Adjustment of Trilateration in Fundamental Figures

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A fundamental figure is a quadrilateral or a central-point triangle which forms the basis of the problem of adjustment in trilateration. Any trilateration may be planned, analyzed and adjusted in fundamental figures. Area equations are used for the formation of the condition equations in the least squares adjustment of the sides of the triangles of fundamental figures. Evaluation of the coefficients of the error equation or correction equation is the first task in the adjustment and can be done systematically as shown. Cases for both geodetic and engineering trilaterations are presented.

•TRIANGULATION has been the conventional method used for the horizontal control of geodetic surveying and for some of the control of engineering surveys since Willebroad Snellins (1591-1626) used it in Holland in 1617. In such an operation, few bases and all of the angles are measured because long-distance measurement to the required accuracy is tedious, time consuming and difficult. The adjustment of triangulation is also complex and requires high technique.

Several years ago, the author tried to simplify the routine work of the adjustment of triangulation in fundamental figures for the convenience of engineers and recommended its use to geodesists (7, 8). Since then, the advancement of the electronic distance measuring instruments such as Shoran, Hiran, Geodimeter, tellurometer and Electrotape and the increasing interest in the adoption of trilateration have caused him to consider applying his ideas on this adjustment to the adjustment of trilateration.

Distance is the basic geometric element in the position science of surveying and geodesy (6). We can not determine horizontal positions by triangulation measurements without at least one known length, but we can determine positions by a trilateration scheme without any angular measurement. We have been using triangulation because we did not have a handy and reliable method of obtaining a great quantity of precise distances. Since the revolution of distance measurement by electronic instruments, trilateration has become increasingly significant. Field operations with electronic instruments have been cautiously carried on. The problem of adjustment has been along the line of traditional triangulation method.

Mechanically, angular measurement with the optical theodolite has its limitations. Even the electrooptical Geodimeter can substitute for the theodolite without difficulty. Other electronic distance measuring instruments are limited very little by weather and can measure long distances (6).

Current literature in the fields of geodesy and surveying generally contains two basic analytical approaches to the adjustment of trilateration: (a) indirect adjustment by calculation of the variation of plane or geodetic coordinates (2, 4, 13, 14, 17); and (b) conditional adjustment by conversion of the lengths of the trilateration into the angular condition equations (1, 11, 15, 16, 18, 19, 20). There are still many other graphical methods, analogue methods (5, 10) and methods for which the Laplace conditions are attached (3, 4), but they are variations of the basic analytical approach.

In this paper, the problem of adjustment of pure trilateration (no angular observations) is attacked by the basic analytical approach of conditional adjustment by area

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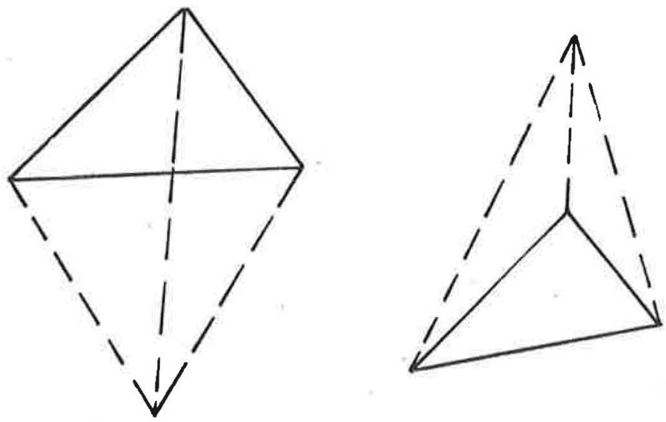

Figure 1. Formation of quadrilateral and central-point triangle.

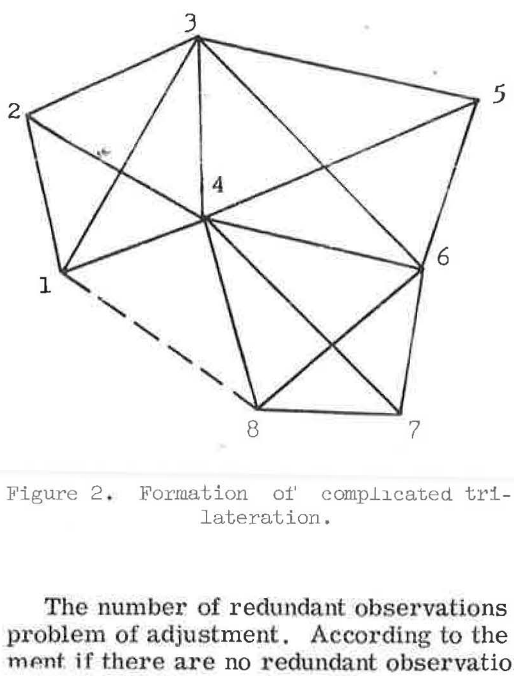

Figure 2. Formation of complicated trilateration.

