Deterministic Aspects of Freeway Operations and Control

DONALD R. DREW, Associate Research Engineer, Texas Transportation Institute

• AMONG the important problems arising from the population explosion is that of congestion. Although this overcrowding manifests itself in virtually every aspect of modern life, nowhere is it as dramatically exhibited as on our streets and highways. The most vigorous attempt to eliminate traffic congestion was the development of the freeway, a concept based on (a) the reduction of vehicle-to-vehicle conflicts, (b) elimination of vehicle-to-pedestrian conflicts, and (c) elimination of delay-producing traffic control devices. Still, practically all major cities are troubled with severe peak hour congestion on newly completed freeways.

Previous studies have shown that a relatively small increase in traffic demand on an already heavily loaded expressway can have a very detrimental effect on the operating conditions for all traffic on the facility. Speeds and volumes are reduced, densities and travel times are increased, and the highway immediately loses much of its efficiency. Theoretically, it seems desirable to either ration or completely deny access to the freeway at certain locations.

The automatic evaluation of freeway traffic flow will be a vital element of any future control system. Research must be directed toward the evaluation of the use of surveillance and sensing equipment, and the simultaneous investigation of those characteristics of traffic flow related to freeway congestion which can be determined and treated by such equipment. The complexities and manifestations of freeway traffic congestion are

comfort. These factors are influenced by such additional variables as traffic demand, traffic composition, lane occupancy, highway geometrics and the drivers' desired speeds. Before it can be decided just what level of efficiency is economically feasible, or stated another way, how much congestion should be tolerated during peak periods, congestion must be defined quantitatively in terms of known and measurable parameters of traffic flow theory.

In recent years, a number of descriptive theories of vehicular traffic have been put forward. These theories are based on mathematical models of two basic types: deterministic and stochastic. Included in the first category are the continuous flow models and individual vehicle models which describe the macroscopic and microscopic properties, respectively, of the traffic flow phenomena. Included in the second group are the probability distribution hypotheses and queueing theory.

GENERALIZATION OF DETERMINISTIC MODELS OF TRAFFIC FLOW

If vehicular traffic is assumed to behave as a one-dimensional compressible fluid of concentration (density), k, and fluid velocity, u, then the conservation of vehicles is explained by

$$\frac{\partial k}{\partial t} + \frac{\partial (ku)}{\partial x} = 0 \qquad (1)$$

Taking the derivative of the product in the second term yields

Paper sponsored by Committee on Freeway Operations.

$$\frac{\partial k}{\partial t} + \frac{u \partial k}{\partial x} + \frac{k \partial u}{\partial x} = 0 \tag{2}$$

It is well established in the theory of traffic flow that vehicular velocity varies inversely with the concentration of vehicles,

$$u = f(k) \tag{3}$$

As a consequence of Eq. 3,

$$\frac{\partial u}{\partial k} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial k} = \frac{du}{dk} = u' \tag{4}$$

Solving for $\partial u/\partial x$ from Eq. 4 and substituting in Eq. 2, one obtains the following equation of continuity for single-lane vehicular traffic flow,

$$\frac{\partial k}{\partial t} + [u + ku'] \frac{\partial k}{\partial x} = 0 \tag{5}$$

Now, if it is assumed that a driver adjusts his velocity at any instant in accordance with the traffic conditions about him as expressed by $k^n \partial k/\partial x$, the acceleration of the traffic stream at a given place and time becomes

$$\frac{du}{dt} = -c^2 k^n \frac{\partial k}{\partial x} \tag{6}$$

Taking the total derivative of $u = f(x, t)$ gives

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt} \tag{7}$$

where $dx/dt = u$ and $dt/dt = 1$. Substituting Eq. 7 in Eq. 6 yields

$$\frac{\partial u}{\partial x} u + \frac{\partial u}{\partial t} + c^2 k^n \frac{\partial k}{\partial x} = 0 \tag{8}$$

