, should check with the secondary stresses, s, by any of the classical methods, i.e.,

$$\left.\begin{array}{l} f_{ij} = s_{ij} \\ f_{ji} = s_{ji} \end{array}\right\} \quad (77)$$

11. With the energy method, either exact or simplified, all bending stresses, f, should satisfy Eq. 72, a criterion for minimum internal strain energy, or least work of deformation, in conformity with which nature works.

12. In the method of panel-load superposition, when the panel load is replaced by unity, the summation of reactions, R, should give

$$\Sigma R = 1 \quad (78)$$

13. With any method of inversion for solving any set of nonhomogeneous linear equations, unless the matrix is singular, i. e.,

$$|a_{ij}| = 0,$$

$$\left[a_{ij}\right]^{-1} \left[a_{ij}\right] = U \quad (79)$$

In European usage, U is denoted as E or I.

14. An overall check of all independent, i, and dependent, d, unknown end moments, M_i and M_d, is given by

$$\Sigma M_i = -\Sigma M_d \quad (80)$$

15. In general, maximum design N, M, Q for all members meeting at a joint or acting through a section give the inequalities:

$$\begin{bmatrix} \Sigma X \\ \Sigma Y \\ \Sigma M \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (81)$$

No attempt should ever be made to check the results in this manner.

Other logical check criteria may be added as suggested by the premises of a particular problem.

CONCLUSIONS

The entire investigation stands for its full justification on the ground that:

1. Engineers have been aware of the "illogical (loath to use the word 'wrong') assumption" in analyzing "rigidly connected trusses" as "frictionlessly pin-connected" for 85 years; why then should the status quo be indefinitely continued by any progressive engineer except the ignorant?
2. Nature has never made the artificial distinction between "primary stresses" and "secondary stresses" as the prevalent engineering parlance has labeled them; they exist by their very nature as axial stresses, bending stresses, and shearing stresses; why then should this misnomer not be dispelled?
3. With the larger versions of modern electronic computers, rigidly connected trusses can be analyzed as a chain of rigid frames in its true nature by the proposed method and associated techniques in not over a few minutes difference in time as compared with the conventional methods; why then should engineers continue to use nineteenth-century methods?

Suffice it to state only a few without exhausting the enumeration.

On the ground as firm as this, the investigators have proceeded:

1. To expound a new exact matrix-energy method for analyzing all stresses in rigidly connected trusses (exact, for the method is compatible with the nature-behavior of any physical system), this being the rational approach for refined analysis required in critical designs or investigations;

2. To synthesize the elegance and efficacy of matrix algebra founded by Authur Cayley with the necessary compatible condition of minimum strain energy in elastic structures propounded by Alberto Castigliano, their technological union constituting the most powerful tool in structural analysis, especially in this electronic age and when the coefficient matrix is symmetrical as proved in Appendix A;

3. To deduce a simplified matrix-energy method for analyzing all stresses in rigidly connected trusses to minimize time and effort for ordinary engineering purposes;

4. To set forth the extended methods of moments, shears, and joints, so that strain-energy expressions of the truss members will be compatible with the exact matrix-energy method expounded;

5. To demonstrate the truth in the validity and thoroughness of the exact matrix-energy method in the case of a determinate truss by comparing results with previous authorities;

6. To reveal the closeness and dependability of the simplified matrix-energy method, again in the case of a determinate truss, by comparing results with previous authorities;

7. To advance the method of panel-load superposition for continuous trusses in lieu of the classical influence-line method of Müller-Breslau, this method being especially adapted to longer span bridges where lane loading plus concentrations govern as a rule;

8. To resort to the method of substitution of an unsymmetrical loading by a set of symmetrical loading and a conjugate set of antisymmetrical loading, as a powerful analytical tool to reduce 2n unknowns to n unknowns, which is indispensable in using the smaller versions of computers;

9. To introduce further a method of transformation from one type of many more unknowns to another type of many fewer unknowns, which enables the solution of a still larger set of equations in a much smaller computer as demonstrated in Appendix B;

