Analysis of Discontinuous Orthotropic Pavement Slabs Subjected to Combined Loads

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A method of solving for the deflected shape of freely discontinuous orthotropic plates and pavement slabs subjected to a variety of loads including transverse loads, in-plane forces and externally applied couples is presented. The method is applicable for plates and pavement slabs with freely variable foundation support including holes in the subgrade.

Anisotropic elasticity governs the behavior of orthotropic plates and pavement slabs and is used to develop the necessary equations. The method is not limited by discontinuities and uses an efficient alternating-direction iteration means of solving the resulting equations. The method allows considerable freedom in configuration, loading, flexural stiffness and boundary conditions. It solves the problem rapidly and should provide a tool for use in later studies of repetitive stochastic loading. Three principal features are incorporated into the method: (a) the plate is defined by a finite-element model consisting of bars, springs, elastic blocks and torsion bars; these are further grouped for analysis into orthogonal systems of beam-column elements; (b) each individual line-element of the two dimensional system is solved rapidly and directly by recursive techniques; and (c) an alternating-direction iterative method is utilized for coordinating the solution of the individual line-elements into the slab solution.

The computer program utilizes the equations and techniques developed and can be used by the reader. Several sample problems illustrate the generality of the method and the use of the computer program. The results compare well with closed-form solutions.

•THE analysis of pavement slabs is a difficult task. To date neither a theoretical approach nor experimental work has solved the problem. In 1926 H. M. Westergaard completed an analysis of stresses in pavement slabs (32) and his equations have become the definitive design equations for pavement slabs in the United States. Many other engineers and mathematicians have attempted to solve this design problem. Unfortunately, limitations of conventional mathematics and of hand solutions have restricted developments. Thus the Westergaard solutions, as well as all others, are subjected to limiting severely assumptions which often are not realistic.

Several large-scale road tests have been conducted in attempts to bridge the gap between theory and reality. These include the Bates Test in 1922, the Maryland Road Test in 1950, and the AASHO Road Test in 1958. All three of these full-scale experiments have added to the knowledge of pavement design. However, only the AASHO Road Test was large enough to provide significant information. Work with the AASHO Road Test data has shown that a mechanistic model of structural behavior is essential in the study of load, environment and performance.

The problem, then, is to develop a mechanistic model for describing slab behavior and to develop better methods for solving these equations. The method should allow

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for freely discontinuous variation of input parameters including bending stiffness and load. Combination loading should be provided for and should include lateral loads, in-plane forces, and applied couples or moments. Freely variable foundation conditions are needed. Such a technique should apply not only to the general slab-on-foundation case, but also for orthotropic plates with various configurations of structural support.

This paper describes such a method of solving for the deflected shape of orthotropic plates and pavement slabs. (Throughout this paper, the term slab is often used as an abbreviation for pavement slab and slab-on-foundation.) From this deflected shape the stresses, deflections, loads, and bending moments can easily be determined. The method developed takes advantage of groundwork laid by others. The finite-element method was developed by Matlock (18, 19, 29), and variations and extensions of his methods have been made by Tucker, Haliburton, Ingram, and Salani (29, 18, 12, 24).

The principal features incorporated into the finite-element method are (a) representation of structural members by a physical model of bars and springs which are grouped for analysis into systems of orthogonal beams, (b) a rapid method for solution of individual beams that serve as line elements of a two-dimensional slab, and (c) an alternating-direction iteration technique for coordinating the solutions of individual beams which ties the system together.

NOTATION

Symbol	Dimensions	Definition
a, b		Temporary bar numbering used in derivations.
a_x, b_x		Coefficient in the matrix equation.
$c_{i,j}$	lb/in.^{2}	Constants used to relate stress to strain in general anisotropic elasticity (i refers to stress component, j refers to strain component).
$C^x_{i,j}$	$\frac{\text{in.-lb}}{\text{rad}}$	Torsional stiffness of slab element i, j about the x-axis.
$C^{x'}_{i,j}$	in.-lb	Torque exerted on the x-beam due to the relative rotation in torsion bar i, j.
$D_{i,j}$	in.-lb	Bending stiffness of an isotropic plate.
$D^x_{i,j}$, D_x	in.-lb	Bending stiffness of an orthotropic plate in the x-direction.
$D^y_{i,j}$, D_y	in.-lb	Bending stiffness of an orthotropic plate in the y-direction.
D_{xy}	in.-lb	$\frac{G_0 t^3}{12}$
D_1	in.-lb	$\frac{E'' t^3}{12}$
E_x	lb/in.^{2}	Modulus of elasticity in x-direction.
E_y	lb/in.^{2}	Modulus of elasticity in y-direction.
E'_x	lb/in.^{2}	$\frac{E_x}{1 - \nu_{xy}\nu_{yx}}$
E'_y	lb/in.^{2}	$\frac{E_y}{1 - \nu_{yx}\nu_{xy}}$
E''	lb/in.^{2}	$\nu_{yx}E'_x = \nu_{xy}E'_y$