From Eq. 4, it is equally apparent that

$$\frac{\partial u}{\partial t} = u' \frac{\partial k}{\partial t} \tag{9}$$

By solving for $\partial u/\partial k$ from Eq. 4 and substituting in Eq. 8, substituting for Eq. 9 in Eq. 8, then dividing through by u', Eq. 8 becomes

$$\frac{\partial k}{\partial t} + \left[u + \frac{c^2 k^n}{u'}\right] \frac{\partial k}{\partial x} = 0 \tag{10}$$

which is the generalized equation of motion. The nontrivial solution of Eqs. 5 and 10 is obtained by equating the quantities within the brackets,

$$(u')^2 = c^2 k^{(n-1)} \tag{11}$$

Finally, because of the inverse relation between velocity and concentration,

$$u' = -ck^{(n-1)/2} \tag{12}$$

Greenberg (1) has solved Eq. 12 for n = -1 obtaining

$$u = c \ln (k_j/k) \tag{13}$$

The solution of Eq. 12 for n > -1 is as follows:

$$u = \frac{-2c}{(n+1)} k^{(n+1)/2} + C_1, \quad n > -1 \tag{14}$$

where the constant of integration is to be evaluated by the boundary conditions inherent in the vehicular velocity-concentration relationship. Thus, since no movement is possible at jam concentration, k_j,

$$C_1 = \frac{2c}{(n+1)} k_j^{(n+1)/2}, \quad n > -1 \tag{15}$$

and

$$u = \frac{2c}{(n+1)} \left[k_j^{(n+1)/2} - k^{(n+1)/2} \right], \quad n > -1 \tag{16}$$

Similarly, the implication exists that a driver is permitted his free speed, u_f, only when there are no other vehicles on the highway (k = 0). Therefore,

$$u_f = \frac{2c}{(n+1)} k_j^{(n+1)/2}, \quad n > -1 \tag{17}$$

and the constant of proportionality takes on the following physical significance:

$$c = \frac{\quad}{2 k_j^{(n+1)/2}}, \quad n > -1 \tag{18}$$

Substitution of Eq. 18 in Eq. 16 yields the generalized equations of state,

$$u = u_f \left[1 - \left(\frac{k}{k_j} \right)^{(n+1)/2} \right], \quad n > -1 \tag{19}$$

$$q = ku = k u_f \left[1 - \left(\frac{k}{k_j} \right)^{(n+1)/2} \right], \quad n > -1 \tag{20}$$

Differentiation of Eq. 20 with respect to k equated to zero gives the optimum concentration, k_m, which is that concentration yielding the maximum flow of vehicles:

$$\frac{dq}{dk} = \left[1 - \frac{(n+3)\, k^{(n+1)/2}}{2 k_j^{(n+1)/2}} \right] u_f = 0$$

$$k_m = \left[(n+3)/2 \right]^{-2/(n+1)} k_j, \quad n > -1 \tag{21}$$

Substituting Eq. 21 in Eq. 19, one obtains the optimum velocity,

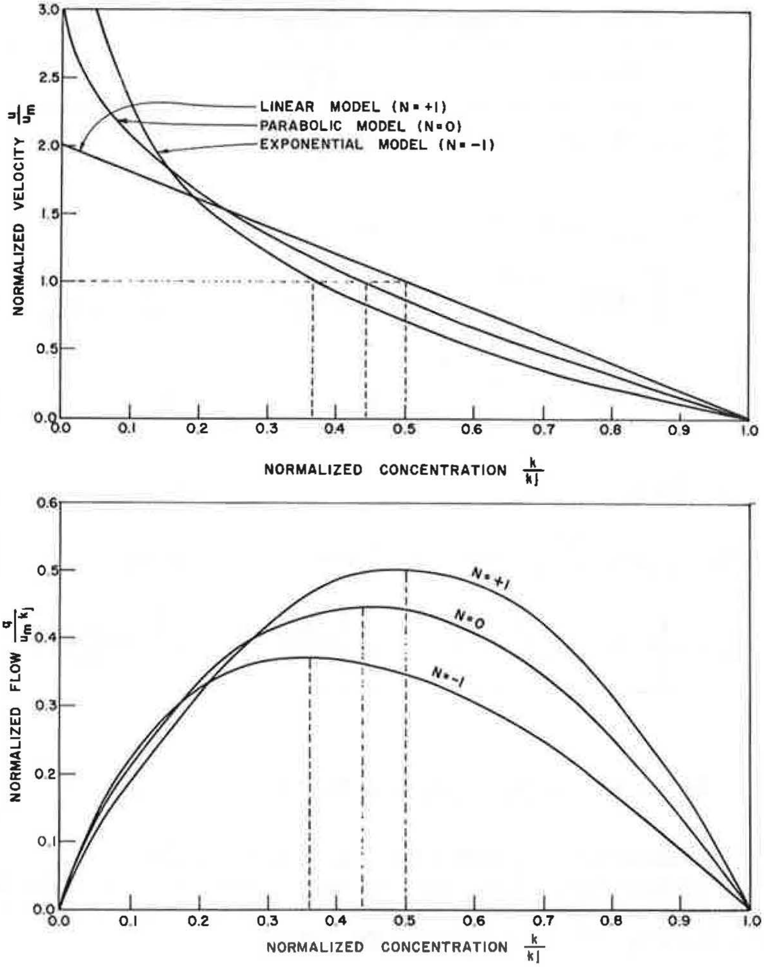

Figure 1. Solution of generalized equation of traffic motion, $\frac{du}{dt} + c^2 k^n \frac{\partial k}{\partial x} = 0$, for N = -1, 0, +1.