10. To elucidate the problem of solving very large sets of simultaneous equations, with reference also to four matrix methods developed by the principal investigator;

11. To apply the proposed simplified matrix-energy method and relevant techniques set forth above to the analysis of a three-span continuous highway bridge truss with inclined upper chords, and to demonstrate the validity and accuracy of the results vs those of the Montana State Highway Commission;

12. To pronounce that the methods advanced make it possible and expedient to determine two maximum stresses states: (a) maximum axial stress plus the larger simultaneous bending stress and simultaneous transverse shear, and (b) the larger of the maximum bending stress plus simultaneous axial stress and simultaneous transverse shear;

13. To indicate that either of these two stress states has the same frequency of occurrence, and that either has the likelihood to dictate the larger requirement for the section of a member; hence,

14. To conclude that either of these two stress states may govern the design, and that both should be computed in cases of important critical designs and investigations; and

15. To establish automatic logical checks for programmed electronic computation throughout its initial, intermediate, sub-final, and final stages.

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Appendix A

INDEPENDENT PROOF FOR SYMMETRY OF COEFFICIENT MATRIX

By generalizing the n unknowns in the matrix-energy method as the column vector $\{X_i\}$, it is seen, from the extended methods of moments, shears, and joints for determining the axial stress in each member of the truss, that the axial stress N_{ij} of any member IJ is represented by

$$N_{ij} = \Sigma q_i X_i + S_{ij} \tag{82}$$

where q_i is the coefficient of the unknown X_i, and S_{ij} corresponds to the conventional primary stress in member IJ.

Knowing the expression of N_{ij}, we conclude, from Eqs. 5 and 6, that the expression of the total strain energy, U, of a given loaded truss is an algebraic expression of the second degree in $\{X_i\}$. After collecting like terms in the expression of U, the total strain energy becomes

$$U = \Sigma c_{ii} X_i^2 + \Sigma c_{ij} X_i X_j + \Sigma c_i X_i + p \tag{83}$$

where c_{ii} is the coefficient of X_i^2, c_{ij} is the coefficient of $X_i X_j$, c_i is the coefficient of X_i, p is the constant term, and $i \neq j$.

By Castigliano's second theorem, we have

$$\frac{\partial U}{\partial X_i} = 0 = 2c_{ii}X_i + c_{ij}X_j + \Sigma c_{ik}X_k + c_i \tag{84}$$

and

$$\frac{\partial U}{\partial X_j} = 0 = 2c_{jj}X_j + c_{ij}X_i + \Sigma c_{kj}X_k + c_j \tag{85}$$

where c_{ik}, c_{jj}, c_{kj}, and c_j are, respectively, the coefficients of X_iX_k, X_j^2, X_kX_j, and X_j in the expression of U; and $i \neq k \neq j$. Therefore, if a_{ij} is the coefficient X_j in the equation obtained from $\frac{\partial U}{\partial X_i} = 0$, and a_{ji} is the coefficient of X_i in the equation obtained from $\frac{\partial U}{\partial X_j} = 0$, Eqs. 84 and 85 yield

$$a_{ij} = a_{ji} = c_{ij} \tag{86}$$

Eq. 86 implies that if the equation obtained from $\frac{\partial U}{\partial X_i} = 0$ is arrayed in the ith row of the set of simultaneous equations, the coefficient matrix $[a_{ij}]$ is always symmetrical about its main diagonal.

Appendix B

DEMONSTRATION OF IDENTICAL RESULTS

The purpose of this demonstration is to show that (a) the simplified matrix-energy method yields identical results as (b) the simplified Manderla's method, (c) the completely transformed energy method, and (d) the partially transformed energy method.

For the sake of mere demonstration, a very simple rigidly connected truss has been chosen as shown in Figure 14 with all dimensions and section properties indicated. The problem is to determine the unknown end moments in all truss members by these four methods and compare their results.

