Some Implications of Viscoelastic Subgrade Behavior

B. B. SCHIMMING, Assistant Professor, Department of Civil Engineering, University of Notre Dame; and
J. VALERA, Research Assistant, Department of Civil Engineering, University of California, Berkeley

The two-dimensional viscoelastic displacement equations for a Voigt solid are approached on a numerical basis to examine the effect of viscosity on the load-settlement-time relation in an ideal soil. Curves indicating the viscosity effect on surface settlement and displacement within the soil mass as a function of time for an applied surface load are presented. In addition, the generality of the method for approaching mixed boundary conditions is demonstrated by the determination of heave in the bottom of an excavation due to an applied surface load.

•THE PROBLEM of determining the stresses and displacements within a homogeneous soil mass due to an imposed surface load has been extensively treated in the literature. It is usually assumed in problems of this type that the deformations and stresses developed within the soil can be computed on the basis of the classical theory of elasticity which assumes that a linear relation exists between stress and strain and that strains are small. The basic solution of this problem is the well-known Boussinesq solution for a single, vertical point load acting on the horizontal surface of a semi-infinite, homogeneous, isotropic elastic solid (1). This solution has been extended to include various surface load configurations by a number of investigators (2, 3, 4, 5, 6, 7, 8). Their results have been useful and of proven practical value.

However, the presence of viscous effects in the mechanical behavior of soil raises a question regarding the influence of this effect on the resulting stress analysis. Instead of being considered elastic, the foundation may, therefore, be more closely approximated by a viscoelastic medium. This problem differs from the corresponding elastic problem, since time appears in the stress-strain relations and, hence, the boundary conditions and the solution must involve the history of the process throughout the time range of interest. The variation of the displacements and the stress distribution with time is sought, and it is found that, in general, the history of loading has a marked influence. This is in contrast to the corresponding elastic problem for which the displacements and stress distribution are functions only of the instantaneous values of the surface displacements and stresses, and not of the loading history (9).

The two elements whose combination represents linear viscoelastic behavior are the linearly elastic spring and the viscous dashpot filled with a Newtonian liquid. The motion of the piston inside the dashpot produces a resisting force in the liquid which is proportional to the velocity of the piston. Figure 1 shows combinations of these elements which form the simplest linear viscoelastic models. If a soil mass is represented as a Voigt material, the displacements will eventually approach the elastic values, whereas with a Maxwell representation, the initial displacements will equal the elastic displacements. Thus, a Voigt solid is of interest with regard to short-term departures from classical elasticity, whereas a Maxwell material relates to long-term departures. More sophisticated models may be constructed by different combinations of the two original elements (10, 11, 12, 13, 14).

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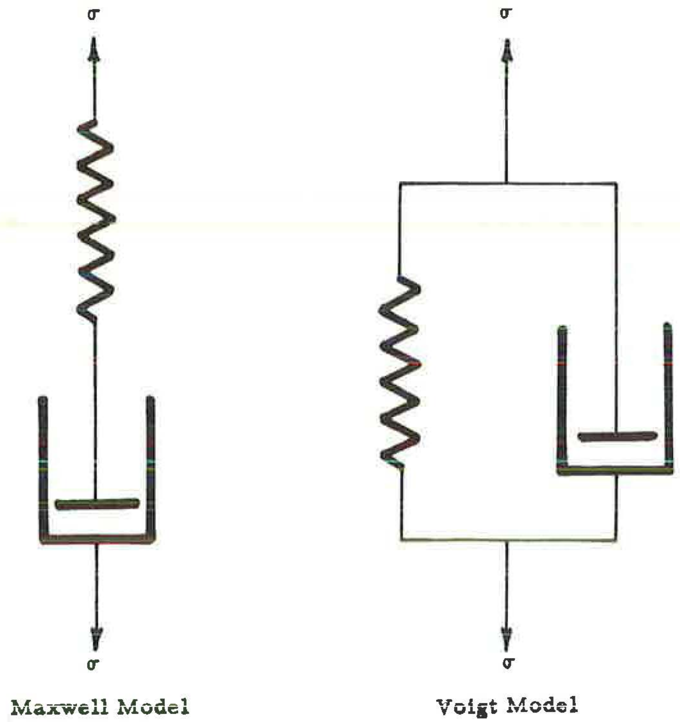


Figure 1. Typical viscoelastic models.