$$u_m = \left[\frac{n+1}{n+3} \right] u_f, \quad n > -1 \tag{22}$$

The maximum flow of vehicles of which the highway lane is capable (capacity) is obtained from the product of Eqs. 21 and 22

$$q_m = \left[\frac{(n+1)}{(1/2)^{2/(n+1)} \, (n+3)^{[2/(n+1)]+1}} \right] u_f \, k_j, \quad n > -1 \tag{23}$$

Some special cases of Eqs. 19 through 23 have proven to be of significance. Greenshields'(2) linear model is obtainable by setting n = 1, and Drew (3) has discussed the case for n = 0. These cases, as well as Greenberg's model, are summarized in Figure 1 and Table 1.

Typical of some of the car-following laws that have been proposed are those that express the performance of a vehicle in terms of its velocity and position with respect to the vehicle immediately preceding it,

TABLE 1

COMPARISON OF MACROSCOPIC MODELS OF TRAFFIC FLOW

Element	General (n > -1)	Exponential (n = -1)	Parabolic (n = 0)	Linear (n = 1)
Eq. of motion	$\frac{du}{dt} + c^2k^n \frac{\partial k}{\partial x} = 0$	$\frac{du}{dt} + \frac{c^2 \partial k}{k \partial x} = 0$	$\frac{du}{dt} + c^2 \frac{\partial k}{\partial x} = 0$	$\frac{du}{dt} + c^2 k \frac{\partial k}{\partial x} = 0$
Constant of proportionality	$c = \left[(n + 1) u_f\right] / 2k_j^{(n + 1)/2}$	u_m	$u_f/2k_j^{1/2}$	u_f/k_j
Eq. of state	$q = ku_f \left[1 - \left(\frac{k}{k_j}\right)^{(n+1)/2}\right]$	$ku_m \ln\left(\frac{k_j}{k}\right)$	$ku_f\left[1 - \left(\frac{k}{k_j}\right)^{1/2}\right]$	$ku_f\left[1 - \frac{k}{k_j}\right]$
Optimum concentration	$k_m = \left[(n + 3)/2\right]^{-2/(n+1)} k_j$	k_j/e	$4k_j/9$	$k_j/2$
Optimum speed	$u_m = \left[(n + 1)/(n + 3)\right] u_f$	c	$u_f/3$	$u_f/2$
Capacity	$q_m = \frac{(n+1)\, u_f k_j}{(1/2)^{2/(n+1)}(n+3)^{[2/(n+1)]+1}}$	$\frac{1}{e} u_m k_j$	$\frac{4}{27} u_f k_j$	$\frac{1}{4} u_f k_j$
Wave vel.	$\frac{dq}{dk} = u_f\left[1 - \frac{(n+3)}{2}\left(\frac{k}{k_j}\right)^{(n+1)/2}\right]$	$u_m\left[\ln\left(\frac{k_j}{k}\right) - 1\right]$	$u_f\left[1 - \frac{3}{2}\left(\frac{k}{k_j}\right)^{1/2}\right]$	$u_f\left[1 - \frac{2k}{k_j}\right]$

$$\ddot{x}_i\,(t + T) = a\,[\dot{x}_{i-1}(t) - \dot{x}_i(t)]\,[x_{i-1}(t) - x_i(t)]^{-m} \qquad (24)$$

Eq. 24 states that the acceleration of a car, $\ddot{x}_i$, at a delayed time, T, is directly proportional to the relative speed of the car, $\dot{x}_i$, with respect to the one ahead, $\dot{x}_{i-1}$, and inversely proportional to the headway of the car, $x_{i-1} - x_i$. Since the right side of Eq. 1 is of the form dy/y^m, integration of Eq. 24 yields

$$\dot{x}_i\,(t + T) = a \ln\,[x_{i-1}(t) - x_i(t)] + C_1,\quad m = 1 \qquad (25)$$

and

$$\dot{x}_i\,(t + T) = (-m + 1)^{-1}\, a\,[x_{i-1}(t) - x_i(t)]^{-m+1} + C_2,\quad m > 1 \qquad (26)$$