ϵ_x	in./in.	Total strain in x-direction.
ϵ_y	in./in.	Total strain in y-direction.
ϵ_z	in./in.	Total strain in z-direction.
$F_{v_{i,j}}$	lb	Vertical forces at joint i, j.
φ		Angular rotation across a plate element.
G	lb/in.^{2}	Shear modulus $= \frac{E}{12(1+\nu)}$.
G_o	lb/in.^{2}	Approximate orthotropic shear modulus $\left(\frac{E_x E_y}{E_y(1+\nu_{xy}) + E_x(1+\nu_{xy})}\right)$
γ	in./in.	Shear strain.
H	in.-lb	$D_1 + 2D_{xy}$.
h_x	in.	The increment length along the x-beams.
h_y	in.	The increment length along the y-beams.
i		An integer used to number mesh points, stations, and bars in the x-direction.
j		An integer used to number mesh points, stations, and bars in the y-direction.
k	lb/in.^{2}/in.	Support modulus of the foundation.
M_a	lb/in.	Moment acting on Bar a about the center of the bar.
M_x	in.-lb	Bending moment acting on an element of the plate in the x-direction.
M_y	in.-lb	Bending moment acting on an element of the plate in the y-direction.
M_{xy}	in.-lb	Twisting moment tending to rotate the element about the x-axis (clockwise-positive).
M_{yx}	in.-lb	Twisting moment tending to rotate the element about the y-axis (clockwise-positive).
$M^{x'}_{i,j}$	in.-lb	The bending moment in the x-beam at Station i, j (equals $h_y M^x_{i,j}$).
$M^{y'}_{i,j}$	in.-lb	The bending moment in the y-beam at Station i, j (equals $h_x M^y_{i,j}$).
ν_{xy}		Poisson's ratio which results in strain in the y-direction if stress is applied in the x-direction.
ν_{yx}		Poisson's ratio which results in strain in the x-direction if stress is applied in the y-direction.
$P^x_{i,j}$	lb	Axial load in the x-beam in Bar i, j (equals $h_y p^x_{i,j}$).

$p^x_{i,j}$	lb/in.	Unit axial load in the slab x-direction at Station i, j.
q	lb/in.^{2}	Distributed lateral load.
Q	lb	Concentrated lateral load.
$Q_{i,j}$	lb	Externally applied load at point i, j.
$Q^x_{i,j}$	lb	Load absorbed internally by the x-beam system at Station i, j.
$Q^y_{i,j}$	lb	Load absorbed internally by the y-beam system at Station i, j.
$\overline{QBMY}_{i,j}$	lb	Load absorbed by the y-beam in bending.
$\overline{QPY}_{i,j}$	lb	Load absorbed by the y-beam due to axial load.
$\overline{QTMY}_{i,j}$	lb	Load absorbed by the y-beam in twisting.
S_f	lb/in.	Fictitious spring stiffness or closure parameter.
S_{fx}	lb/in.	Fictitious spring representing the x-beams.
S_{fy}	lb/in.	Fictitious spring representing the y-beams.
$S_{i,j}$	lb/in.	Elastic restraint used to represent the foundation in the finite-element model.
σ_x	lb/in.^{2}	Stress applied in the x-direction.
σ_y	lb/in.^{2}	Stress applied in the y-direction.
σ_z	lb/in.^{2}	Stress applied in the z-direction.
t	in.	Slab thickness.
$T^x_{i,j}$	in./lb	External torque applied to Bar i on the j^{th} x-beam.
$T^y_{i,j}$	in./lb	External torque applied to Bar j on the i^{th} y-beam.
τ	lb/in.^{2}	Shear stress.
$V^x_{a,j}$	lb	Shear in Bar a of the j^{th} x-beam.
$w_{i,j}$	in.	Lateral deflection.
$w^x_{i,j}$	in.	Deflection of the j^{th} x-beam at Station i.
$w^y_{i,j}$	in.	Deflection of the i^{th} y-beam at Station j.
x, y, z		Standard Cartesian coordinate directions.