In recent years it has been shown that under certain circumstances the stress analysis of a system including linear viscoelastic components can be treated in terms of the analogous elastic problem having the same geometry and boundary conditions. In the case of a system, for which the geometry does not change and the boundary conditions remain of the same type during the loading process, application of the Laplace transform removes the time dependence and transforms the problem into an associated elastic problem. Thus, the determination of transformed variables, e.g., of stress or displacement, becomes an elasticity problem and standard methods of inversion of the Laplace transform determine the stresses and displacements as a function of time. Hoskin and Lee (15) and Lee (16) have determined stress as a function of time for a viscoelastic Maxwell foundation acted on by a uniformly loaded elastic plate. Pister (17) obtained a solution to the same problem where the plate was also viscoelastic. In all these cases, Laplace transforms were employed, so that it was first necessary to know the elastic solution to the problem before the viscoelastic solution could be found. If the elastic solution is unknown, the solution to the corresponding viscoelastic problem cannot be obtained with this approach.

In view of this situation, a desirable alternative would be to approach the viscoelastic problem directly on a numerical basis, thus eliminating the requirement of an elastic solution which may not be available for a particular foundation problem.

The ultimate objective of this presentation is to examine the effect of the Voigt solid viscosity parameters on the short-term displacements of a soil mass subjected to boundary forces. Of particular interest is the development of lateral strains beneath a uniform surface load as a function of time which gives rise to displacements of the type schematically indicated in Figure 2. The importance of this effect has been recently described by Lambe (18).

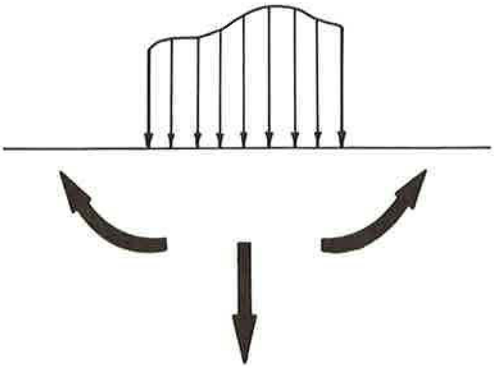
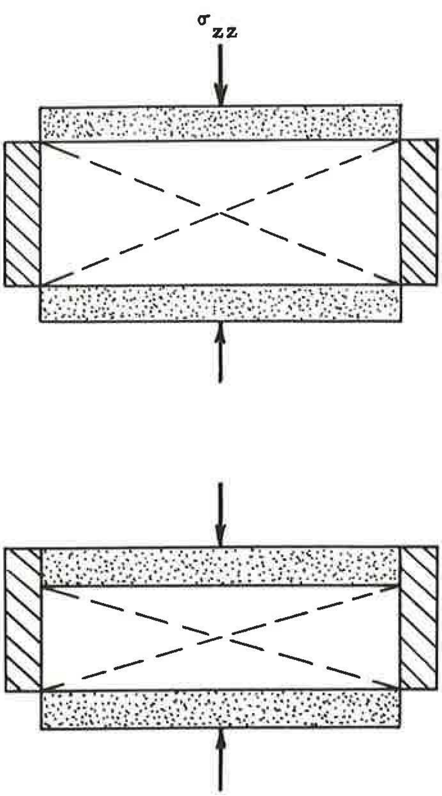


Figure 2. Schematic representation of displacements.



$$\sigma_{zz} = \lambda(\epsilon_{zz} + \cancelto{0}{\epsilon_{yy}} + \cancelto{0}{\epsilon_{xx}}) + 2\mu\,\epsilon_{zz}$$

$$\sigma_{zz} = (\lambda + 2\mu)\epsilon_{zz}$$

Figure 3. One-dimensional stress strain.

DESCRIPTION OF VOIGT SOLID

The development of the two-dimensional field equations for a soil requires a description of both the dilatational and distortional components of behavior. The inability of the conventional one-dimensional test to provide this description is demonstrated with respect to the consolidation test in Figure 3. Even if the soil is considered to be elastic and the appropriate stress-strain relations are written in terms of Lame's constants, the resulting response involves both volume change and shear, thus disallowing their individual determination. Any realistic viscoelastic description of a soil will eventually necessitate the development of appropriate laboratory tests for the determination of the fundamental mechanical constants involved.

The following development is most conveniently accomplished by formulating the elastic equation and then converting to the viscoelastic condition. In the absence of an exact stress-strain-time relation, a Voigt solid which is known to be similar to soil in some respects was assumed.

