Effect of Variations in Poisson's Ratio on Soil Triaxial Testing

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The effect of variations in Poisson's ratio on the apparent elastic modulus and the apparent Poisson's ratio as determined from a conventional soil triaxial test is investigated. These parameters, the apparent elastic modulus and the apparent Poisson's ratio, are not the same as the true elastic constants appearing in Hooke's law. However, assuming that the soil behaves elastically, the true elastic constants, Young's modulus of elasticity and Poisson's ratio, can be determined by applying corrections to the apparent values obtained in triaxial testing. Curves showing the relationships between these true and apparent parameters are shown. Numerical solutions are presented for three values of Poisson's ratio and two values of height to diameter ratio.

•THE ANALYTIC problem of stresses and deformations in a short, elastic, right circular cylinder loaded between two flat loading plates has been considered several times in the literature (1, 2, 4, 5, 6). Because many materials are tested in this manner to evaluate their elastic properties, this problem must be thoroughly understood.

In the normal compression test, a cylinder tends to expand laterally as it is shortened by an axial load in a testing machine; however, frictional forces between the loading plates and the cylinder ends tend to prohibit this expansion. Exactly how the axial load is transmitted from the loading plate to the cylinder is unknown. It is known, however, that the ends of the cylinder are kept plane and that they are restricted to some large extent from radial expansion.

REVIEW OF LITERATURE AND BOUNDARY CONSIDERATIONS

Filon (1) first considered this problem. He assumed that the ends of the cylinder were kept plane and that no point on the ends could move in a radial direction, as if the ends were glued. He was unable to meet exactly these boundary conditions. Pickett (2) pointed out that Filon's solution allows all end points except those on the periphery to be displaced toward the center.

Considering the same boundary conditions that Filon attempted to meet, Pickett (2) solved this problem using a multiple Fourier technique. His graphical solution appears to be correct with possibly some numerical inaccuracies due to the slowly converging infinite series and the difficulty in hand computation (3).

Because using Pickett's solution was cumbersome for determining numerical values, D'Appolonia and Newmark (4) attacked the problem using a framework analogy. Their solution agrees reasonably well with Pickett's solution. However, it does not exactly meet all boundary conditions. Because their framework mesh size was rather large, some inaccuracies were to be expected.

Assuming slightly different boundary conditions, Balla (5) solved a similar problem. He assumed that the cylinder ends were kept plane, that the end shear stress distribution was linear, and that the radial displacement on the periphery at the ends varied inversely with a friction factor. When Balla's friction factor is the maximum value, the radial displacement at the periphery is zero.

Paper sponsored by Committee on Mechanics of Earth Masses and Layered Systems.

Because the true end conditions are unknown, it is important that the end conditions in the standard compression test be investigated further. The use of porous stones for the end plates in the standard soil triaxial test, however, makes it difficult to imagine any slippage in this case. Thus, for this test the assumption of the boundary conditions—plane ends and no radial displacement on these ends—appears reasonable. These boundary conditions are used in this paper.

All of these solutions, except Balla's, have considered an elastic material with a Poisson's ratio of $\frac{1}{4}$; Balla used $\frac{1}{3}$. Considering the available solutions which may be applied to the triaxial test, two pertinent questions arise:

1. Are present interpretations valid, since measured values of Poisson's ratio have been reported which cover a considerable range of values (0.1 to above 0.5); and
2. How do variations in Poisson's ratio affect stresses in an elastic material compressed between two rough rigid loading plates?

It is the purpose of this paper to investigate the effect of Poisson's ratio on the "apparent modulus of elasticity" and the "apparent Poisson's ratio" estimated from a conventional triaxial test. The apparent modulus of elasticity is the average vertical stress (the total load acting on the loading plates divided by the cylinder cross-sectional area) divided by the average vertical unit strain (the change in height due to load divided by the original height). The apparent Poisson's ratio is the ratio of the average lateral unit strain at the cylinder mid-height (the increase in diameter divided by the original diameter) to the average vertical unit strain.

The difference between the apparent modulus and the true modulus, for the conditions assumed here, has been pointed out before in the literature. Edelman (6) reported that a cylinder with a height to diameter ratio equal to 1 and a Poisson's ratio of $\frac{1}{3}$ would have a measured apparent modulus approximately 5 percent larger than the true modulus. For a Poisson's ratio of $\frac{1}{4}$, D'Appolonia and Newmark (4) noted an error of approximately the same magnitude. In soil mechanics literature reviewed by the author, this phenomenon has not been mentioned. This is not at all surprising since the error reported seemed to be negligible when compared to the rather large variations usually observed in soil testing.

NOTATION

r, z, θ = cylindrical coordinate independent variables;
u, w = displacements in r and z directions, respectively;
$\epsilon_r, \epsilon_z, \epsilon_\theta$ = unit normal strains in subscript direction;
γ_{rz} = unit shear strain in r-z plane;
$\sigma_r, \sigma_z, \sigma_\theta$ = unit normal stress in the subscript direction;
τ_{rz} = unit shear stress in r-z plane;
D = diameter of cylinder;
H = height of cylinder;
P = total axial load on cylinder;
Δ = shortening in height due to load;
μ = Poisson's ratio;
E = Young's modulus of elasticity;
σ_1 = P divided by cylinder cross-sectional area; and
σ_3 = lateral pressure acting on cylinder at $r = D/2$.