THEORY OF ELASTIC PLATES AND SLABS

A review of the various theories involved in the analysis of plate and slab bending will be helpful in understanding the problem at hand. A brief discussion of the biharmonic equation is presented in this section. A discussion of generalized Hooke's Law which leads into the derivation of the equation of bending for orthotropic plates follows. The effect of in-plane forces applied to the plate in combined loadings is presented next. Finally, elastic foundations are discussed as related to pavement support.

General Plate Theory

The bending of a plate depends greatly on its thickness as compared with its other dimensions. Timoshenko (28) distinguishes three kinds of plate bending: (a) thin plates with small deflections, (b) thin plates with large deflections, and (c) thick plates.

For thin plates with small deflections (i.e., the deflection is small in comparison with thickness), a satisfactory approximate theory of bending of a plate by lateral loads can be developed by making the following assumptions:

1. There is no deformation in the middle plane of the plate. This plane remains neutral during bending.
2. Planes of the plate lying initially normal to the middle surface of the plate remain normal to the middle surface of the plate after bending.
3. The normal stresses in the direction transverse to the plate can be disregarded. (This assumption is necessary in the analysis of bending of the plate as will be seen later; approximate corrections can be made to account for pressures directly under the transverse load.)

With these assumptions, all components of stress can be expressed in terms of the deflected shape of the plate. This function has to satisfy a linear partial differential equation which, together with the boundary conditions, completely defines the deflection w. The solution of this differential equation gives all necessary information for calculating the stresses at any point in the plate.

Timoshenko (28) develops the theory of bending of plates very thoroughly from the simplest problem of bending in a long rectangular plate subjected to transverse load to the very complex problems of thick plates with various boundary conditions.

The Isotropic Plate Equation

Structural plates and pavement slabs are normally subjected to loads applied perpendicular to their surface, i.e., lateral loads. Timoshenko and others have derived a differential equation which describes the deflection surface of such plates, the biharmonic equation. With one minor change, Timoshenko's equation is given below. This change is to reverse the sense of the z-axis and make "up" positive. This new coordinate system is consistent with recent beam-column developments (18). The equation becomes

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} = q \tag{1}$$

in which M_x is the bending moment acting on an element of the plate in the x-direction, M_y is the bending moment acting on an element of the plate in the y-direction, M_{xy} is a twisting moment tending to rotate the element about the x-axis (clockwise positive), and M_{yx} is a twisting moment tending to rotate the element about the y-axis. Observing that $M_{xy} = -M_{yx}$ for equilibrium ($\tau_{xy} = \tau_{yx}$), the equation can be condensed into the following form.

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = q \tag{2}$$

To evaluate this equation, it is safe to assume that expressions for moment derived for pure bending can also be used for laterally loaded plates. This assumption is equivalent to neglecting the effect on bending of the shearing forces and the compressive stress in the z-direction produced by the lateral load. Errors introduced into these solutions by such assumptions are negligible provided the thickness of the plate is small in comparison with the other dimensions of the plate.

The equations for moment are derived in Ref. 11 for the general case. For the special case of isotropy, they can be stated

$$M_x = D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \tag{3}$$

$$M_y = D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \tag{4}$$

$$M_{xy} = -M_{yx} = -D\,(1 - \nu)\,\frac{\partial^2 w}{\partial x \partial y} \tag{5}$$

where D is the bending stiffness of the plate, ν is the Poisson's ratio, and other terms have been previously defined.

Substituting these expressions into Eq. 2 obtains

$$D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = q \tag{6}$$

The Generalized Hooke's Law

To obtain the relations between the components of stress and the components of deformation in an elastic body, it is necessary to choose some mathematical model which reflects the elastic properties of the body. In these derivations it is always assumed that the components of strain are linear functions of the components of stress. In other words, it is assumed that a continuous body satisfies the generalized Hooke's Law.