equations of direct linear measurements. There are no angles, no coordinates and, hence, no trigonometric functions involved in the plane trilateration case in this approach. In the development of this method, both geodetic and engineering purposes are considered.

FUNDAMENTAL FIGURES

A simple triangle is the basic element of trilateration as it is in triangulation, but unlike triangulation, there is no redundant observation when the three sides are measured instead of three angles. When a new point is attached to a triangle to form a two-triangle trilateration, there is still no redundant observation unless the attachment is led to all three vertices to form a four-triangle overlapped quadrilateral or central-point triangle as shown in Figure 1.

The number of redundant observations is equal to the number of conditions in the problem of adjustment. According to the theory of adjustment, there can be no adjustment if there are no redundant observations. The quadrilateral or the central-point triangle, each having one geometrical condition, starts the problem of adjustment in trilateration. We shall call them the fundamental figures.

For one geometric condition, unlike the traditional method of forming equations by using angles in terms of triangle sides indirectly, there is only one way to form the area equation in terms of sides directly for a fundamental figure. Therefore, it is unique and consistent in adjustment and accuracy. This is another meaning of fundamental figures.

The fundamental figures are the fundamental units of more complicated trilateration in the geometric consideration of the formation of the figures and also, as we shall see later, in the mathemetical handling of a large number of equations. As shown in Figure 2, 1 to 8 is a waste measurement unless we make another measurement from 1 to 7 or 8 to 2 to form one more fundamental figure, or from 1 to 6 or 8 to 3 to form two more fundamental figures, in addition to the original three fundamental figures.

Of course, we can measure from 1 to 6, 1 to 7, 8 to 2 and 8 to 3 to form six more fundamental figures. If 1 and 8 are known fixed points, the original three fundamental figures are sufficient for the adjustment of the observed sides to fix the six unknown points.

The use of fundamental figures as an index to identify the number of conditions involved and to study the accuracy of the figures of the more complicated geodetic trilateration, with or without restraints and with or without Laplace orientation, can be developed further. In this introductory paper, however, only the theory of using area equations of fundamental figures and the possible application of these equations to the systematic adjustment of geodetic and engineering trilaterations are presented.

AREA CONDITION EQUATIONS

The area equations to be used in this new approach of adjustment of trilateration are, for plane triangles,

$$f_p = \sqrt{s(s - \ell_1)(s - \ell_2)(s - \ell_3)} \tag{1a}$$

or as given earlier (9)

$$f_p = \sqrt{[(\ell_2 + \ell_3)^2 - \ell_1^2][\ell_1^2 - (\ell_2 - \ell_3)^2]}/4 \tag{1b}$$

and for spherical triangles, according to Lhuilier's formula,

$$\begin{aligned} f_s &= R^2E \\ &= 4R^2 \operatorname{arc\,tan} \sqrt{\tan \tfrac{1}{2}\left(\frac{s}{R}\right) \tan \tfrac{1}{2}\left(\frac{s - \ell_1}{R}\right) \tan \tfrac{1}{2}\left(\frac{s - \ell_2}{R}\right) \tan \tfrac{1}{2}\left(\frac{s - \ell_3}{R}\right)} \\ &= 4R^2 \operatorname{arc\,tan} c \end{aligned} \tag{2}$$

where

ℓ_1, ℓ_2, ℓ_3 = sides in straight-line distances for plane triangles or in reduced spherical distances for spherical triangles,

$s = \tfrac{1}{2}(\ell_1 + \ell_2 + \ell_3)$;

R = mean radius of earth's sphere = 3, 959 mi = 6, 371 km;

E = spherical excess; and

$$c = \sqrt{\tan \tfrac{1}{2}\left(\frac{s}{R}\right) \tan \tfrac{1}{2}\left(\frac{s - \ell_1}{R}\right) \tan \tfrac{1}{2}\left(\frac{s - \ell_2}{R}\right) \tan \tfrac{1}{2}\left(\frac{s - \ell_3}{R}\right)}. \tag{3}$$