THEORY

The fundamental equations to be satisfied are as follows (Fig. 1):

Equations of equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (1)$$

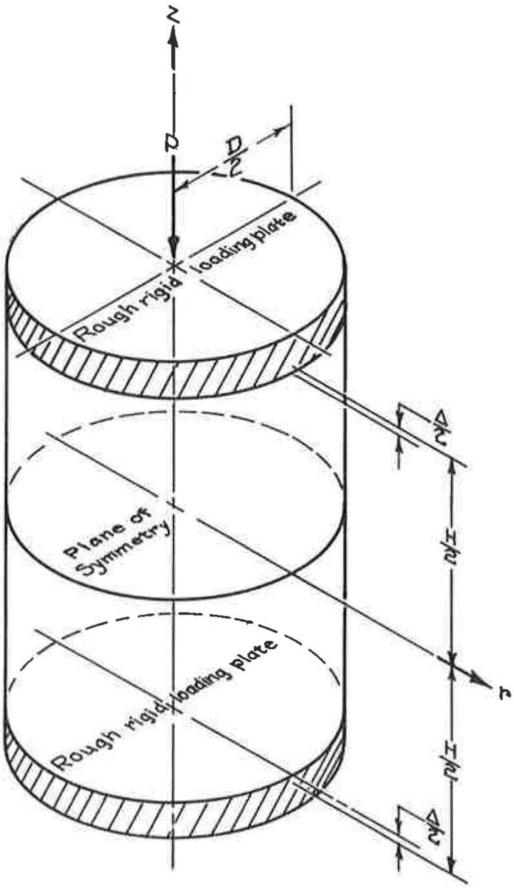

Figure 1. Loading system.

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0 \qquad (2)$$

Equations representing Hooke's law

$$\epsilon_z = \frac{\partial w}{\partial z} = \frac{\sigma_z - \mu(\sigma_r + \sigma_\theta)}{E} \qquad (3)$$

$$\epsilon_z = \frac{\partial u}{\partial r} = \frac{\sigma_r - \mu(\sigma_z + \sigma_\theta)}{E} \qquad (4)$$

$$\epsilon_\theta = \frac{u}{r} = \frac{\sigma_\theta - \mu(\sigma_z + \sigma_r)}{E} \qquad (5)$$

$$\gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} = \frac{2(1+\mu)\,\tau_{rz}}{E} \qquad (6)$$

Hooke's law can be rewritten in the following form:

$$\sigma_z = \frac{E\mu}{(1+\mu)(1-2\mu)} \left[\left(\frac{1-\mu}{\mu}\right)\frac{\partial w}{\partial z} + \frac{\partial u}{\partial r} + \frac{u}{r}\right] \qquad (7)$$

$$\sigma_r = \frac{E\mu}{(1+\mu)(1-2\mu)} \left[\frac{\partial w}{\partial z} + \left(\frac{1-\mu}{\mu}\right)\frac{\partial u}{\partial r} + \frac{u}{r}\right] \qquad (8)$$

$$\sigma_\theta = \frac{E\mu}{(1+\mu)(1-2\mu)} \left[\frac{\partial w}{\partial z} + \frac{\partial u}{\partial r} + \left(\frac{1-\mu}{\mu}\right)\frac{u}{r}\right] \qquad (9)$$

$$\tau_{rz} = \frac{E}{2(1+\mu)} \left[\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right] \qquad (10)$$

By combining Eqs. 7 through 10 with Eqs. 1 and 2, the equilibrium equations can be rewritten as follows:

$$\frac{\partial^2 w}{\partial r\, \partial z} + 2(1-\mu)\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] + (1-2\mu)\frac{\partial^2 u}{\partial z^2} = 0 \qquad (11)$$

$$(1 - 2\mu)\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right] + 2(1 - \mu)\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 u}{\partial r\,\partial z} + \frac{1}{r}\frac{\partial u}{\partial z} = 0 \qquad (12)$$

The assumed boundary conditions are:

$$\sigma_r = 0, \text{ when } r = D/2^* \qquad (13a)$$

$$\tau_{rz} = 0, \text{ when } r = D/2 \qquad (14)$$

$$w = \Delta/2, \text{ when } z = H/2 \qquad (15a)$$

$$u = 0, \text{ when } z = H/2 \qquad (16)$$

On writing the first two boundary conditions in terms of u and w,

$$\frac{\partial w}{\partial z} + \left(\frac{1 - \mu}{\mu}\right)\frac{\partial u}{\partial r} + \frac{u}{r} = 0, \text{ when } r = D/2 \qquad (17a)$$

$$\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} = 0, \text{ when } r = D/2 \qquad (18)$$

The problem is reduced to finding the solution to Eqs. 11 and 12 (considering u and w only) which satisfies the boundary conditions set in Eqs. 15a, 16, 17a, and 18. Since stresses are not considered in the solution, they must be determined using Hooke's law (Eqs. 7 through 10) after the solution for u and w is obtained.

If the equilibrium equations (Eqs. 11 and 12) and boundary equations (Eqs. 15a, 16, 17a, and 18) are rewritten in terms of u and w, for a given height to diameter ratio (H/D), the solution to these equations in not dependent on the modulus of elasticity (E) but only on Poisson's ratio (μ) and the change in height (Δ). The only way E enters the solution is through Hooke's law in the stress calculations (Eqs. 7 through 10). At any point in the cylinder, the displacements and the strains are directly proportional to Δ. Therefore, the stresses at any point are directly proportional to $E\Delta$ and the constant of proportionality for each stress component is a function of r, z, and μ.

Considering the principle of superposition in elasticity theory, it is evident that confining pressure has no influence on the apparent modulus of elasticity or the apparent Poisson's ratio. The measurements made in testing to determine these quantities are the changes in total load and the corresponding changes in diameter at mid-height while holding the confining pressure constant.