The relationships between stress and strain for a homogeneous, isotropic elastic solid may be written as:

$$\sigma_{xx} = \lambda\Delta + 2\mu\epsilon_{xx}, \quad \sigma_{yy} = \lambda\Delta + 2\mu\epsilon_{yy}, \quad \sigma_{zz} = \lambda\Delta + 2\mu\epsilon_{zz},$$

$$\tau_{yz} = \mu\epsilon_{yz}, \quad \tau_{zx} = \mu\epsilon_{zx},$$

$$\tau_{xy} = \mu\epsilon_{xy} \qquad (1)$$

where

$$\text{Lame's coefficient } \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},$$

$$\text{Lame's coefficient } \mu = \frac{E}{2(1+\nu)}, \text{ and}$$

$$\Delta = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

Substitution of the stress-strain relationships into the equilibrium equations in the x-y plane yields in the absence of body forces:

$$(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 v}{\partial x \partial y} + \lambda \frac{\partial^2 w}{\partial x \partial z} = 0 \tag{2}$$

and

$$\mu \frac{\partial^2 v}{\partial x^2} + (\lambda + 2\mu) \frac{\partial^2 v}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 w}{\partial y \partial z} = 0 \tag{3}$$

where u, v and w are the displacements in the x, y and z directions, respectively.

For a condition of plane strain $\frac{\partial w}{\partial z} = 0$. Eqs. 2 and 3 reduce to:

$$(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 v}{\partial x \partial y} = 0 \tag{4}$$

and

$$\mu \frac{\partial^2 v}{\partial x^2} + (\lambda + 2\mu) \frac{\partial^2 v}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 u}{\partial x \partial y} = 0 \tag{5}$$

Eqs. 4 and 5 can be modified to express the behavior of a Voigt solid by an appropriate change of coefficients (19). Substitution of $\lambda + \lambda' \frac{\partial}{\partial t}$ for λ and $\mu + \mu' \frac{\partial}{\partial t}$ for μ into Eq. 4 yields:

$$\left[\lambda + \lambda' \frac{\partial}{\partial t} + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right)\right] \frac{\partial^2 u}{\partial x^2} + \left(\mu + \mu' \frac{\partial}{\partial t}\right) \frac{\partial^2 u}{\partial y^2} + \left(\lambda + \lambda' \frac{\partial}{\partial t} + \mu + \mu' \frac{\partial}{\partial t}\right) \frac{\partial^2 v}{\partial x \partial y} = 0 \tag{6}$$

where λ' and μ' are the viscosity coefficients corresponding to the Lame coefficients. On rearranging terms, Eq. 6 becomes:

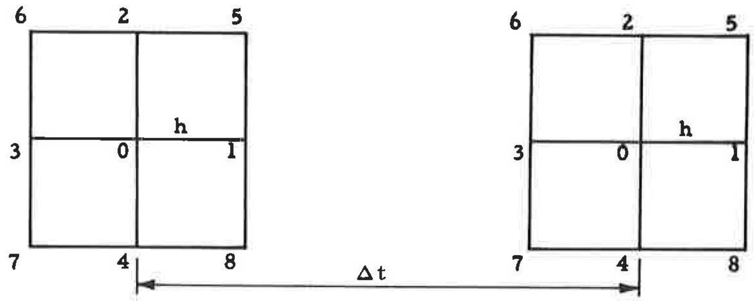
$$(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + (\lambda' + 2\mu') \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) + \mu \frac{\partial^2 u}{\partial y^2} + \mu' \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial y^2}\right) + (\lambda + \mu) \frac{\partial^2 v}{\partial x \partial y} + (\lambda' + \mu') \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial x \partial y}\right) = 0 \tag{7}$$

Proceeding in the same manner for Eq. 5, Eq. 8 is obtained:

$$\mu \frac{\partial^2 v}{\partial x^2} + \mu' \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial x^2}\right) + (\lambda + 2\mu) \frac{\partial^2 v}{\partial y^2} + (\lambda' + 2\mu') \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial y^2}\right) + (\lambda + \mu) \frac{\partial^2 u}{\partial x \partial y} + (\lambda' + \mu') \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = 0 \tag{8}$$

Eqs. 7 and 8 are the equations of equilibrium in terms of displacements for a two-dimensional linear viscoelastic Voigt medium for the particular case where a plane strain condition exists.