For the fundamental figures, the conditional equations in terms of areas as shown in Figures 3a and 3b are for a quadrilateral

$$\omega_Q = f_I - f_{II} + f_{III} - f_{IV} = 0 \tag{4}$$

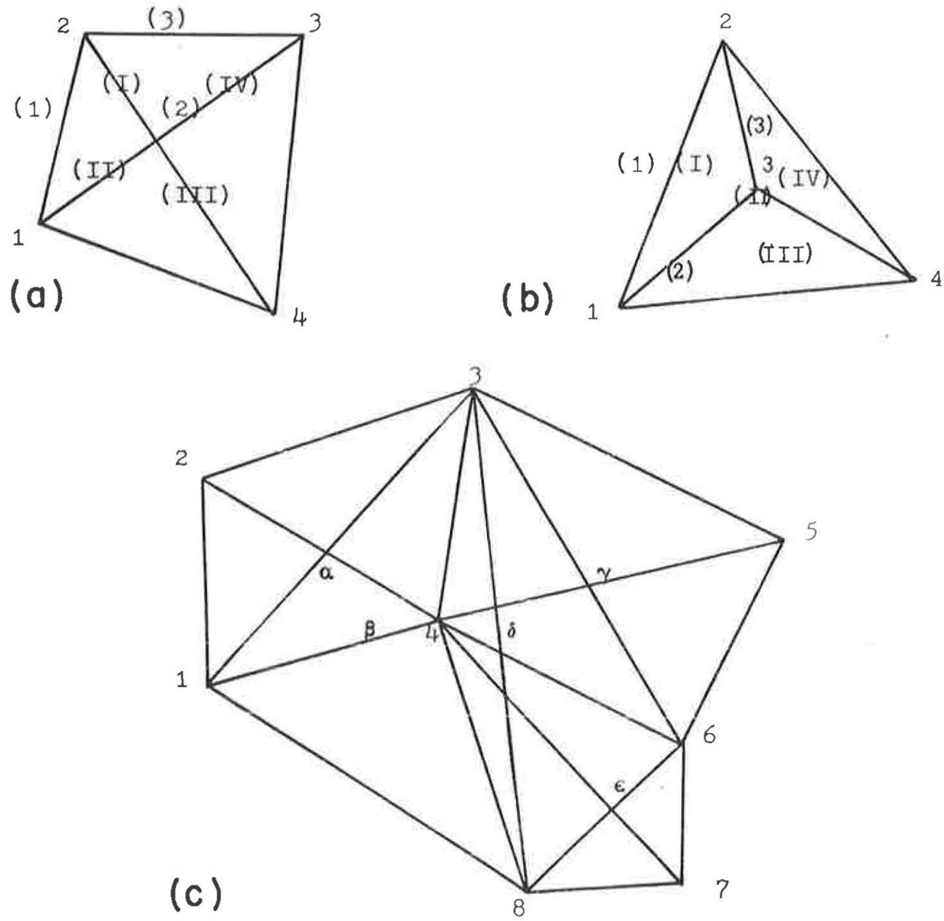

Figure 3. Area conditions.

and for a central-point triangle,

$$\omega_T = f_I + f_{II} - f_{III} - f_{IV} = 0 \tag{5}$$

For a more complicated figure of trilateration, the condition matrix of areas can be easily written according to a number of r fundamental figures (e.g., Fig. 3c, with $r = \epsilon$)

$$\begin{vmatrix} \omega_\alpha \\ \omega_\beta \\ \omega_\gamma \\ \omega_\delta \\ \omega_\epsilon \end{vmatrix} = \begin{vmatrix} f_{\alpha\text{-}I} - f_{\alpha\text{-}II} + f_{\alpha\text{-}III} - f_{\alpha\text{-}IV} \\ -f_{\beta\text{-}I} + f_{\beta\text{-}II} - f_{\beta\text{-}III} - f_{\beta\text{-}IV} \\ f_{\gamma\text{-}I} - f_{\gamma\text{-}II} + f_{\gamma\text{-}III} - f_{\gamma\text{-}IV} \\ f_{\delta\text{-}I} - f_{\delta\text{-}II} + f_{\delta\text{-}III} - f_{\delta\text{-}IV} \\ f_{\epsilon\text{-}I} - f_{\epsilon\text{-}II} + f_{\epsilon\text{-}III} - f_{\epsilon\text{-}IV} \end{vmatrix} = W_o = 0 \tag{6}$$

where

$f_{\alpha\text{-III}} = f_{\beta\text{-I}}$,

$f_{\beta\text{-IV}} = f_{\delta\text{-II}}$,

$f_{\gamma\text{-I}} = f_{\delta\text{-I}}$, and

$f_{\delta\text{-IV}} = f_{\epsilon\text{-II}}$ (see Table 2).