For the most general case of a homogeneous anisotropic body, the equations which express Hooke's Law in Cartesian coordinates x, y, z have the form

$$\begin{aligned}
\epsilon_x &= S_{11}\sigma_x + S_{12}\sigma_y + S_{13}\sigma_z + S_{14}\tau_{yz} + S_{15}\tau_{xz} + S_{16}\tau_{xy} \\
\epsilon_y &= S_{21}\sigma_x + S_{22}\sigma_y + S_{23}\sigma_z + S_{24}\tau_{yz} + S_{25}\tau_{xz} + S_{26}\tau_{xy} \\
\epsilon_z &= S_{31}\sigma_x + S_{32}\sigma_y \;.\;.\;.\;. \\
\gamma_{yz} &= S_{41}\sigma_x \qquad .\;.\;.\;.\;.\;. \\
\gamma_{xz} &= S_{51}\sigma_x \qquad .\;.\;.\;.\;.\;. \\
\gamma_{xy} &= S_{61}\sigma_x + S_{62}\,\sigma_y + \;.\;.\;.\; + S_{66}\tau_{xy}
\end{aligned} \tag{7}$$

These equations contain 36 coefficients $S_{i,j}$, the so-called elastic constants. Solving these equations for stress components obtains an equivalent form for the equations in terms of stress:

$$\begin{aligned}
\sigma_x &= c_{11}\epsilon_x + c_{12}\epsilon_y + c_{13}\epsilon_z + c_{14}\gamma_{yz} + c_{15}\gamma_{xz} + c_{16}\gamma_{xy} \\
\sigma_y &= c_{21}\epsilon_x + c_{22}\epsilon_y + \;.\;.\;.\; + c_{26}\gamma_{xy}
\end{aligned}$$

$$\sigma_z = c_{31}\epsilon_x + c_{32}\epsilon_y \quad . \quad . \quad . \quad .$$

$$\tau_{yz} = c_{41}\epsilon_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\tau_{xz} = c_{51}\epsilon_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\tau_{xy} = c_{61}\epsilon_x + c_{62}\epsilon_y + \quad . \quad . \quad . \quad + c_{66}\gamma_{xy} \tag{8}$$

Several authors (8) have called the constants $S_{i,j}$ the coefficients of deformation, and the constants $c_{i,j}$ the moduli of elasticity. The moduli of elasticity can be uniquely expressed in terms of the coefficients of deformation when the value of their determinants are different from zero. It has been shown by others that the number of elastic constants in the most general case of anisotropy is reduced to 21 if the deformations of the elastic body can be considered to occur isothermally, that is, the temperature of each element remains constant during the deformation process.

Since

$$S_{12} = S_{21}$$

$$. \quad . \quad .$$

$$S_{56} = S_{65} \tag{9}$$

and likewise,

$$c_{12} = c_{21}$$

$$. \quad . \quad .$$

$$c_{56} = c_{65} \tag{10}$$

Eq. 1 can be written in terms of 21 coefficients as follows:

$$\epsilon_x = S_{11}\sigma_x + S_{12}\sigma_y + S_{13}\sigma_z + S_{14}\tau_{yz} + S_{15}\tau_{xz} + S_{16}\tau_{xy}$$

$$\epsilon_y = S_{12}\sigma_x + S_{22}\sigma_y + \quad . \quad . \quad . \quad + S_{26}\tau_{xy}$$

$$\epsilon_z = S_{13} \quad . \quad . \quad .$$

$$. \quad . \quad .$$

$$\gamma_{xy} = S_{16}\sigma_x + S_{26}\sigma_y + \quad . \quad . \quad . \quad + S_{66}\tau_{xy} \tag{11}$$

and, likewise Eq. 2 can be written in terms of 21 moduli.

$$\sigma_x = c_{11}\epsilon_x + c_{12}\epsilon_y + c_{13}\epsilon_z + c_{14}\gamma_{yz} + c_{15}\gamma_{xz} + c_{16}\gamma_{xy}$$

$$\sigma_y = c_{12}\epsilon_x + c_{22}\epsilon_y + \quad . \quad . \quad . \quad + c_{26}\gamma_{xy}$$

$$\sigma_z = c_{13}\epsilon_x + \quad . \quad . \quad .$$

. . .

$$\tau_{xy} = c_{16}\epsilon_x + c_{26}\epsilon_y + \ . \ . \ . + c_{66}\gamma_{xy} \qquad (12)$$

The problem of determining 21 coefficients to describe the behavior of an elastic body is still formidable. Fortunately, conditions of elastic symmetry permit still further reduction of this number. If the internal structure of a material possesses symmetry of any kind, the same symmetry can be observed in its elastic properties. F. Neumann (8) set forth a principle for crystals which establishes a connection between symmetry of construction and elastic symmetry. In general, this principle says that a material has the same kind of symmetry with regard to physical properties as it has in its crystallography. This principle can be expanded to include bodies which are not crystalline but which possess a symmetry of structure such as wood, plywood, and reinforced concrete.