Eqs. 7 and 8 can now be reduced to finite difference form. A typical difference approximation to Eq. 7 along with the grid notation is given in Figure 4. By this tech-



$$A_1(u_3 - 2u_0 + u_1)_{t_0+\Delta t} + B_1(u_2 - 2u_0 + u_4)_{t_0+\Delta t} + A_2(v_5 - v_6 + v_7 - v_8)_{t_0+\Delta t} = A_3(u_3 - 2u_0 + u_1)_{t_0} + B_2(u_2 - 2u_0 + u_4)_{t_0} + A_4(v_5 - v_6 + v_7 - v_8)_{t_0}$$

where
$$A_1 = (\lambda\Delta t + 2\mu\Delta t + 2\lambda' + 4\mu')$$
$$A_3 = (-\lambda\Delta t - 2\mu\Delta t + 2\lambda' + 4\mu')$$
$$A_2 = 1/4(\lambda\Delta t + \mu\Delta t + 2\lambda' + 2\mu')$$
$$A_4 = 1/4(-\lambda\Delta t - \mu\Delta t + 2\lambda' + 2\mu')$$
$$B_1 = \mu\Delta t + 2\mu' \qquad B_2 = -\mu\Delta t + 2\mu'$$

Figure 4. Finite difference approximation.

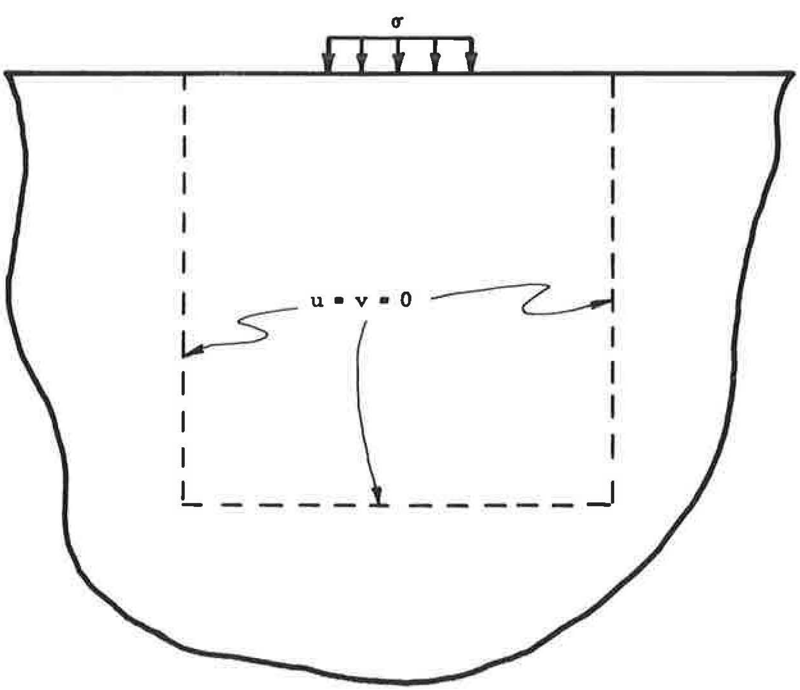


Figure 5. Statement of boundary conditions.

nique, the governing partial differential equations that are often impossible to solve in closed form can be transformed into a set of linear simultaneous algebraic equations. This transformation involves expressing the pertinent equations in terms of values of the desired function at finite points in the system under analysis. By utilizing a digital computer, the resulting equations can then be solved by a matching procedure with respect to the independent time variable.

For the particular case of a uniform surface load, the dependent displacement variables must be defined on the boundary of the region beneath the load which requires an approximation in the case of a numerical approach to a half space. Thus, the displacements were set equal to zero along the dashed lines in Figure 5. The effect of this assumption will be subsequently discussed.

DISPLACEMENT RESULTS

Figure 6 is a graph of the displacement of the surface at the centerline of the load vs the logarithm of time for various values of the viscosity coefficients. It is of interest to note the significant role played by μ' which is equivalent to a shear viscosity. The largest value of $\mu' = 5,000 \times E$ displays the most retarded displacement. This effect is considerably larger than the role played by λ'. Thus, for the configuration of this problem, the resistance to lateral strains beneath the applied load as dictated by a