THEORY OF ADJUSTMENT

The relations of the one-column matrices of the observed lengths of the side ℓ's, their error e's or correction v's and the most probable values ℓ_o's of the sides are

$$L - L_o = E = -V$$

and

$$L + V = L_o \tag{7}$$

The condition matrix W_o is a function of area and in turn a function of length. Through expansion by Taylor's theorem and omission of the terms of and over second order, the condition matrix W_o in terms of a number of n observed lengths and their corrections becomes:

$$\begin{aligned} W_o(\ell_o) &= W_o(\ell + v) \\ &= W(\ell) + \frac{\partial W}{\partial L} V \\ &= W + B'V = 0 \end{aligned} \tag{8}$$

where B' is a transposed matrix of B,

$$\underset{r\times 1}{W} = \begin{vmatrix} \omega_\alpha \\ \omega_\beta \\ \cdot \\ \cdot \\ \cdot \\ \omega_\rho \end{vmatrix} \tag{9}$$

$$\underset{n\times 1}{V} = \begin{vmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_n \end{vmatrix} \tag{10}$$

$$\underset{r\times n}{B'} = \frac{\partial W}{\partial L} = \begin{vmatrix} \frac{\partial \omega_\alpha}{\partial \ell_1} & \frac{\partial \omega_\alpha}{\partial \ell_2} & \cdots & \frac{\partial \omega_\alpha}{\partial \ell_n} \\ \frac{\partial \omega_\beta}{\partial \ell_1} & \frac{\partial \omega_\beta}{\partial \ell_2} & \cdots & \frac{\partial \omega_\beta}{\partial \ell_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial \omega_\rho}{\partial \ell_1} & \frac{\partial \rho}{\partial \ell_2} & \cdots & \frac{\partial \omega_\rho}{\partial \ell_n} \end{vmatrix} = \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \rho_1 & \rho_2 & \cdots & \rho_n \end{vmatrix} \tag{11}$$

and there are only six non-zero α's, β's, ... , ρ's in each row for the six sides of each fundamental figure. By the method of least squares using Causs' correlate k's and differentiating the conditional minimum,

$$\Phi = V'PV - 2K'(W + B'V) = \text{min.}$$

$$\frac{\partial \Phi}{\partial V} = V'P - K'B' = 0$$

we obtain

$$V = \overline{P}BK \tag{12}$$

where $\overline{P}$ is the inverse matrix of weight. By substituting into Eq. 8, we obtain the normal equation matrix

$$B'\overline{P}BK = MK = -W \tag{13}$$

and then

$$K = -\overline{M}W \tag{14}$$

Knowing $\overline{P}$ and B, we can compute

$$M = B'\overline{P}B \tag{15}$$

By solving for K, V can be evaluated.

EVALUATION OF B MATRIX

In computing M, the matrix B has to be evaluated if $\overline{P}$ has been already assigned or assumed to be a unit matrix, as is justified in the case of trilateration where the three sides of each triangle are measured with equal accuracy.

As stated in Eq. 10, the elements of B are derivatives of ω's, the condition equations of areas, with respect to the sides of each fundamental figure. For the ith side of the αth fundamental figures, e.g., a quadrilateral as in Figure 3a, the differential coefficient is:

$$\alpha_i = \frac{\partial \omega_\alpha}{\partial \ell_i} = \frac{\partial f_I}{\partial \ell_i} - \frac{\partial f_{II}}{\partial \ell_i} + \frac{\partial f_{III}}{\partial \ell_i} - \frac{\partial f_{IV}}{\partial \ell_i} \tag{16}$$

and, in turn, this coefficient is reduced to the problem of evaluating the derivative of the area of the Ith triangle with sides 1, 2 and 3 with respect to the ith side, e.g., side 1. This has been derived from Eqs. 1a and 2 as:

$$\frac{\partial f_I}{\partial \ell_i} = \frac{(s_I - \ell_i)(s_I - \ell_2)(s_I - \ell_3) + [(s_I - \ell_i) - (s_I - \ell_2) + (s_I - \ell_i)(s_I - \ell_3) - (s_I - \ell_2)(s \ - \ell_3)]s_I}{4f_I} \tag{17a}$$

or in a new simplified form for a plane triangle (9):

$$\frac{\partial f_I}{\partial \ell_i} = \ell_i\,(\ell_2^2 + \ell_3^2 - \ell_i^2)/8f_I \tag{17b}$$

and for a spherical triangle:

$$\begin{aligned} \frac{\partial f_I}{\partial \ell_i} = \frac{R^2}{2(1+c_I^2)c_I} \Bigg\{ & \tan\tfrac{1}{2}\left(\frac{s_I - \ell_i}{R}\right) \tan\tfrac{1}{2}\left(\frac{s_I - \ell_2}{R}\right) \tan\tfrac{1}{2}\left(\frac{s_I - \ell_3}{R}\right) \left[1 + \tan^2\tfrac{1}{2}\left(\frac{s_I}{R}\right)\right] + \\ & \tan\tfrac{1}{2}\left(\frac{s_I - \ell_i}{R}\right) \tan\tfrac{1}{2}\left(\frac{s_i - \ell_2}{R}\right) \left[1 + \tan^2\tfrac{1}{2}\left(\frac{s_I - \ell_3}{R}\right)\right] \tan\tfrac{1}{2}\left(\frac{s_I}{R}\right) + \\ & \tan\tfrac{1}{2}\left(\frac{s_I - \ell_i}{R}\right) \left[1 + \tan^2\tfrac{1}{2}\left(\frac{s_I - \ell_2}{R}\right)\right] \tan\tfrac{1}{2}\left(\frac{s_I - \ell_3}{R}\right) \tan\tfrac{1}{2}\left(\frac{s_I}{R}\right) - \\ & \left[1 + \tan^2\tfrac{1}{2}\left(\frac{s_I - \ell_i}{R}\right)\right] \tan\tfrac{1}{2}\left(\frac{s_I - \ell_2}{R}\right) \tan\tfrac{1}{2}\left(\frac{s_I - \ell_3}{R}\right) \tan\tfrac{1}{2}\left(\frac{s_I}{R}\right) \Bigg\} \end{aligned} \tag{18}$$

If spherical excess E's instead of area f's are used in forming the condition equation ω for spherical triangles, computations will be saved for the factor R^2 in all related equations.

By deduction, any $\partial\omega/\partial\ell$ can be written or computed from equations similar to the forms of Eqs. 16, 17a and 18 for any side of the triangle in a fundamental figure. Thus, the B matrix can be formed.

EXAMPLE

The method discussed in the last sections can be accomplished systematically either by a desk calculator or by an electronic digital computer. An example for the analysis of a trilateration net based on Figure 3c is given in Tables 1 through 4. The numbering system of the points, the sides, the triangles and the fundamental figures or the condition equations, which are being tested in a computer program, is self-explanatory in the tables. The results of the adjustment of the trilateration net of Figure 3c according to the basic principle presented in this paper are also shown. The detailed sample computation of the adjustment of a plane quadrilateral has been given earlier (9).

TABLE 1

SIDES OF TRILATERATION NET

Observed Side	Side No.	Distance (ft)	Figure-Triangle-Side No.
1-2	1	6,973.270	α-I-1 = α-II-1
1-3	2	10,201.010	α-I-2 = α-III-1 = β-I-1 = β-II-1
1-4	3	4,474.850	α-II-2 = α-III-2 = β-I-2 = β-III-1
1-8	4	7,061.170	β-II-2 = β-III-2
3-2	5	9,418.530	α-I-3 = α-IV-1
4-2	6	8,376.250	α-II-3 = α-IV-2
3-4	7	6,599.990	α-III-3 = α-IV-3 = β-I-3 = β-IV-1 = γ-I-1 = γ-II-1 = δ-I-1 = δ-II-1
3-5	8	9,101.230	γ-I-2 = γ-III-1
3-6	9	11,054.280	γ-II-2 = γ-III-2 = δ-I-2 = δ-III-1
3-8	10	12,443.930	β-II-3 = β-IV-2 = δ-II-2 = δ-III-2
4-5	11	10,972.020	γ-I-3 = γ-IV-1
4-6	12	8,540.750	γ-II-3 = γ-IV-2 = δ-I-3 = δ-IV-1 = ε-I-1 = ε-II-1
4-7	13	10,142.850	ε-I-2 = ε-III-1
4-8	14	6,362.350	β-III-3 = β-IV-3 = δ-II-3 = δ-IV-2 = ε-II-2 = ε-III-2
6-5	15	6,601.030	γ-III-3 = γ-IV-3
6-7	16	5,288.990	ε-I-3 = ε-IV-1
6-8	17	7,103.720	δ-III-3 = δ-IV-3 = ε-II-3 = ε-IV-2
7-8	18	4,970.630	ε-III-3 = ε-IV-3