If an anisotropic body possesses elastic symmetry, the equations of the generalized Hooke's Law are simplified. The simplifications can be thought of as follows: When viewed from the center of the symmetric coordinate system of the body, equal elastic properties are seen in both the positive and negative directions of any axis of symmetry. As a result, elastic bodies possessing symmetry have a smaller number of independent elastic constants than 21. The final number depends on the number of axes or planes of symmetry present in the body.

Three Planes of Elastic Symmetry

The case of interest involves three planes of elastic symmetry passing through each point of a body orthogonally, that is, the planes occur at right angles to each other. If the axes of the coordinate system are directed perpendicular to these planes, the following equations of the generalized Hooke's Law for an orthotropic body can be derived.

$$\begin{aligned}
\epsilon_x &= S_{11}\sigma_x + S_{12}\sigma_y + S_{13}\sigma_z \\
\epsilon_y &= S_{12}\sigma_x + S_{22}\sigma_y + S_{23}\sigma_z \\
\epsilon_z &= S_{13}\sigma_x + S_{23}\sigma_y + S_{33}\sigma_z \\
\gamma_{yz} &= S_{44}\tau_{yz} \\
\gamma_{xz} &= S_{55}\tau_{xz} \\
\gamma_{xy} &= S_{66}\tau_{xy} \qquad (13)
\end{aligned}$$

Since the constants $S_{i,j}$ are redundant with $S_{j,i}$, it can be observed that there are nine independent elastic constants remaining.

Plane Stress Case

For the particular case of thin plates in bending, σ_z is taken to be zero (plane stress), and the following equations are obtained:

$$\begin{aligned}
\epsilon_x &= S_{11}\sigma_x + S_{12}\sigma_y \\
\epsilon_y &= S_{12}\sigma_x + S_{22}\sigma_y \\
\gamma_{xy} &= S_{66}\tau_{xy} \qquad (14)
\end{aligned}$$

These equations are derived directly from an orthotropic plane stress element (shown in Ref. 11). The corresponding elements for stress in terms of strain are also developed in Ref. 11, and can be stated

$$\sigma_x = \frac{E_x}{1 - \nu_{xy}\nu_{yx}} (\epsilon_x - \nu_{yx}\epsilon_y) = E'_x\epsilon_x - E''\epsilon_y \tag{15}$$

$$\sigma_y = \frac{E_x}{1 - \nu_{xy}\nu_{yx}} (\epsilon_y - \nu_{xy}\epsilon_x) = E'_y\epsilon_y - E''\epsilon_x \tag{16}$$

$$\tau_{xy} = G\,\gamma_{xy} \tag{17}$$

where

$$E'_x = \frac{E_x}{1 - \nu_{xy}\nu_{yx}} \tag{18}$$

$$E'_y = \frac{E_y}{1 - \nu_{xy}\nu_{yx}} \tag{19}$$

$$E'' = \nu_{yx}E'_x = \nu_{xy}E'_y \tag{20}$$

Isotropic Elasticity

Hooke's Law for standard isotropic conditions can be stated

$$\sigma_x = \frac{E}{1 - \nu^2} (\epsilon_x + \nu\epsilon_y) \tag{21}$$

$$\sigma_y = \frac{E}{1 - \nu^2} (\nu\epsilon_x + \epsilon_y) \tag{22}$$

$$\tau_{xy} = G\,\gamma_{xy} \tag{23}$$

where γ_{xy} is the shearing strain, τ_{xy} is the corresponding shearing stress, and the shear modulus is

$$G = \frac{E}{2(1 + \nu)} \tag{24}$$

Other terms have been previously defined. By comparing these equations with Eq. 14, it can be seen that four elastic constants are required to describe the behavior of thin orthotropic plates, whereas two independent elastic constants are required for isotropic plates. The orthotropic constants are E'_x, E'_y, E'', and G_o. The shear modulus, G_o, is an independent constant, and a method for determining it is discussed in Ref. 11. It can be approximated by relating the other three constants as follows:

$$G_o = \frac{E_x E_y}{E_y (1 + \nu_{xy}) + E_x (1 + \nu_{yx})} \tag{25}$$

Orthotropic Elastic Plate Equations

The complete derivation of the differential equation of equilibrium is given in Ref. 11. Utilizing the elastic constants previously described, this equation can be stated

$$D_x \frac{\partial^4 w}{\partial x^4} + 2(D_1 + 2D_{xy}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q \tag{26}$$

where

$$D_x = \frac{E'_x t^3}{12} \tag{27}$$

$$D_y = \frac{E'_y t^3}{12} \tag{28}$$

$$D_1 = \frac{E'' t^3}{12} \tag{29}$$

$$D_{xy} = \frac{G_o t^3}{12} \tag{30}$$

since

$$H = D_1 + 2D_{xy} \tag{31}$$

Then Eq. 26 reduces to

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q \tag{32}$$

For the particular case of isotropy, this equation collapses to the equation

$$D \left[\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = q \tag{33}$$

where

$$D = \frac{E t^3}{12 (1 - \nu^2)} \tag{34}$$

Since for isotropy

$$E'_x = E'_y = \frac{E}{1 - \nu^2} \tag{35}$$

$$E'' = \frac{\nu E}{1 - \nu^2} \tag{36}$$

$$G = \frac{E}{2(1 + \nu)} \tag{37}$$

Therefore, it can be seen that

$$D_x = D_y = \frac{Et^3}{12(1 - \nu^2)} = D \tag{38}$$

$$H = D_1 + 2D_{xy} = \frac{t^3}{12}(E'' + 2G) = \frac{t^3}{12}\left(\frac{\nu E}{1 - \nu^2} + 2\frac{E}{2(1 + \nu)}\right)$$

$$= \frac{Et^3}{12(1 - \nu^2)} = D \tag{39}$$

These equations were derived for structurally orthotropic materials. A great deal of work today deals with geometrically orthotropic plates. The same equations are used in such cases, but an equivalent thickness t is derived as appropriate to account for the variation in moment of inertia. Hoppmann and Huffington treat this problem in Ref. 10.

Pavement Slabs

Solutions of pavement slabs, or slabs-on-foundation as they are sometimes called, are of particular interest in this paper. For these developments it is satisfactory to assume that the intensity of the reaction of the foundation on the slab is proportional to the deflection w of the slab. This intensity is then given by the expression kw, where the constant k, expressed in pounds per square inch per inch of deflection, is called the "support modulus of the foundation." Determination of numerical values for this modulus depends largely on the properties of the foundation, but a discussion of these properties is beyond the scope of this paper. Such determinations, however, have been made by Terzahgi (27).

Although a great deal of work has been done on the pavement slab problem, probably the most significant accomplishments to date were made by Westergaard (31, 34), particularly with reference to the design problems encountered in concrete pavement. His analysis was done in the early 1920's and relates to three special-case loadings as follows: (a) load applied near the corner of a large rectangular slab (corner load); (b) load applied near the edge of a slab, but at a considerable distance from any corner (edge load); and (c) load applied at the interior of a large slab at considerable distance from any edge (interior load).

Other theoretical work was done by Gerald Pickett, et al. (21) in 1951 in studies of deflections and moments for concrete pavements. This study added additional solutions to those available for use by the practicing engineer. Perhaps more significantly, the results of these solutions were made available for practical use by incorporating them into influence charts similar to those developed by N. M. Newmark (20).

Additional contributions include work by Spangler (25), Teller and Sutherland (26), Kelley (13), and others who have, in the past 25 years, conducted experimental studies on pavement slabs to correlate deflections under static load with those predicted by theory. The evaluation of such work is beyond the scope of this paper, but it is discussed briefly under the heading of needed research.

Nonuniform Conditions

Unfortunately for the designer, most pavement slabs do not meet the stringent assumptions imposed by Westergaard. First, the slabs must in reality be finite (Fig. 1b). Second, uniform support is hard to obtain since local loss of support under the pavement due to pumping or settlement of the foundation is common (Fig. 1c). Richart and Zia (22) have treated this problem by applying a general method developed by Brotchie (4). Their solution relates specifically to a large slab-on-foundation spanning a circular void. They provide the designer with several curves useful in evaluating this specific condition. They do not treat, however, the more general cases of (a) smaller slabs spanning a void of irregular shape, (b) the problem of random placement of a void near a corner or edge, or (c) in the more general case, several voids under a single slab.