Confining pressure can be introduced as a consideration in the solution by changing only boundary Eqs. 13a and 15a:

$$\sigma_r = \sigma_3, \text{ when } r = D/2 \qquad (13b)$$

$$w = 0, \text{ when } z = H/2 \qquad (15b)$$

On writing Eq. 13b in terms of u and w, Eq. 17 is replaced with the following:

$$\frac{\partial w}{\partial z} + \left(\frac{1 - \mu}{\mu}\right)\frac{\partial u}{\partial r} + \frac{u}{r} = \frac{\sigma_3(1 + \mu)(1 - 2\mu)}{\mu E}, \text{ when } r = D/2 \qquad (17b)$$

These boundary conditions result in a total load P of less than σ_3 times the cylinder cross-sectional area. The resultant solution is not for the condition of the usual tri-

*Introducing confining pressure as a boundary consideration at this point will only tend to make the problem more difficult. It will be treated later and the reasons for delay will be apparent.

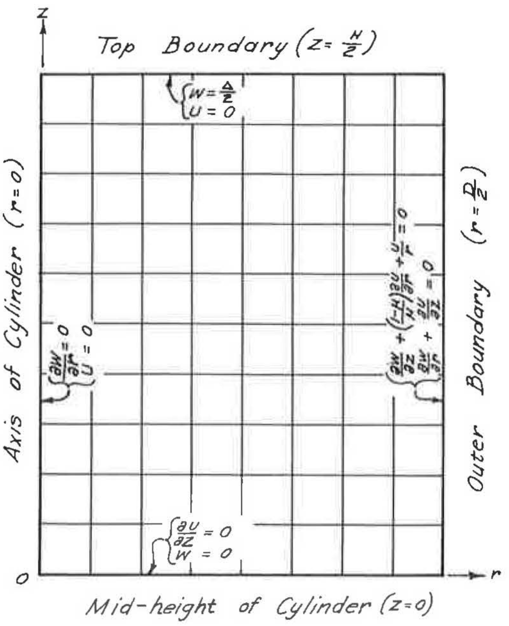

Figure 2. Equation system.

TABLE 1

H/D	μ	Apparent μ	Apparent E/True E
1	0.15	0.151	1.013
1	0.25	0.260	1.038
1	0.33	0.356	1.071
1	0.40	0.452	1.117
1	0.45	0.537	1.167
1	0.48	0.608	1.229
2	0.15	0.152	1.006
2	0.25	0.257	1.018
2	0.33	0.345	1.034
2	0.40	0.429	1.055
2	0.45	0.499	1.078
2	0.48	0.563	1.119

axial test in which the confining pressure acts on both the sides and loading plates. However, employing the principle of superposition the solutions for the unconfined cylinder and the confined cylinder can be combined for this condition or any other particular condition desired.

PROCEDURE

The procedure used to study this problem, finite difference equations solved by iterative methods, is the same as that used by Dingwall and Scrivner (7) to solve a rather complex embankment problem. This approximate procedure can be quite accurate, as Dingwall and Scrivner pointed out: "... any desired degree of accuracy may be obtained by making the mesh size sufficiently small." To perform the iterative calculations on a small mesh size for this research, an IBM 709 computer was used to obtain a set of solutions for various Poisson's ratios.

Rather than use a single stress function, as used by Dingwall and Scrivner, Eqs. 11 and 12 were used to solve for u and w, respectively, at all interior nodal points. The boundary conditions, Eqs. 15a, 16, 17a, and 18 along with symmetry conditions were used on appropriate boundaries (Fig. 2). Basically the procedure consists of successively changing the values of u and w at each nodal point as required to satisfy the appropriate equations written in finite difference form, until the changes become negligible. In writing the finite difference equations at any nodal point, it was assumed that the functions representing u and w in the neighborhood of the point could be approximated by a parabola.

Solutions for unconfined cylinders of six different Poisson's ratios, 0.15, 0.25, 0.33, 0.40, 0.45, and 0.48, were obtained for both H/D equal to 1 and H/D equal to 2. Stresses and displacements from these solutions for three different Poisson's ratios, 0.25, 0.40, and 0.45, for both H/D equal to 1 and H/D equal to 2 are shown in Appendices A and B. Solutions for confined cylinders of three different Poisson's ratios, 0.25, 0.40, and 0.45, were obtained for H/D equal to 2. Stresses and displacements from these solutions are shown in Appendix C. To obtain the solutions reported here a square grid was used with a mesh size of D/64. Convergence was assumed when changes in the maximum deformations were confined to the sixth significant figure; all other changes in deformations were smaller.

The total force, P, was calculated for each z by numerical integration. The value of P at z = H/4 for each value of μ remained approximately constant as the mesh size

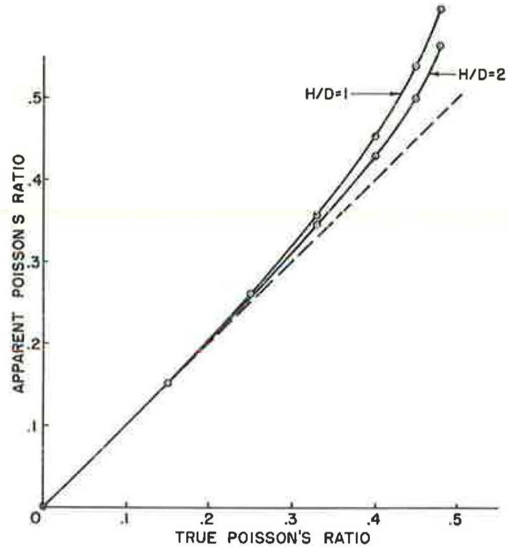

Figure 3. Apparent Poisson's ratio vs true Poisson's ratio.