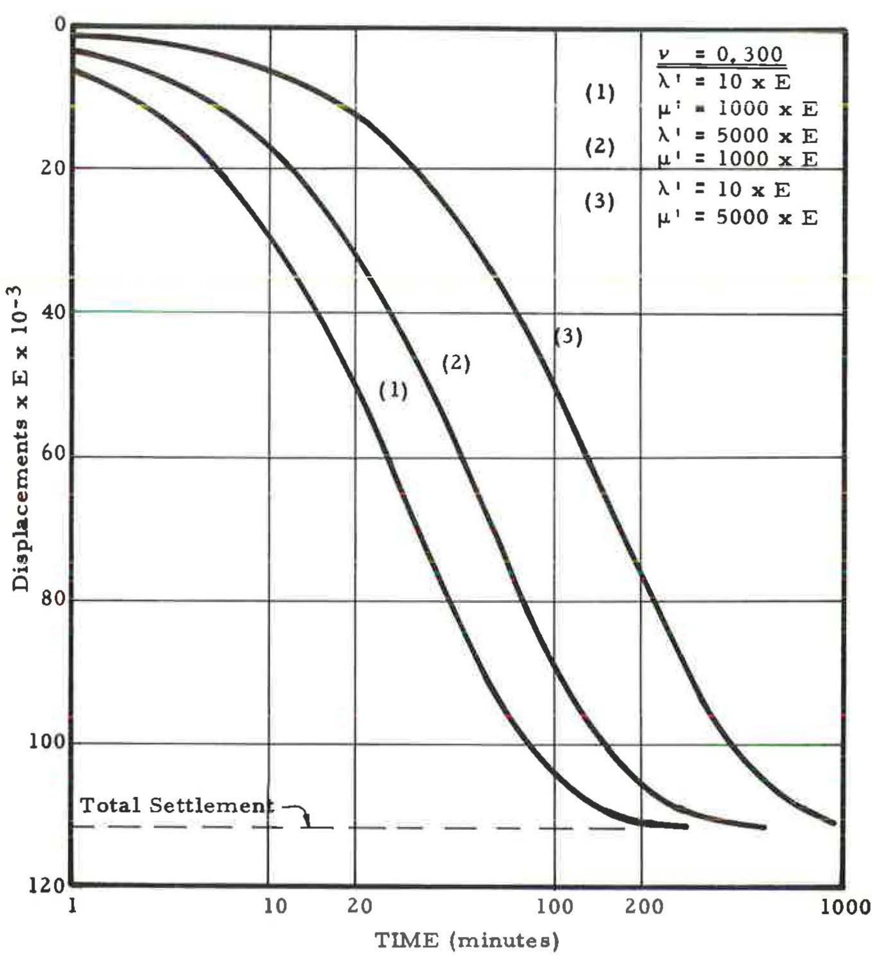


Figure 6. Vertical displacement at centerline of load.

large shear viscosity can significantly alter the deflection time relationship. The ultimate settlement is, of course, independent of the viscosity coefficient for a Voigt material, depending only on the elastic constants which are the same for all the cases in Figure 6.

Figures 7 and 8 present vertical displacements as a function of depth and time. Again, the shear viscosity effect is graphically displayed. In Figure 7, which involves the smaller shear viscosity, an actual heave or bulge initially occurs adjacent to the applied load. Gradually, as volume change takes place, the top surface subsides, removing the heave until the elastic equilibrium position is reached.