The constants of integration are evaluated by observing that the velocity of a car approaches zero as its headway approaches the effective length of each car, L;

$$C_1 = a \ln L \qquad (27)$$

$$C_2 = -(-m + 1)^{-1}\, a\, L^{-m+1},\quad m > 1 \qquad (28)$$

Substituting for C_1 and C_2, Eqs. 25 and 26 become

$$\dot{x}_i\,(t + T) = a \ln L^{-1}\,[x_{i-1}(t) - x_i(t)],\quad m = 1 \qquad (29)$$

TABLE 2

COMPARISON OF MICROSCOPIC MODELS OF TRAFFIC FLOW

Element	General (m > 1)	m = 1	m = 3/2	m = 2
Eq. of motion	$\ddot{x}_i = \frac{a(\dot{x}_{i-1} - \dot{x}_i)}{(x_{i-1} - x_i)^m}$	$\ddot{x}_i = \frac{a(\dot{x}_{i-1} - \dot{x}_i)}{(x_{i-1} - x_i)}$	$\ddot{x}_i = \frac{a(\dot{x}_{i-1} - \dot{x}_i)}{(x_{i-1} - x_i)^{3/2}}$	$\ddot{x}_i = \frac{a(\dot{x}_{i-1} - \dot{x}_i)}{(x_{i-1} - x_i)^2}$
Constant of proportionality	$a = (m-1) u_f k_j^{-(m-1)}$	u_m	$u_f/2k_j^{1/2}$	u_f/k_j
Eq. of state	$q = ku_f \left[1 - \left(\frac{k}{k_j}\right)^{m-1}\right]$	$ku_m \ln\left(\frac{k_j}{k}\right)$	$ku_f \left[1 - \left(\frac{k}{k_j}\right)^{1/2}\right]$	$ku_f \left[1 - \left(\frac{k}{k_j}\right)\right]$
Macroscopic counterpart (see Table 1)	n = 2m - 3	n = -1	n = 0	n = +1

$$\dot{x}_i(t+T) = (m-1)^{-1} a \left\{ L^{-(m-1)} - [x_{i-1}(t) - x_i(t)]^{-(m-1)} \right\}, \quad m > 1 \tag{30}$$

Eq. 29 is due to Gazis, Herman and Potts (4) who showed that the traffic equation of state could be derived from the microscopic car-following law just as the gas equation of state can be derived from the microscopic law of molecular interaction. Since the space headway is the reciprocal of concentration, k, Eqs. 29 and 30 become

$$u = a \ln (k_j/k) \tag{31}$$

and

$$u = (m-1)^{-1} a \left(k_j^{m-1} - k^{m-1}\right), \quad m > 1 \tag{32}$$

The constant of proportionality is evaluated at $u = u_f$ and $k = 0$, giving

$$a = \left[\frac{(m-1)}{k_j^{m-1}}\right] u_f, \quad m > 1 \tag{33}$$

Special cases of Eq. 32, as well as the relationship of the macroscopic parameters c and n to the microscopic parameters a and m are given in Table 2.

APPLICATION OF DETERMINISTIC MODELS

The applicability of these deterministic models to freeway traffic was tested on the Gulf Freeway in Houston, Texas (Fig. 2). Time-lapse aerial photography with a 60 percent overlap was utilized to insure a given point on the freeway appearing on three consecutive photos (Fig. 3). Six flight runs were made in the direction of the traffic being studied, inbound during the morning peak. Since a given vehicle appeared on at least three consecutive photos, individual vehicular speeds, accelerations, and space headways were measured. The observations were compared (on a lane basis) to the three macroscopic models in Table 1 and the three microscopic models in Table 2.

Regression analyses based on the macroscopic hypotheses of Eqs. 13 and 19 (n = 0 and n = +1) are summarized in Table 3. Statistical tests were, in general, highly significant on each of the three freeway lanes, as well as on the total traffic on all three lanes. The microscopic analyses, however, were inconclusive. A constant of proportionality, a, was calculated for every freeway vehicle based on its performance and position with respect to the vehicle in front of it. The physical significance of a is indicated in Table 2 for the three microscopic models tested. The values obtained were