TABLE 2

TRIANGLES OF A TRILATERATION NET

Triangle Point No.	Triangle No.	Area (sq ft)	Figure-Triangle No.	Side Used (No.)
1, 2, 3	1	31,767,883.8270	α-I	1, 2, 5
1, 2, 4	2	15,597,596.3250	α-II	1, 3, 6
1, 3, 4	3	10,754,874.0270	α-III = β-I	2, 3, 7
1, 3, 8	4	36,014,781.3490	β-II	2, 4, 10
1, 4, 8	5	13,984,375.4720	β-III	3, 4, 14
2, 3, 4	6	26,924,985.4690	α-IV	5, 6, 7
3, 4, 5	7	29,996,444.6060	γ-I	7, 8, 11
3, 4, 6	8	28,148,476.0840	γ-II = δ-I	7, 9, 12
3, 4, 8	9	11,288,330.9630	β-IV = δ-II	7, 10, 14
3, 5, 6	10	30,020,311.5120	γ-III	8, 9, 15
3, 6, 8	11	39,010,020.0090	δ-III	9, 10, 17
4, 5, 6	12	28,172,289.2010	γ-IV	11, 12, 15
4, 6, 7	13	22,580,657.0760	ε-I	12, 13, 16
4, 6, 8	14	22,145,704.9730	δ-IV = ε-II	12, 14, 17
4, 7, 8	15	12,698,281.8090	ε-III	13, 14, 18
6, 7, 8	16	13,133,108.2150	ε-IV	16, 17, 18

TABLE 3

INFORMATION ON FIGURES OF TRILATERATION NET

Figure Point No.	Figure No.	Triangle Used (No.)	Side Used (No.)	ω
1, 2, 3, 4	α	1, 2, 3, 6	1, 2, 3, 5, 6, 7	-176.0600
1, 3, 4, 8	β	3, 4, 5, 9	2, 3, 4, 7, 10, 14	12,689.1130
3, 4, 5, 6	γ	7, 8, 10, 12	7, 8, 9, 11, 12, 15	53.7890
3, 4, 6, 8	δ	8, 9, 11, 14	7, 9, 10, 12, 14, 17	4,059.9150
4, 6, 7, 8	ε	13, 14, 15, 16	12, 13, 14, 16, 17, 18	-125.6970

TABLE 4

ADJUSTMENT OF A TRILATERATION NET

Observed Side	α	β	γ	δ	ε	Correction v
1-2	-1,632.533	-	-	-	-	0.197
1-3	3,463.391	-8,362.605	-	-	-	0.021
1-4	-3,094.385	9,448.741	-	-	-	-0.122
1-8	-	-4,451.629	-	-	-	0.234
2-3	-1,277.335	-	-	-	-	0.154
2-4	2,313.738	-	-	-	-	-0.279
3-4	-2,643.833	17,269.190	51.578	6,649.493	-	-0.337
3-5	-	-	66.738	-	-	-0.039
3-6	-	-	-85.728	3,223.571	-	0.185
3-8	-	-9,717.541	-	-9,047.636	-	0.129
4-5	-	-	-86.388	-	-	0.050
4-6	-	-	71.117	-4,387.762	-1,870.126	-0.169
4-7				-	3,873.248	-0.118
4-8	-	12,799.095	-	8,148.742	-2,477.302	-0.253
5-6	-	-	51.571	-	-	-0.030
6-7	-	-	-	-	-1,952.866	0.059
6-8	-	-	-	2,633.135	2,674.465	0.029
7-8	-	-	-	-	-3,263.620	0.099
ω	-176.0600	12,689.1130	53.7890	4,059.9150	-125.6970	-

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