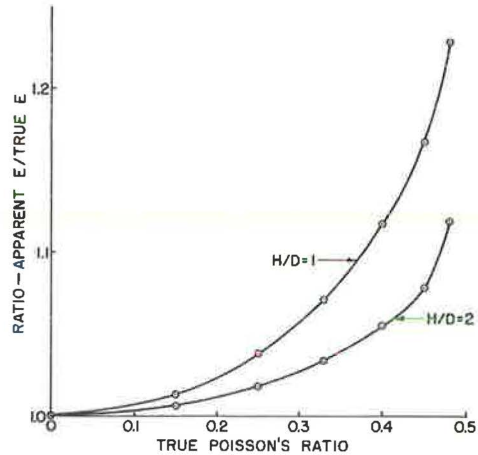

Figure 4. Ratio-apparent E/true E vs Poisson's ratio.

was decreased; at other values of z it varied somewhat. The average vertical stress, σ_1, used to compute the apparent moduli (Table 1) and the stress ratios (Appendices A, B and C) for each μ, was obtained by dividing P at z = H/4 by the cross-sectional area.

DISCUSSION

The numerical values (Appendices A and B) for the shear stress (τ_{rz}) on the ends of the cylinder are not large when compared to the corresponding vertical normal stress (σ_z). Thus, for the conventional triaxial test it appears that sufficient friction would exist to restrain end expansion, and the end conditions assumed here are justified.

The author was unable to obtain a solution to this problem for a cylinder with a Poisson's ratio equal to ½. It appeared (Eqs. 7 through 9) that for μ equal to ½, the normal stresses would be infinite. In addition, the computer time required for convergence on the small mesh became prohibitive as μ approached the limit of ½. Therefore, for practical considerations the results shown in Table 1 and plotted in Figures 3 and 4 do not show results for values of Poisson's ratios larger than 0.48. However, it is evident that the apparent modulus of elasticity and the apparent Poisson's ratio are both increasing very rapidly as μ approaches ½. Figures 3 and 4 show that for H/D equal to 2 and μ less than ⅓ the corrections in the commonly measured elastic parameters are less than 5 percent. However, when μ equals 0.48, the correction for E is approximately 12 percent and for μ approximately 17 percent. These corrections appear to become quite large as μ approaches ½.

The following is an example of how the true elastic parameters could be estimated from the data reported here. If in a triaxial test with H/D equal to 2 an apparent Poisson's ratio of 0.52 is measured, the true μ is 0.44 according to Figure 3. Then from Figure 4 the ratio of the apparent to true modulus is 1.07. Thus, the true elastic modulus is the measured apparent modulus divided by 1.07.

CONCLUSIONS

1. The boundary conditions—plane ends and no radial displacement on these ends—appear to be realistic assumptions for an ideal elastic soil tested in a conventional triaxial test.

2. The elastic parameters measured in triaxial testing are not the elastic parameters (E and μ) expressed in Hooke's law.

3. The true elastic parameters (E and μ) expressed in Hooke's law can be obtained from the conventional triaxial test by applying corrections to the measured apparent parameters (Figs. 3 and 4), providing that the material tested behaves elastically.

4. The upper limit of $\frac{1}{2}$ for Poisson's ratio in the theory of elasticity should be studied further as related to rigid boundaries.

ACKNOWLEDGMENTS

The author wishes to acknowledge the assistance given him by F. H. Scrivner, Texas Transportation Institute, and thank him particularly for his suggestions, comments, and advice during all phases of this research. Appreciation is expressed to W. A. Dunlap, L. E. Stark and other members of the Pavement Design Department for their help and cooperation. Appreciation is also due S. A. Sims and G. N. Williams, staff members of the Data Processing Center. The author is especially indebted to the School of Engineering, Texas A& M University, for providing the facilities of the Data Processing Center which made this study possible.

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Appendix A

STRESSES AND DISPLACEMENTS FOR UNCONFINED COMPRESSION, H/D = 1

TABLE A-1

$\mu = 0.25,\ \sigma_3 = 0,\ H/D = 1$

2z/H	2r/D					2r/D				
	0	0.25	0.50	0.75	1	0	0.25	0.50	0.75	1
	u/Δ					w/Δ				
1	0	0	0	0	0	-0.500	-0.500	-0.500	-0.500	-0.500
0.75	0	0.015	0.032	0.054	0.085	-0.394	-0.393	-0.392	-0.385	-0.357
0.50	0	0.026	0.053	0.084	0.115	-0.271	-0.270	-0.266	-0.257	-0.240
0.25	0	0.032	0.064	0.096	0.127	-0.137	-0.137	-0.134	-0.128	-0.121
0	0	0.033	0.067	0.099	0.130	0	0	0	0	0
	τ_{rz}/σ_1					σ_z/σ_1				
1	0	0.048	0.101	0.173	0.686	0.892	0.893	0.898	0.932	3.130
0.75	0	0.039	0.076	0.092	0	0.980	0.982	0.994	1.022	0.924
0.50	0	0.020	0.030	0.018	0	1.039	1.038	1.033	1.006	0.908
0.25	0	0.006	0.007	0.000	0	1.065	1.059	1.038	0.993	0.925
0	0	0	0	0	0	1.072	1.064	1.037	0.989	0.933
	σ_r/σ_1					σ_θ/σ_1				
1	0.297	0.298	0.299	0.311	1.043	0.297	0.298	0.299	0.311	1.043
0.75	0.177	0.170	0.146	0.082	0	0.177	0.174	0.163	0.138	0.066
0.50	0.083	0.076	0.053	0.018	0	0.083	0.079	0.066	0.041	0.005
0.25	0.030	0.026	0.015	0.004	0	0.030	0.027	0.017	0.003	-0.013
0	0.014	0.011	0.005	0.001	0	0.014	0.011	0.003	-0.007	-0.017