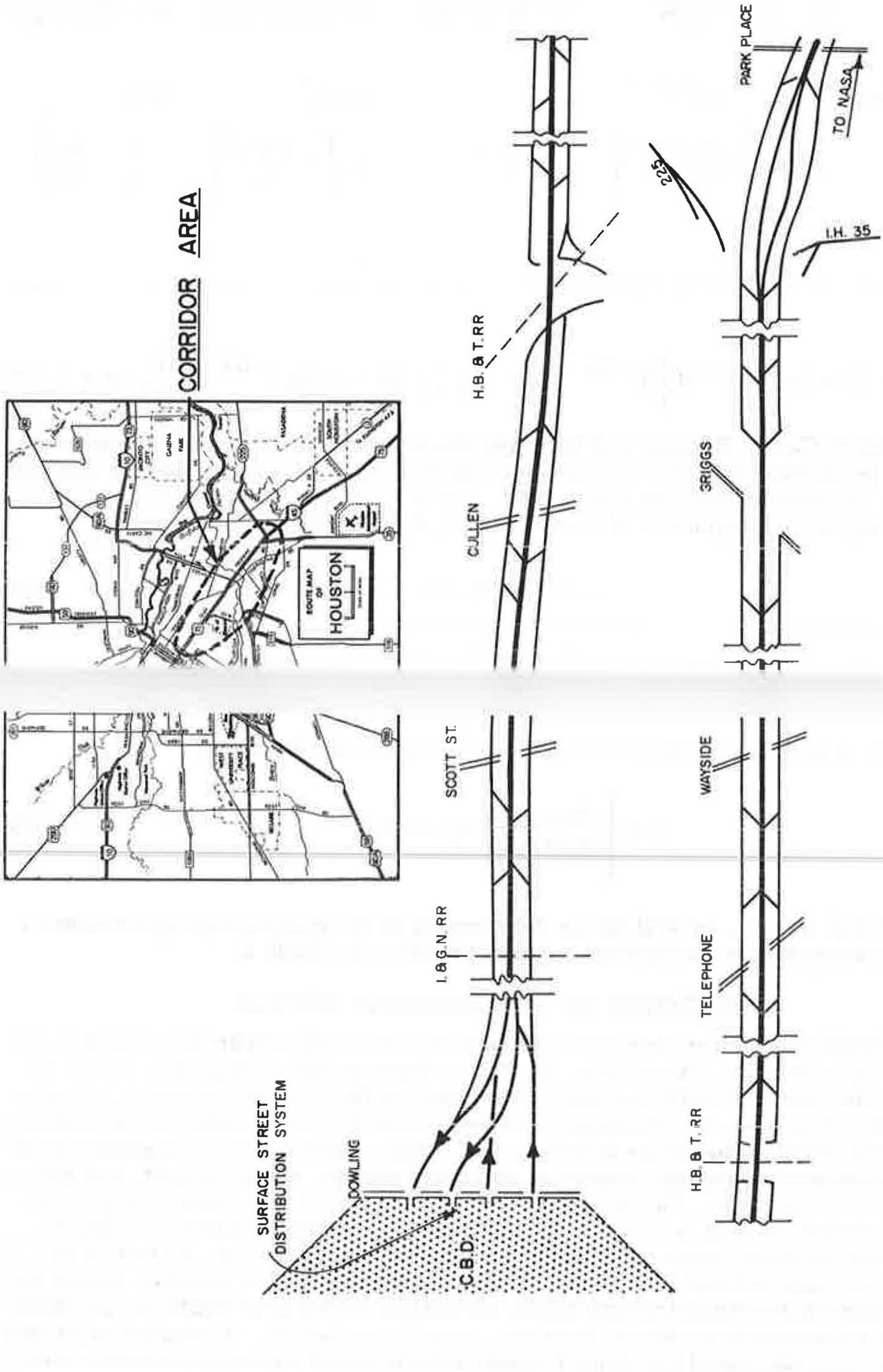

Figure 2. Study [illegible], Gulf Freeway, Houston.

TABLE 3

REGRESSION ANALYSES OF EQUATIONS OF STATE (3 LANE TOTAL)