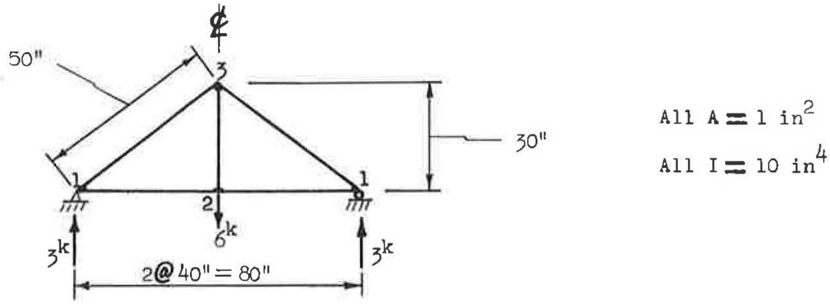

Figure 14.

Solution by Simplified Matrix-Energy Method

1. Choice of Independent Unknowns—The three statically independent unknowns are

$$\lfloor M_{12} \quad M_{21} \quad M_{31} \rfloor = \lfloor X_1 \quad X_2 \quad X_3 \rfloor$$

2. Determination of Axial Stresses—Axial stresses are determined by using the extended methods of moments, shears, and joints. Thus, for N_{12}, taking the free body diagram to the left of member 2-3, by $\Sigma M_3 = 0$, we have

$$\lfloor -N_{12} \quad X_2 \quad X_3 \quad 3 \rfloor \{30 \quad 1 \quad 1 \quad 40\} = 0$$

from which

$$N_{12} = 0.03\,(X_2 + X_3) + 4$$

Similarly,

$$N_{13} = -0.015X_1 - 0.0416X_2 - 0.026X_3 - 5$$

and

$$N_{23} = 0.05(X_1 + X_2) + 6$$

3. Tabulation of Constants for Strain-Energy Expressions:

Member	L/A	L/16
1-2	40	0.6
1-3	50	0.83
2-3	30	0.5

4. Formulation of Strain Energy:

$$E\,U = \sum\left[\frac{1}{2}\,\frac{L_{ij}}{A_{ij}}\,N_{ij}^2 + \frac{L_{ij}}{6I_{ij}}\left(M_{ij}^2 - M_{ij}\,M_{ji} + M_{ji}^2\right)\right]$$

Specific equations are given in Table 16.

5. Simultaneous Equations and Their Solution—Three simultaneous equations will be formed by taking partial derivatives of one-half the strain energy in the truss with respect to each independent unknown as stated in the simplified matrix-energy method. For example,

$$\frac{1}{2}E\,\frac{\partial U}{\partial X_1} = 0 = 0.6(2X_1 - X_2) + 50(-0.015)(-5) +$$

$$0.83(2X_1 + X_3) + \frac{1}{2}(30)(0.05)(6)$$

TABLE 16

Member	E Times Strain Energy
1-2	$\frac{1}{2}(40)\left[0.03(X_2 + X_3) + 4\right]^2 + 0.6(X_1^2 - X_1X_2 + X_2^2)$
1-3	$\frac{1}{2}(50)\left[-0.015X_1 - 0.0416X_2 - 0.026X_3 - 5\right]^2 + 0.83(X_1^2 + X_1X_3 + X_3^2)$
2-3	$\frac{1}{2}(30)\left[0.05(X_1 + X_2) + 6\right]^2 + 0$
	Use only one-half of the strain energy in this member by virtue of symmetry.