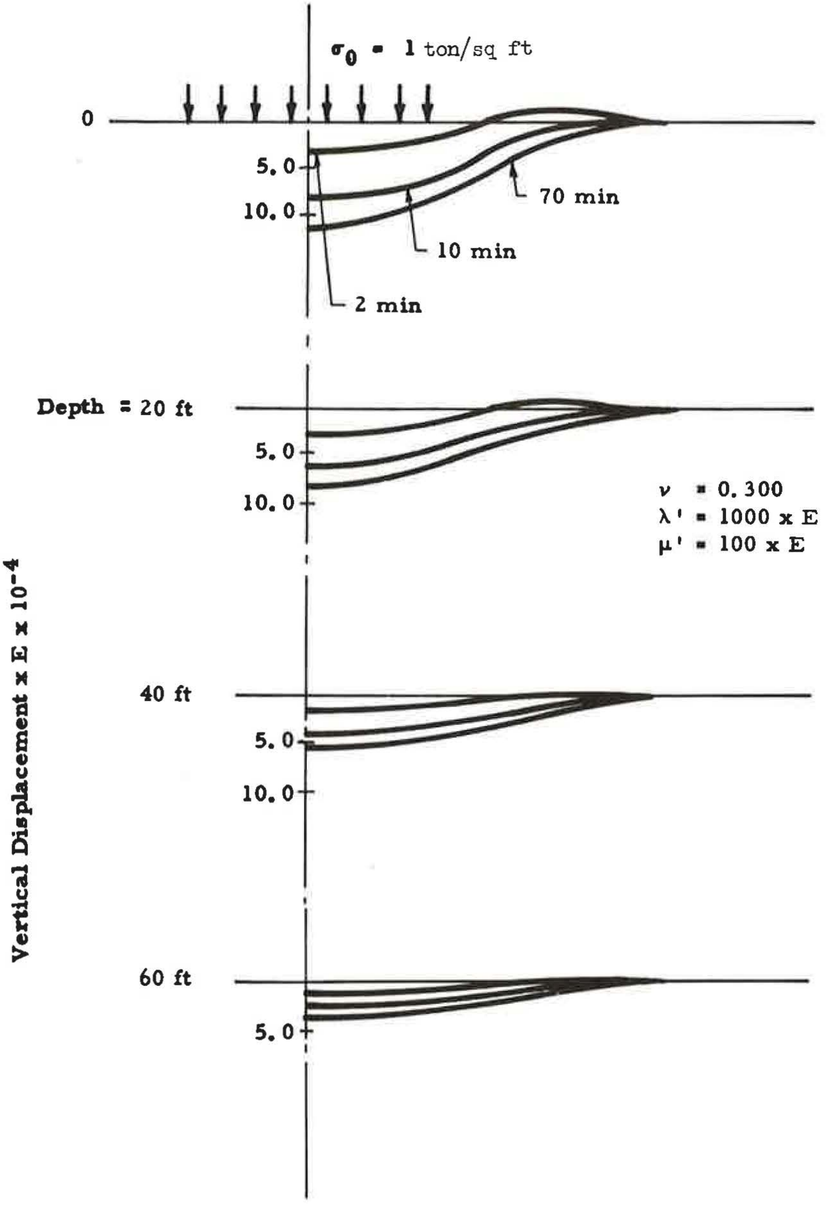


Figure 7. Vertical displacement as function of depth and time.