Station	$u = a - bk$			$u = a - bk^{1/2}$		$\ln k = a - bu$	
	b	a	t	b	a	b	a
306-288	0.129	52.8	7.52**	3.32	71.0	0.044	6.20
299-281	0.115	50.9	32.16**	3.14	69.2	0.050	6.35
292-274	0.112	52.0	18.37**	3.20	72.0	0.047	6.41
286-268	0.132	54.3	40.60**	3.47	74.6	0.046	6.35
280-262	0.131	53.3	30.60**	3.34	72.2	0.048	6.38
273-255	0.142	55.3	13.49**	3.74	78.1	0.041	6.26
267-249	0.141	56.0	11.87**	3.89	81.3	0.038	6.25
261-243	0.102	46.7	4.98**	2.09	54.4	0.069	6.77
254-236	0.143	58.0	7.79**	4.03	84.8	0.035	6.20
248-230	0.173	66.1	20.77**	4.66	95.5	0.032	6.18
241-223	0.175	64.6	11.53**	4.56	92.6	0.032	6.15
235-217	0.181	63.0	10.93**	4.67	91.9	0.032	6.09
229-211	0.167	59.7	4.85**	4.32	86.6	0.032	6.06
223-205	0.182	64.5	15.84**	4.91	96.5	0.030	6.08
216-198	0.205	67.8	10.00**	5.33	101.5	0.028	6.00
210-192	0.176	62.3	8.82**	4.45	89.2	0.035	6.17
204-186	0.190	65.5	19.03**	4.86	95.4	0.032	6.12
197-179	0.176	64.3	14.32**	4.55	92.6	0.033	6.20
190-172	0.197	66.4	7.99**	5.11	99.0	0.028	6.04
183-166	0.200	65.9	4.97**	5.01	97.0	0.027	5.97
176-158	0.181	63.2	7.12**	4.57	91.8	0.032	6.16
169-151	0.179	63.0	11.14**	4.31	88.5	0.036	6.28
162-144	0.154	60.0	8.78**	3.59	79.7	0.043	6.51
155-137	0.157	60.1	4.18*	3.78	82.3	0.034	6.23
148-130	0.167	61.3	6.06**	4.03	85.1	0.035	6.25
141-123	0.158	61.2	5.03**	3.68	82.2	0.038	6.41
134-118	0.140	58.3	4.00*	3.18	76.0	0.042	6.50
128-110	0.145	58.1	3.60*	3.19	75.4	0.041	6.44
121-103	0.051	45.6	0.90	1.08	51.2	0.029	5.91
115-97	0.153	57.8	3.93*	3.14	73.7	0.049	6.71
108-90	0.222	66.4	4.85**	4.75	91.7	0.034	6.12
101-83	0.194	64.8	2.67	4.14	86.8	0.028	5.94
95-77	0.165	61.9	1.65	3.35	78.7	0.023	5.67
89-71	0.176	64.0	1.47	3.79	84.2	0.021	5.62
82-64	0.126	58.0	2.87*	2.75	72.7	0.048	6.81
76-58	0.121	58.0	3.16*	2.67	71.7	0.053	6.99
69-51	0.127	57.7	2.73*	2.98	74.8	0.043	6.62
63-45	0.114	55.7	2.80*	2.67	71.1	0.049	6.86
56-38	0.110	55.5	1.95	2.48	69.1	0.039	6.51
50-32	0.131	57.6	3.79*	2.88	73.1	0.051	6.91
44-26	0.122	54.2	3.37*	2.62	67.9	0.053	6.88
37-19	0.137	54.2	4.60**	2.96	69.8	0.053	6.75