TABLE A-2

$\mu = 0.40,\ \sigma_3 = 0,\ H/D = 1$

2z/H	2r/D					2r/D				
	0	0.25	0.50	0.75	1	0	0.25	0.50	0.75	1
	u/Δ					w/Δ				
1	0	0	0	0	0	-0.500	-0.500	-0.500	-0.500	-0.500
0.75	0	0.029	0.060	0.101	0.154	-0.420	-0.419	-0.416	-0.406	-0.359
0.50	0	0.048	0.099	0.152	0.203	-0.299	-0.298	-0.291	-0.275	-0.246
0.25	0	0.058	0.116	0.172	0.221	-0.155	-0.153	-0.148	-0.138	-0.126
0	0	0.061	0.121	0.177	0.226	0	0	0	0	0
	τ_{rz}/σ_1					σ_z/σ_1				
1	0	0.078	0.165	0.280	1.236	0.862	0.859	0.853	0.875	6.975
0.75	0	0.061	0.119	0.142	0	1.002	1.002	1.006	1.025	0.811
0.50	0	0.029	0.042	0.021	0	1.092	1.086	1.066	1.004	0.824
0.25	0	0.008	0.007	-0.005	0	1.127	1.114	1.071	0.985	0.867
0	0	0	0	0	0	1.136	1.120	1.069	0.981	0.885
	σ_r/σ_1					σ_θ/σ_1				
1	0.574	0.572	0.569	0.583	4.650	0.574	0.572	0.569	0.583	4.650
0.75	0.333	0.320	0.274	0.156	0	0.333	0.325	0.297	0.233	0.048
0.50	0.154	0.140	0.099	0.037	0	0.154	0.144	0.112	0.053	-0.033
0.25	0.055	0.047	0.028	0.009	0	0.055	0.046	0.023	-0.012	-0.049
0	0.025	0.020	0.010	0.003	0	0.025	0.018	-0.002	-0.028	-0.051

TABLE A-3

μ = 0.45, σ_3 = 0, H/D = 1

2z/H	2r/D					2r/D				
	0	0.25	0.50	0.75	1	0	0.25	0.50	0.75	1
	u/Δ					w/Δ				
1	0	0	0	0	0	-0.500	-0.500	-0.500	-0.500	-0.500
0.75	0	0.035	0.073	0.120	0.180	-0.434	-0.434	-0.431	-0.419	-0.367
0.50	0	0.059	0.119	0.182	0.239	-0.315	-0.314	-0.307	-0.288	-0.254
0.25	0	0.071	0.140	0.206	0.262	-0.164	-0.163	-0.157	-0.146	-0.130
0	0	0.074	0.146	0.212	0.268	0	0	0	0	0
	τ_{rz}/σ_1					σ_z/σ_1				
1	0	0.091	0.186	0.315	1.378	0.864	0.858	0.843	0.853	11.966
0.75	0	0.068	0.133	0.161	0	1.021	1.017	1.011	1.016	0.757
0.50	0	0.032	0.049	0.026	0	1.120	1.110	1.081	1.004	0.790
0.25	0	0.009	0.009	-0.005	0	1.160	1.143	1.090	0.988	0.845
0	0	0	0	0	0	1.171	1.152	1.090	0.985	0.868
	σ_r/σ_1					σ_θ/σ_1				
1	0.707	0.702	0.690	0.698	9.791	0.707	0.702	0.690	0.698	9.791
0.75	0.406	0.390	0.334	0.201	0	0.406	0.394	0.356	0.273	0.032
0.50	0.189	0.173	0.124	0.051	0	0.189	0.175	0.134	0.060	-0.053
0.25	0.069	0.059	0.035	0.011	0	0.069	0.058	0.027	-0.020	-0.068
0	0.034	0.027	0.012	0.003	0	0.034	0.024	-0.003	-0.039	-0.069

Appendix B

STRESSES AND DISPLACEMENTS FOR UNCONFINED COMPRESSION, H/D = 2

TABLE B-1

μ = 0.25, σ_3 = 0, H/D = 2

2z/H	2r/D					2r/D				
	0	0.25	0.50	0.75	1	0	0.25	0.50	0.75	1
	u/Δ					w/Δ				
1	0	0	0	0	0	-0.500	-0.500	-0.500	-0.500	-0.500
0.875	0	0.007	0.015	0.026	0.042	-0.448	-0.448	-0.447	-0.443	-0.430
0.750	0	0.012	0.025	0.035	0.056	-0.389	-0.388	-0.386	-0.381	-0.372
0.625	0	0.015	0.030	0.044	0.062	-0.325	-0.324	-0.322	-0.318	-0.313
0.500	0	0.016	0.032	0.047	0.064	-0.260	-0.259	-0.257	-0.254	-0.252
0.375	0	0.016	0.033	0.048	0.064	-0.194	-0.194	-0.192	-0.191	-0.190
0.250	0	0.016	0.033	0.049	0.064	-0.129	-0.129	-0.128	-0.127	-0.127
0.125	0	0.016	0.032	0.048	0.064	-0.064	-0.064	-0.064	-0.064	-0.064
0	0	0.016	0.033	0.048	0.064	0	0	0	0	0
	τ_{rz}/σ_1					σ_z/σ_1				
1	0	0.046	0.098	0.170	0.683	0.886	0.887	0.895	0.932	3.118
0.875	0	0.037	0.072	0.089	0	0.969	0.973	0.969	1.021	0.926
0.750	0	0.016	0.024	0.013	0	1.022	1.024	1.026	1.007	0.915
0.625	0	0.001	-0.002	-0.008	0	1.039	1.037	1.026	0.997	0.943
0.500	0	-0.005	-0.010	-0.011	0	1.035	1.031	1.018	0.995	0.967
0.375	0	-0.005	-0.009	-0.008	0	1.025	1.022	1.011	0.996	0.984
0.250	0	-0.004	-0.006	-0.005	0	1.016	1.014	1.007	0.999	0.993
0.125	0	-0.002	-0.003	-0.002	0	1.011	1.009	1.005	1.000	0.998
0	0	0	0	0	0	1.009	1.008	1.004	1.000	0.999
	σ_r/σ_1					σ_θ/σ_1				
1	0.295	0.296	0.298	0.311	1.039	0.295	0.296	0.298	0.311	1.039
0.875	0.179	0.172	0.147	0.082	0	0.179	0.176	0.165	0.140	0.068
0.750	0.087	0.079	0.054	0.018	0	0.087	0.083	0.070	0.045	0.010
0.625	0.033	0.028	0.016	0.005	0	0.033	0.030	0.021	0.008	-0.006
0.500	0.007	0.006	0.003	0.001	0	0.007	0.006	0.002	-0.003	-0.008
0.375	-0.002	-0.002	-0.002	0.000	0	-0.002	-0.003	-0.004	-0.006	-0.007
0.250	-0.005	-0.004	-0.003	-0.001	0	-0.005	-0.005	-0.005	-0.005	-0.004
0.125	-0.005	-0.004	-0.003	-0.001	0	-0.005	-0.005	-0.005	-0.004	-0.003
0	-0.005	-0.004	-0.002	-0.001	0	-0.005	-0.005	-0.004	-0.004	-0.003