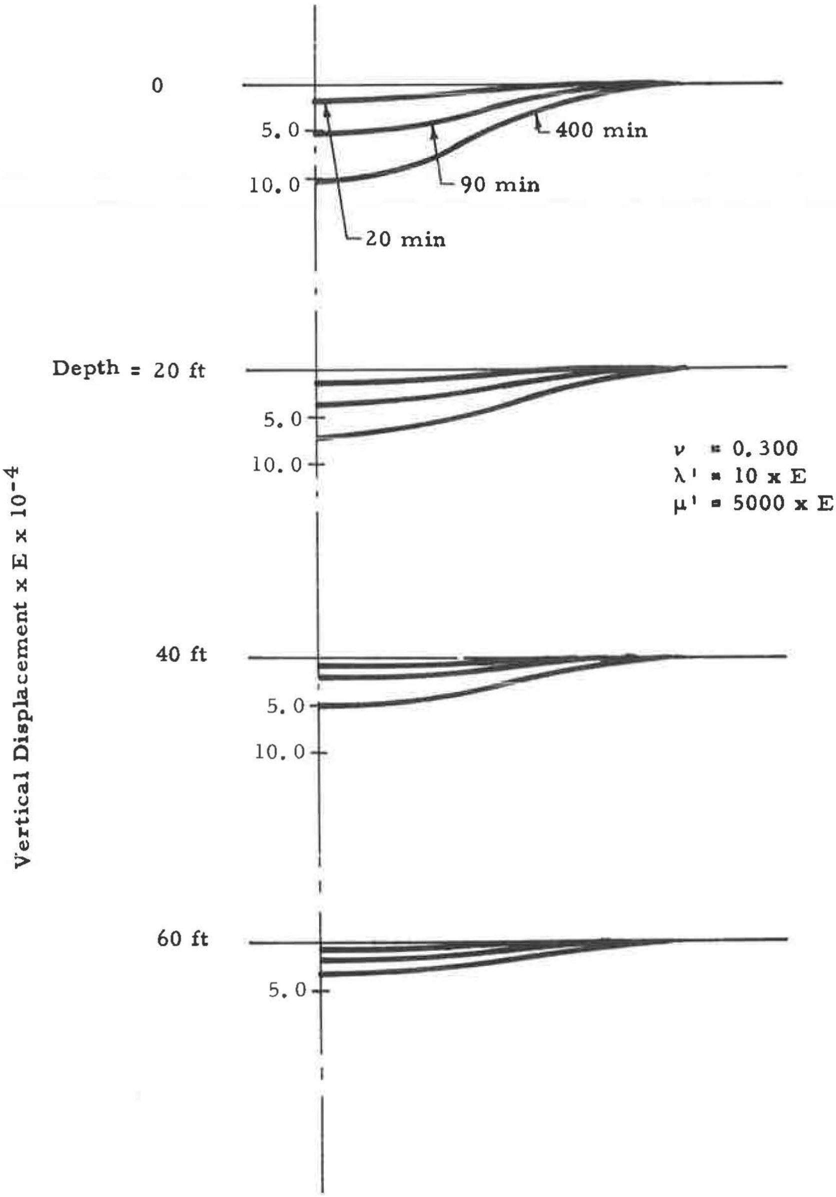


Figure 8. Vertical displacement as function of depth and time.

In contrast, a large shear viscosity, as indicated in Figure 8, shows no initial heave at the surface. This essentially means that the development of lateral strains is retarded which, in turn, retards the surface displacement. This is, of course, consistent with the settlement time plots in Figure 6.

To examine the boundary effect assumption, the vertical stresses at two different depths are compared in Figure 9. One distribution is the elastic case as presented by Jurgenson (20), and the other is the final viscoelastic value. The agreement appears to be acceptable in view of the numerical approximation involved. In addition, the motivation for the study was not to develop quantitatively a specific solution, but also to examine in a qualitative manner the effect of the time-dependent viscoelastic parameters on the form of the solution.

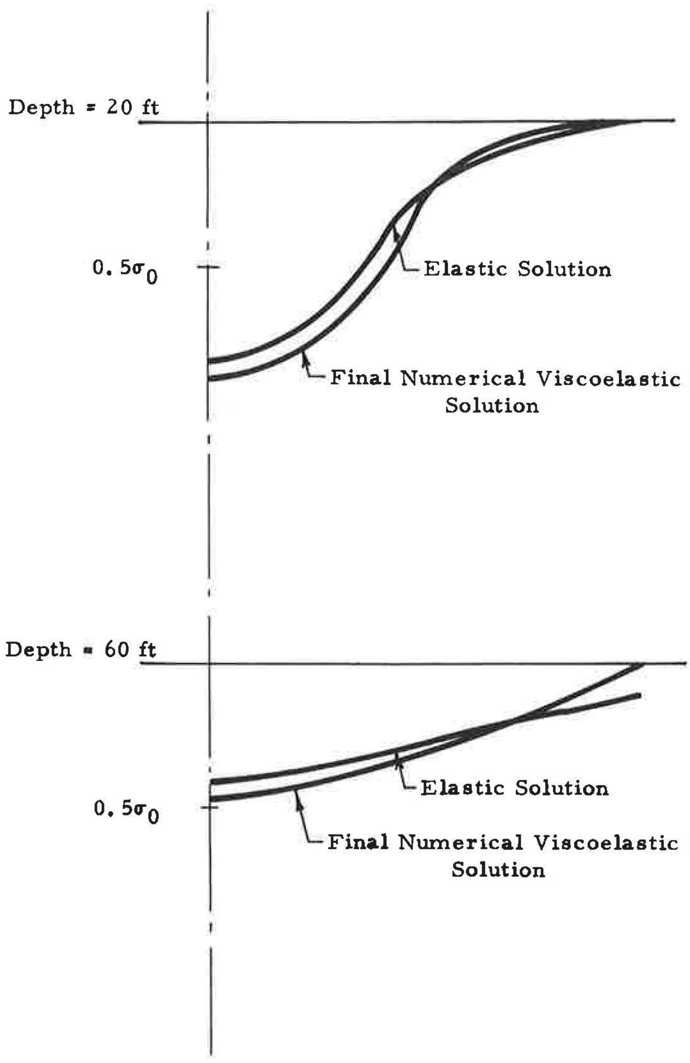


Figure 9. Vertical Stress distribution.