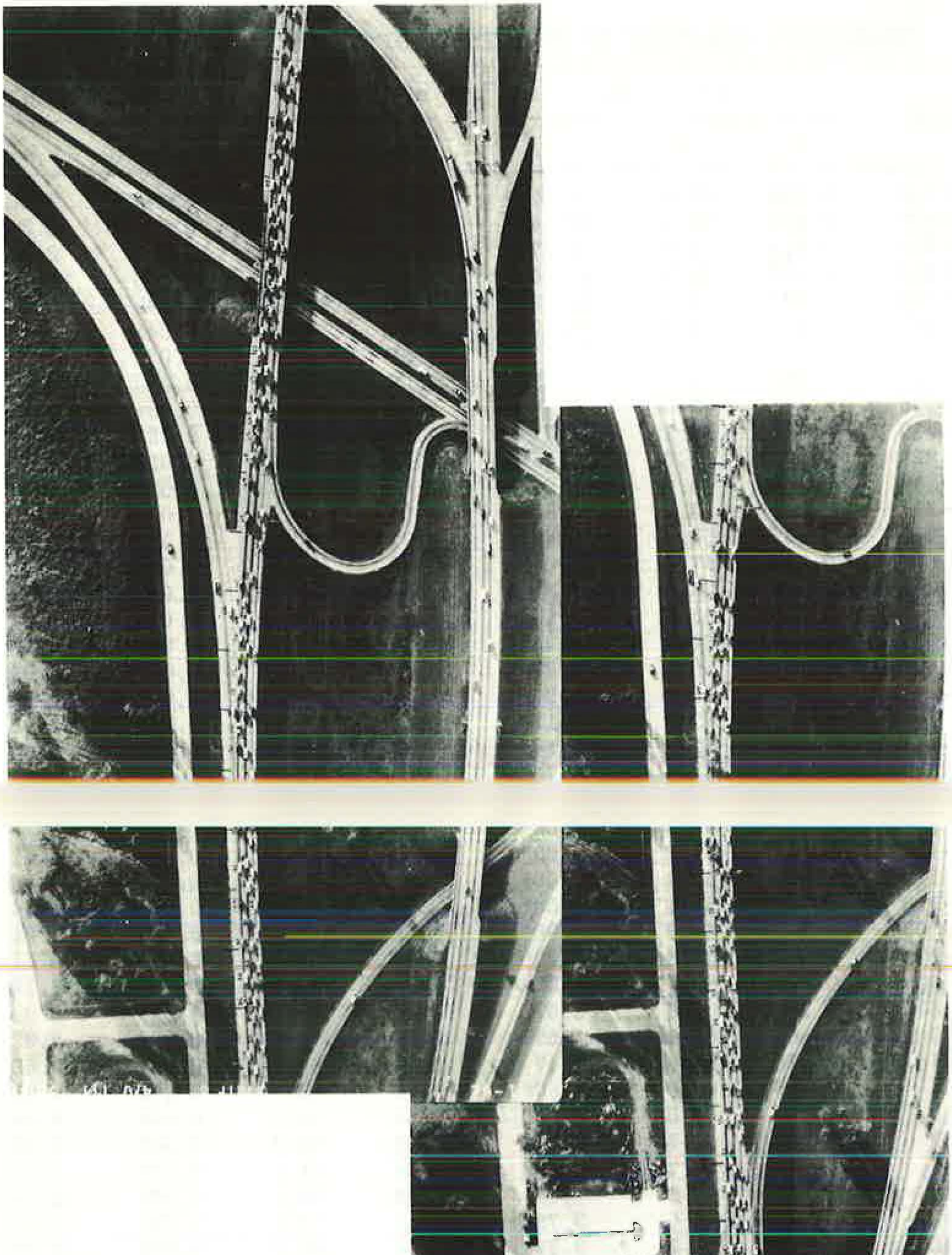

Figure 3. [illegible]-lapse photography.

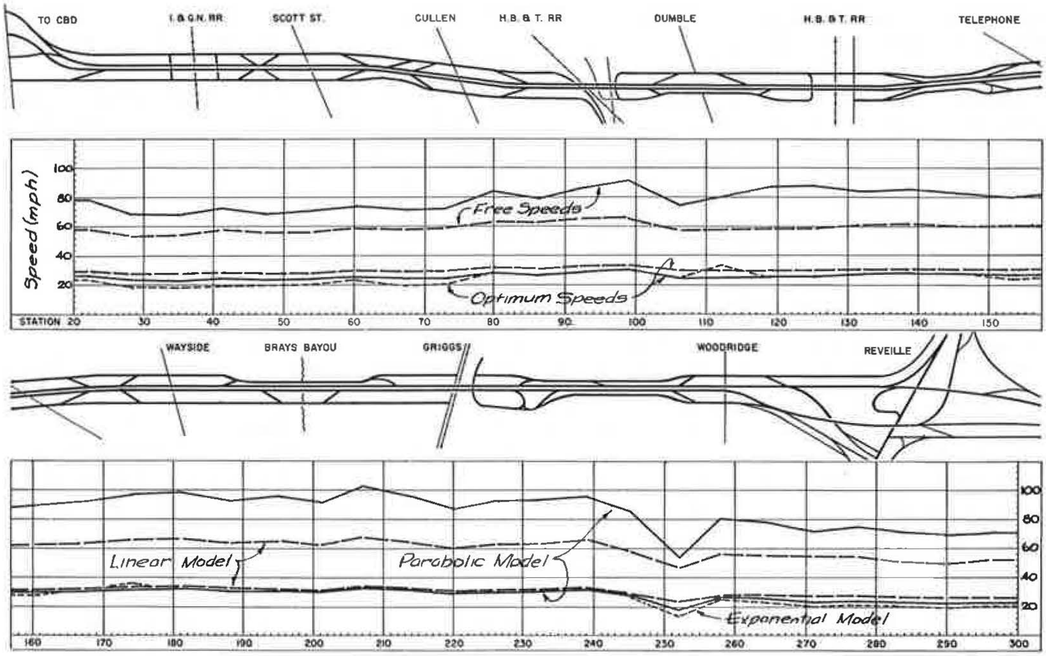

Figure 4. Speed profiles (total inbound traffic).