TABLE B-2

μ = 0.4, σ_3 = 0, H/D = 2

2z/H	2r/D					2r/D				
	0	0.25	0.50	0.75	1	0	0.25	0.50	0.75	1
	u/Δ					w/Δ				
1	0	0	0	0	0	-0.500	-0.500	-0.500	-0.500	-0.500
0.875	0	0.013	0.027	0.046	0.072	-0.463	-0.463	-0.461	-0.456	-0.434
0.750	0	0.022	0.045	0.070	0.094	-0.409	-0.408	-0.403	-0.395	-0.381
0.625	0	0.026	0.052	0.079	0.103	-0.344	-0.342	-0.338	-0.330	-0.323
0.500	0	0.028	0.055	0.081	0.106	-0.275	-0.273	-0.270	-0.265	-0.261
0.375	0	0.028	0.055	0.082	0.107	-0.205	-0.204	-0.202	-0.199	-0.197
0.250	0	0.028	0.055	0.081	0.107	-0.136	-0.136	-0.134	-0.133	-0.132
0.125	0	0.027	0.054	0.081	0.107	-0.068	-0.068	-0.067	-0.067	-0.066
0	0	0.027	0.054	0.081	0.107	0	0	0	0	0
	τ_{rz}/σ_1					σ_z/σ_1				
1	0	0.073	0.157	0.273	1.223	0.843	0.842	0.843	0.870	6.889
0.875	0	0.056	0.112	0.136	0	0.975	0.978	0.992	1.020	0.812
0.750	0	0.022	0.032	0.012	0	1.054	1.054	1.047	1.002	0.835
0.625	0	-0.002	-0.010	-0.019	0	1.075	1.069	1.046	0.989	0.895
0.500	0	-0.010	-0.019	-0.021	0	1.065	1.057	1.032	0.990	0.994
0.375	0	-0.010	-0.016	-0.014	0	1.047	1.040	1.022	0.996	0.975
0.250	0	-0.007	-0.010	-0.008	0	1.032	1.028	1.016	1.001	0.992
0.125	0	-0.003	-0.005	-0.004	0	1.024	1.021	1.013	1.004	1.001
0	0	0	0	0	0	1.021	1.018	1.012	1.005	1.004
	σ_r/σ_1					σ_θ/σ_1				
1	0.562	0.561	0.562	0.580	4.599	0.562	0.561	0.562	0.580	4.599
0.875	0.333	0.320	0.272	0.155	0	0.333	0.325	0.299	0.236	0.053
0.750	0.159	0.145	0.101	0.038	0	0.159	0.150	0.120	0.063	0.000
0.625	0.058	0.050	0.030	0.010	0	0.058	0.052	0.032	0.002	-0.032
0.500	0.011	0.009	0.004	0.002	0	0.011	0.008	0.001	-0.014	-0.025
0.375	-0.006	-0.005	-0.004	-0.001	0	-0.006	-0.007	-0.010	-0.014	-0.016
0.250	-0.010	-0.009	-0.006	-0.002	0	-0.010	-0.011	-0.011	-0.013	-0.013
0.125	-0.011	-0.009	-0.006	-0.002	0	-0.011	-0.011	-0.010	-0.010	-0.008
0	-0.010	-0.009	-0.006	-0.002	0	-0.010	-0.010	-0.009	-0.008	-0.005