After taking similar derivatives with respect to X_2 and X_3, the following matrix equations result:

$$\begin{bmatrix} 3 & -0.6 & 0.83 \\ -0.6 & 1.3 & 0 \\ 0.83 & 0 & 1.6 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = -\begin{bmatrix} 8.25 \\ 20.25 \\ 12 \end{bmatrix}$$

Being non-singular, the matrix has the solution:

$$\lfloor X_1 \quad X_2 \quad X_3 \rfloor = -\lfloor 5.5 \quad 17.9375 \quad 4.45 \rfloor$$

<u>Solution by Simplified Manderla's Method</u>

1. Unknowns—The same unknown moment vector $\lfloor X_1 \quad X_2 \quad X_3 \rfloor$ used in the energy method will be found here. Since these unknowns may be expressed in terms of tangent rotations, they result in one unknown reference τ vector $\lfloor \tau_{12} \quad \tau_{21} \quad \tau_{31} \rfloor$ at joints 1, 2, and 3. Using τ and $\Delta\alpha$ as defined for the method of transformation, the $\Delta\alpha$'s may be evaluated by the usual angle change formula. Thus,

$$E\,\Delta\alpha_{13} = \left\lfloor \left(f_a - f_b\right) \quad \left(f_a - f_c\right) \right\rfloor \left\{\cot\gamma \quad \cot\beta\right\}$$

where f is the unit stress and $\lfloor \alpha\ \beta\ \gamma \rfloor$ and $\lfloor a\ b\ c \rfloor$ are as defined in Figure 15.

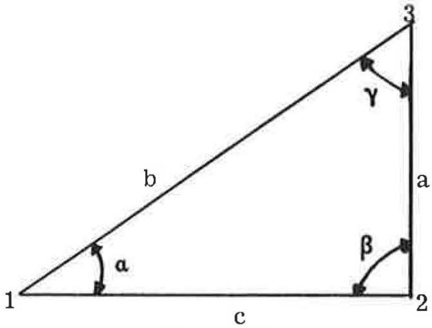

Figure 15.

2. Tabulation of Unit Stresses:

Member	N/A (ksi)
3-1′, 1-3	-5.0
2-1′, 1-2	4.0
2-3	6.0

3. Expressing All τ's in Terms of Reference τ's—In joint 1, the reference τ is τ_{12}. Then,

$$E\,\Delta\alpha_{13} = \lfloor (6+5) \quad (6-4) \rfloor \begin{Bmatrix} \frac{3}{4} & 0 \end{Bmatrix} = 8.25$$

Therefore, at joint 1,

$$E\begin{bmatrix} \tau_{12} \\ \tau_{13} \end{bmatrix} = \begin{bmatrix} E\tau_{12} \\ E\tau_{12} + 8.25 \end{bmatrix}$$

At joints 2 and 3, by symmetry,

$$E\begin{bmatrix} \tau_{23} \\ \tau_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying the angle change formula, we have then:

$$\begin{array}{cc} \text{At joint} & \text{At joint} \\ 2 & 3 \end{array}$$
$$E\begin{bmatrix} \tau_{21}, & \tau_{31} \\ \tau_{23} & \tau_{32} \\ \tau_{21} & \tau_{31} \end{bmatrix} = \begin{bmatrix} 20.25 & -12 \\ 0 & 0 \\ -20.25 & 12 \end{bmatrix}$$

4. Formulation of Equations and Evaluation of Unknown End Moments—The unknown end moments may be expressed as a function of τ's; thus,

$$X_k = M_{ij} = \frac{2EI_{ij}}{L_{ij}}\left(2\tau_{ij} + \tau_{ji}\right)$$

By joint equilibrium, at joint 1,

$$\Sigma M_{ij} = 0 = M_{12} + M_{13}$$

where j represents far end joints. On substituting M_{ij} in terms of τ's, we get

$$\frac{2(10)}{40}\left(2E\tau_{12} - 20.25\right) + \frac{2(10)}{50}\left[2\left(E\tau_{12} + 8.25\right) - 12\right] = 0$$

from which

$$E\tau_{12} = 4.625$$

All the unknown end moments may now be found by back substitution, resulting in:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = -\begin{bmatrix} 5.5 \\ 17.9375 \\ 4.45 \end{bmatrix}$$

These results are identical with those of the energy method.

It should be noted that due to inherent simplicity, no simultaneous equations need be solved here in using Manderla's method. An unsymmetrical truss under any loading will result in a set of n equations in n unknowns, where n equals the number of joints in the truss.