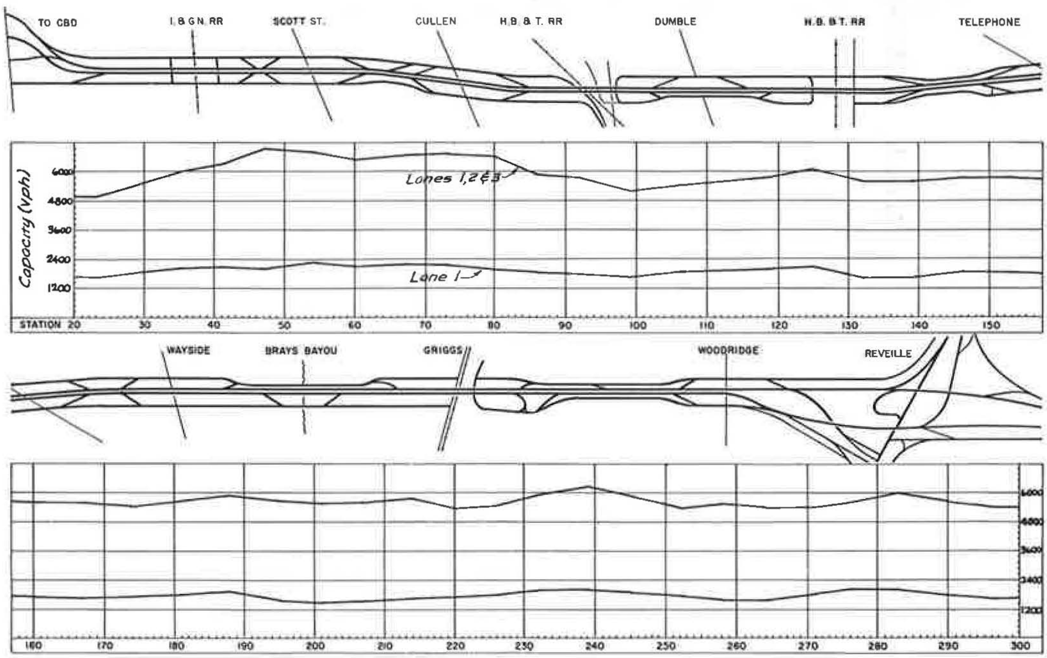

Figure 5. Capacity profiles (linear model).

extremely variable; approximately one-eighth of the values were negative indicating that, even under conditions of heavy traffic, the opportunity for changing lanes reduces a driver's necessity to respond to the performance of the car in front of him.

Essential to the development of freeway control techniques is the determination of suitable control parameters. Among the many techniques for controlling freeway traffic, ramp metering at entrance ramps and changeable advisory speed limit signs located on the freeway itself offer the most promise. Capacity, q_m, and optimum speed, u_m, represent two ideal control parameters. Figures 4 and 5 show continuous speed and capacity profiles for the outside lane of the 6-mi stretch of the Gulf Freeway. Free speeds, u_f, are also shown in Figure 4 for the linear and parabolic models ($u_f = \infty$ for the exponential model).

Because control of vehicles entering the freeway, as against control of vehicles already on the freeway, offers a more positive means of preventing congestion, considerable emphasis is being placed on the technique of ramp metering. Entrance ramp metering may be oriented to either the freeway capacity or freeway demand. A capacity-oriented ramp control system restricts the volume rate on the entrance ramps to prevent the flow rates at downstream bottlenecks from exceeding the capacities of the bottlenecks. Figure 5 shows a capacity profile for traffic on all three inbound lanes of the Gulf Freeway. Bottleneck sections along with their respective control capacities are evident.

REFERENCES

1. Greenberg, H. An Analysis of Traffic Flow. Operations Res., Vol. 7, 1959.
2. Greenshields, B. D. A Study in Highway Capacity. Highway Research Board Proc., Vol. 14, pp. 448-477, 1934.
3. Drew, Donald R. A Study of Freeway Traffic Congestion. Doctoral Diss., Texas A & M Univ., 1964.
4. Gazis, D. C., Herman, R., and Potts, R. B. Car Following Theory of Steady State Traffic Flow. Operations Res., Vol. 7, 1959.