TABLE B-3

μ = 0.45, σ_3 = 0, H/D = 2

2z/H	2r/D					2r/D				
	0	0.25	0.50	0.75	1	0	0.25	0.50	0.75	1
	u/Δ					w/Δ				
1	0	0	0	0	0	-0.500	-0.500	-0.500	-0.500	-0.500
0.875	0	0.015	0.017	0.029	0.081	-0.472	-0.471	-0.469	-0.463	-0.440
0.750	0	0.025	0.043	0.069	0.107	-0.420	-0.419	-0.414	-0.404	-0.388
0.625	0	0.030	0.057	0.087	0.117	-0.356	-0.354	-0.348	-0.340	-0.330
0.500	0	0.032	0.063	0.093	0.122	-0.285	-0.284	-0.279	-0.273	-0.268
0.375	0	0.032	0.064	0.094	0.124	-0.213	-0.212	-0.209	-0.206	-0.203
0.250	0	0.032	0.064	0.095	0.124	-0.142	-0.141	-0.139	-0.137	-0.136
0.125	0	0.032	0.063	0.094	0.125	-0.071	-0.070	-0.070	-0.069	-0.068
0	0	0.032	0.063	0.094	0.125	0	0	0	0	0
	τ_{rz}/σ_1					σ_z/σ_1				
1	0	0.082	0.175	0.300	1.342	0.827	0.824	0.818	0.836	11.648
0.875	0	0.061	0.122	9.150	0	0.971	0.972	0.979	0.997	0.750
0.750	0	0.024	0.035	0.014	0	1.058	1.055	1.044	0.989	0.796
0.625	0	-0.002	-0.011	-0.022	0	1.083	1.076	1.048	0.981	0.872
0.500	0	-0.011	-0.022	-0.023	0	1.075	1.066	1.038	0.988	0.933
0.375	0	-0.011	-0.018	-0.017	0	1.059	1.051	1.030	0.999	0.973
0.250	0	-0.007	-0.012	-0.010	0	1.045	1.040	1.025	1.007	0.996
0.125	0	-0.004	-0.006	-0.004	0	1.037	1.033	1.024	1.003	1.008
0	0	0	0	0	0	1.034	1.031	1.023	1.014	1.011
	σ_r/σ_1					σ_θ/σ_1				
1	0.677	0.674	0.670	0.684	9.531	0.677	0.674	0.670	0.684	9.531
0.875	0.399	0.383	0.328	0.196	0	0.399	0.389	9.354	0.275	0.038
0.750	0.193	0.176	0.126	0.051	0	0.193	0.181	0.143	0.072	-0.038
0.625	0.073	0.063	0.039	0.014	0	0.073	0.064	0.039	0.001	-0.043
0.500	0.015	0.012	0.006	0.002	0	0.015	0.011	-0.002	-0.018	-0.032
0.375	-0.008	-0.007	-0.006	-0.002	0	-0.008	-0.009	-0.014	-0.019	-0.021
0.250	-0.015	-0.013	-0.009	-0.003	0	-0.015	-0.015	-0.016	-0.016	-0.014
0.125	-0.016	-0.014	-0.009	-0.004	0	-0.016	-0.015	-0.015	-0.013	-0.009
0	-0.016	-0.014	-0.009	-0.004	0	-0.016	-0.015	-0.014	-0.012	-0.008

Appendix C

STRESS AND DISPLACEMENTS FOR CONFINING PRESSURE WITH NO AXIAL COMPRESSION

TABLE C-1

$\mu = 0.25$, $\sigma_1/\sigma_3 = 0.455$, H/D = 2

2z/H	2r/D					2r/D				
	0	0.25	0.50	0.75	1	0	0.25	0.50	0.75	1
	$uE/D\sigma_3$					$wE/D\sigma_3$				
1	0	0	0	0	0	0	0	0	0	0
0.875	0	0.035	0.076	0.130	0.209	0.055	0.053	0.048	0.030	-0.038
0.750	0	0.062	0.128	0.203	0.280	0.070	0.067	0.055	0.029	-0.014
0.625	0	0.076	0.153	0.232	0.309	0.062	0.058	0.046	0.025	0.003
0.500	0	0.081	0.162	0.241	0.319	0.047	0.044	0.034	0.021	0.010
0.375	0	0.082	0.163	0.243	0.321	0.032	0.030	0.024	0.016	0.012
0.250	0	0.082	0.163	0.243	0.321	0.019	0.018	0.015	0.011	0.010
0.125	0	0.082	0.162	0.242	0.321	0.009	0.009	0.007	0.006	0.005
0	0	0.081	0.162	0.242	0.321	0	0	0	0	0
	τ_{rz}/σ_3					σ_z/σ_3				
1	0	0.117	0.250	0.436	1.738	0.750	0.746	0.728	0.638	-4.938
0.875	0	0.094	0.186	0.228	0	0.535	0.525	0.485	0.399	0.642
0.750	0	0.041	0.062	0.033	0	0.398	0.393	0.388	0.434	0.668
0.625	0	0.002	-0.007	-0.023	0	0.354	0.360	0.387	0.463	0.601
0.500	0	-0.013	-0.025	-0.028	0	0.365	0.375	0.408	0.469	0.539
0.375	0	-0.014	-0.023	-0.021	0	0.392	0.401	0.427	0.465	0.497
0.250	0	-0.009	-0.015	-0.012	0	0.416	0.422	0.439	0.460	0.473
0.125	0	-0.005	-0.007	-0.006	0	0.430	0.434	0.445	0.457	0.461
0	0	0	0	0	0	0.435	0.438	0.447	0.456	0.458
	σ_r/σ_3					σ_θ/σ_3				
1	0.250	0.249	0.243	0.213	-1.646	0.250	0.249	0.243	0.213	-1.646
0.875	0.548	0.565	0.629	0.797	1.000	0.548	0.555	0.581	0.647	0.828
0.750	0.782	0.802	0.865	0.957	1.000	0.782	0.792	0.825	0.888	0.977
0.625	0.921	0.932	0.961	0.990	1.000	0.921	0.928	0.949	0.982	1.018
0.500	0.985	0.988	0.995	0.999	1.000	0.985	0.988	1.018	1.011	1.022
0.375	1.008	1.007	1.005	1.001	1.000	1.008	1.009	1.012	1.015	1.017
0.250	1.013	1.011	1.007	1.002	1.000	1.013	1.013	1.013	1.013	1.011
0.125	1.012	1.011	1.006	1.002	1.000	1.012	1.012	1.011	1.008	1.007
0	1.012	1.010	1.006	1.001	1.000	1.012	1.011	1.010	1.008	1.006