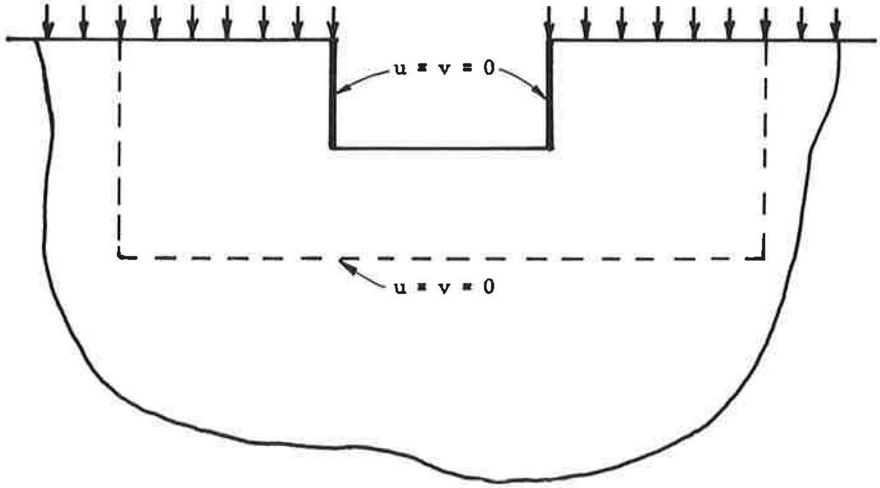


Figure 10. Description of excavation problem.

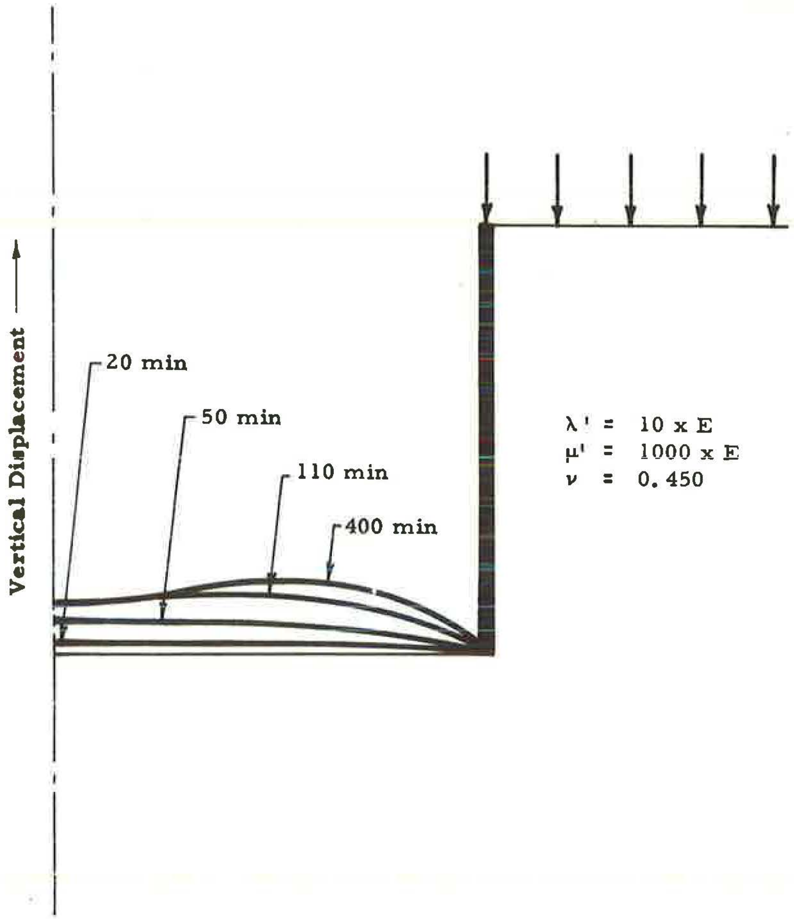


Figure 11. Displacements at bottom of excavation.

A final comment regarding the flexibility of the numerical procedure employed can be made with reference to Figure 10. The upward displacement of the bottom of an excavation due to the application of a uniform load at the surface is desired. The sides of the excavation are assumed to be fixed against any movement ($u = v = 0$).

The resulting heave at the bottom of the cut as a function of time is shown in Figure 11. The particular shape of these curves is, of course, a function of the geometry of the excavation. A very wide excavation with respect to depth would probably exhibit very little heave at the centerline.

CONCLUSIONS

A particular viscoelastic material, a Voigt solid, was chosen to represent the behavior of a soil for the purpose of analyzing the effect of viscoelastic behavior on certain common foundation situations. No illusion is entertained regarding the adequacy of a Voigt solid. A more complete equation of state is, of course, required. However, the importance of considering not only the time-dependent volume change characteristics of soil but also the time-dependent distortional properties has been qualitatively demonstrated. A one-dimensional theory cannot be expected to describe the time-dependent settlement of isolated surface loads located on deep layers of soil.

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