Solution by Method of Complete Transformation

In this method, the unknown end moments will be written in terms of τ. There result, in this particular case, three equations involving one unknown τ_{12}. Any one of these equations will yield the correct value of $E\tau_{12}$.

1. End Moments as Functions of τ:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} M_{12} \\ M_{21} \\ M_{31} \end{bmatrix} = 2 \begin{bmatrix} \frac{10}{40}\left[2(E\tau_{12}) - 20.25\right] \\ \frac{10}{40}\left[2(-20.25) + E\tau_{12}\right] \\ \frac{10}{50}\left[2(-12) + (E\tau_{12} + 8.25)\right] \end{bmatrix}$$

2. Transformation of Energy Equations—Making use of these equations, the set of simultaneous equations of the matrix-energy method may be transformed into a new set as functions of the mono-unknown $E\tau_{12}$:

$$\begin{bmatrix} 3 & -0.6 & 0.83 \\ -0.6 & 1.3 & 0 \\ 0.83 & 0 & 1.6 \end{bmatrix} \overset{X_i \,=}{\begin{bmatrix} E\tau_{12} - 10.125 \\ 0.5\,E\tau_{12} - 20.25 \\ 0.4\,E\tau_{12} - 6.3 \end{bmatrix}} = -\begin{bmatrix} 8.25 \\ 20.25 \\ 12 \end{bmatrix}$$

The solution of any one of these equations yields $E\tau_{12} = 4.625$. Back substitution into the column vector X_i yields:

$$\lfloor X_1 \quad X_2 \quad X_3 \rfloor = -\lfloor 5.5 \quad 17.9375 \quad 4.45 \rfloor$$

which are identical with the results of the simplified energy method and the simplified Manderla's method.

There is no advantage in using the completely transformed energy method since, after the end moments have been expressed in terms of the reference τ's, it is much easier to obtain relations between the end moments by using joint equilibrium equations than by using the simplified energy method.

Solution by Method of Partial Transformation

This method becomes useful in case it is necessary to reduce the number of simultaneous equations by a small number so that an existing program may be used.

The number of unknowns that will be reduced by substituting for part of X_i's depends on the configuration of the truss and the end moments chosen as unknowns. In the present simple case, X_1 and X_3 will be transformed resulting in two simultaneous equations with $E\tau_{12}$ and X_2 as unknowns, thus

$$\begin{bmatrix} 3 & -0.6 & 0.83 \\ -0.6 & 1.3 & 0 \end{bmatrix} \begin{array}{c} X_i \\ \begin{bmatrix} E\tau_{12} - 10.125 \\ X_2 \\ 0.4\ E\tau_{12} - 6.3 \end{bmatrix} \end{array} = - \begin{bmatrix} 8.25 \\ 20.25 \end{bmatrix}$$

which by simplification becomes:

$$\begin{bmatrix} 10 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} E\tau_{12} \\ X_2 \end{bmatrix} = \begin{bmatrix} 82.125 \\ -81 \end{bmatrix}$$

whose solution gives

$$\lfloor E\tau_{12} \quad X_2 \rfloor = \lfloor 4.625 \quad -17.9375 \rfloor$$

By back substitution into the X_i vector, the complete solution becomes

$$\lfloor X_1 \quad X_2 \quad X_3 \rfloor = - \lfloor 5.5 \quad 17.9375 \quad 4.45 \rfloor$$

which is identical with all preceding solutions.

Conclusions

This demonstration has conclusively shown that the simplified energy method yields results identical with the simplified Manderla's method which is the forerunner of the modern slope-deflection method. Thus, the simplified energy method has the same accuracy as the classical methods in the determination of secondary stresses, but it has dispensed with the unjustified assumption of analyzing rigidly connected trusses as ideally pin jointed in the determination of primary stresses, especially if its exact counterpart is applied.

It has also been demonstrated that the completely transformed and the partially transformed energy methods are equally valid, and each has its particular usefulness in reducing more or less unknowns to enable solution of a larger set of equations by a smaller computer.