TABLE C-2

$\mu = 0.4$, $\sigma_1/\sigma_3 = 0.762$, H/D = 2

2z/H	2r/D					2r/D				
	0	0.25	0.50	0.75	1	0	0.25	0.50	0.75	1
	$uE/D\sigma_3$					$wE/D\sigma_3$				
1	0	0	0	0	0	0	0	0	0	0
0.875	0	0.018	0.038	0.065	0.101	0.036	0.036	0.033	0.025	-0.005
0.750	0	0.030	0.063	0.098	0.132	0.047	0.046	0.040	0.027	0.008
0.625	0	0.037	0.074	0.110	0.144	0.044	0.042	0.035	0.025	0.014
0.500	0	0.039	0.077	0.114	0.149	0.034	0.033	0.028	0.021	0.016
0.375	0	0.039	0.077	0.114	0.150	0.025	0.024	0.020	0.016	0.014
0.250	0	0.039	0.077	0.114	0.150	0.016	0.015	0.013	0.011	0.010
0.125	0	0.038	0.076	0.113	0.150	0.008	0.007	0.006	0.006	0.005
0	0	0.038	0.076	0.113	0.150	0	0	0	0	0
	τ_{rz}/σ_3					σ_z/σ_3				
1	0	0.054	0.116	0.202	0.904	0.878	0.878	0.878	0.858	-3.601
0.875	0	0.042	0.082	0.100	0	0.780	0.778	0.768	0.747	0.901
0.750	0	0.016	0.023	0.009	0	0.721	0.722	0.725	0.760	0.884
0.625	0	-0.002	-0.006	-0.015	0	0.706	0.710	0.728	0.770	0.839
0.500	0	-0.008	-0.014	-0.015	0	0.714	0.719	0.738	0.769	0.803
0.375	0	-0.007	-0.012	-0.011	0	0.727	0.732	0.746	0.765	0.780
0.250	0	-0.005	-0.008	-0.006	0	0.738	0.741	0.750	0.761	0.767
0.125	0	-0.002	-0.004	-0.003	0	0.745	0.747	0.753	0.759	0.761
0	0	0	0	0	0	0.747	0.749	0.753	0.758	0.759
	σ_r/σ_3					σ_θ/σ_3				
1	0.585	0.585	0.585	0.572	-2.401	0.585	0.585	0.585	0.572	-2.401
0.875	0.755	0.764	0.799	0.886	1.000	0.755	0.760	0.780	0.826	0.961
0.750	0.883	0.893	0.926	0.973	1.000	0.883	0.890	0.912	0.954	1.007
0.625	0.957	0.963	0.976	0.993	1.000	0.957	0.962	0.976	0.999	1.024
0.500	0.992	0.993	0.997	0.999	1.000	0.992	0.994	1.001	1.010	1.019
0.375	1.004	1.004	1.003	1.001	1.000	1.004	1.005	1.008	1.011	1.012
0.250	1.007	1.007	1.004	1.001	1.000	1.007	1.008	1.008	1.008	1.007
0.125	1.008	1.007	1.004	1.001	1.000	1.008	1.008	1.007	1.006	1.005
0	1.008	1.007	1.004	1.001	1.000	1.008	1.007	1.007	1.006	1.004

TABLE C-3

$\mu = 0.45,\ \sigma_1/\sigma_3 = 0.874,\ H/D = 2$

2z/H	2r/D					2r/D				
	0	0.25	0.50	0.75	1	0	0.25	0.50	0.75	1
	$uE/D\sigma_3$					$wE/D\sigma_3$				
1	0	0	0	0	0	0	0	0	0	0
0.875	0	0.009	0.047	0.034	0.051	0.022	0.022	0.021	0.017	0.002
0.750	0	0.016	0.035	0.051	0.068	0.030	0.029	0.026	0.020	0.009
0.625	0	0.019	0.039	0.058	0.075	0.028	0.027	0.024	0.018	0.012
0.500	0	0.021	0.041	0.060	0.079	0.024	0.023	0.020	0.016	0.013
0.375	0	0.021	0.041	0.061	0.080	0.018	0.017	0.015	0.013	0.011
0.250	0	0.021	0.041	0.061	0.081	0.011	0.011	0.010	0.009	0.008
0.125	0	0.021	0.041	0.061	0.081	0.005	0.005	0.004	0.004	0.004
0	0	0.021	0.041	0.061	0.081	0	0	0	0	0
	τ_{rz}/σ_3					σ_z/σ_3				
1	0	0.028	0.060	0.104	0.460	0.934	0.935	0.937	0.932	-2.771
0.875	0	0.021	0.042	0.051	0	0.885	0.885	0.883	0.879	0.963
0.750	0	0.008	0.012	0.006	0	0.855	0.857	0.861	0.880	0.946
0.625	0	0	-0.003	-0.007	0	0.847	0.849	0.859	0.881	0.919
0.500	0	-0.003	-0.007	-0.007	0	0.848	0.851	0.861	0.878	0.897
0.375	0	-0.004	-0.006	-0.006	0	0.853	0.856	0.863	0.874	0.883
0.250	0	-0.002	-0.004	-0.003	0	0.857	0.859	0.864	0.870	0.875
0.125	0	-0.001	-0.002	-0.002	0	0.860	0.861	0.864	0.868	0.870
0	0	0	0	0	0	0.861	0.862	0.865	0.868	0.869
	σ_r/σ_3					σ_θ/σ_3				
1	0.765	0.765	0.767	0.763	-2.267	0.765	0.765	0.767	0.763	-2.267
0.875	0.860	0.865	0.884	0.930	1.000	0.860	0.863	0.875	0.904	0.986
0.750	0.931	0.937	0.954	0.981	1.000	0.931	0.935	0.948	0.973	1.012
0.625	0.973	0.976	0.985	0.995	1.000	0.973	0.976	0.985	0.998	1.014
0.500	0.994	0.995	0.997	0.999	1.000	0.994	0.995	1.000	1.006	1.011
0.375	1.002	1.002	1.002	1.001	1.000	1.002	1.003	1.007	1.007	1.008
0.250	1.005	1.005	1.003	1.001	1.000	1.005	1.005	1.006	1.006	1.005
0.125	1.006	1.005	1.004	1.002	1.000	1.006	1.006	1.005	1.005	1.004
0	1.006	1.005	1.004	1.002	1.000	1.006	1.006	1.005	1.004	1.003