Leonards and Harr (16) also treat nonuniform subgrade support. They evaluate the effect of curling on a circular slab. The circular slab is not used in pavement design, but the methods developed may be useful in treating differential temperature effects in the future.

In the methods of this paper, the foundation is represented by the modulus of support k. This approach provides the basis for future consideration of nonlinear elastic foundation support (17). The freely discontinuous inputs allowed by the method provide the capability of varying k anywhere under the slab.

Cracks and Other Discontinuities

The theories described thus far relate to homogeneous materials. No provision has been made for cracks or other discontinuities (Fig. 1d). Other authors have treated this subject in some detail for special cases. These include Ang (1, 2), Williams (36), Reissner (23), and Knowles and Wang (14). Many authors, including those listed, discuss "Reissner bending of plates." This phrase refers to the equations for bending of elastic plates developed by Eric Reissner of MIT. Classical theory, that discussed at the beginning of this chapter, meets the so-called Kirchhoff boundary conditions at free edges, these being a vanishing bending couple and a vanishing sum of transverse force and edgewise rate-of-change of twisting couple at all free plate edges. These two con-

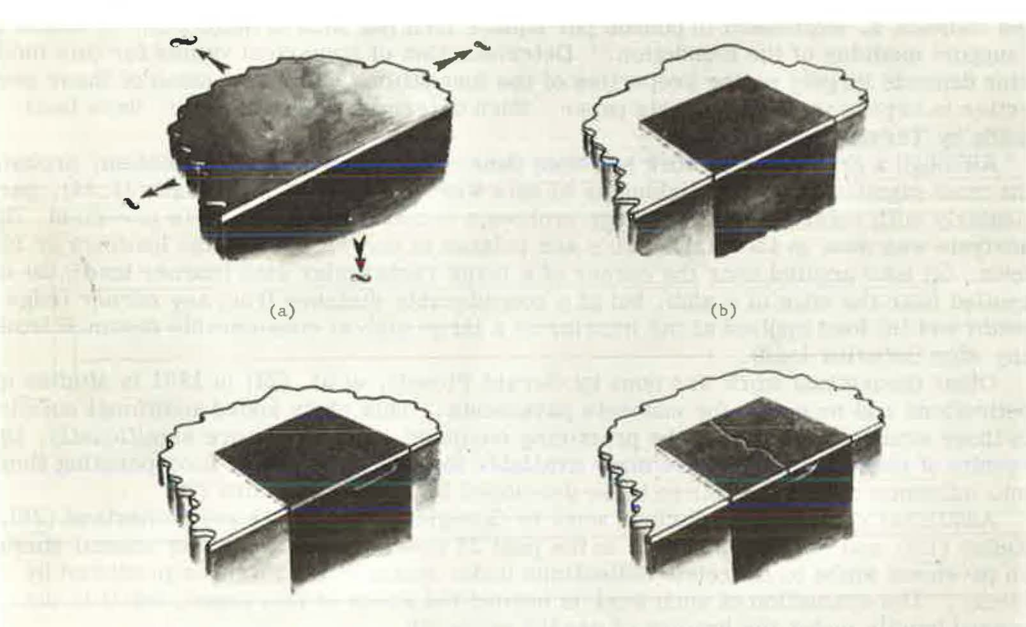


Figure 1. Comparison of real and infinite pavement slabs.

ditions are actually a compression of three independent conditions: (a) vanishing transverse force, (b) vanishing bending couple, and (c) vanishing twisting couple at free edges. "Reissner bending" includes differential equations fulfilling these three boundary conditions.

Reissner's studies further show that stresses near a finite crack in infinite plates are somewhat greater than those calculated by classical theory. This is accentuated near the base of the crack where extremely high stress concentrations might be expected. Such cracks are beyond the scope or application of this work and will not be treated.

The discontinuities of interest are those which occur across the entire slab cross-section at any particular location. Ang, Williams, and Reissner indicate that stress distribution can be predicted reasonably outside a distance half of the plate thickness from the edge of a crack. This is acceptable for the application of the method discussed herein, since this distance also approximates a half increment length in the finite analogy. Furthermore, such accuracy is quite adequate for structural plates and pavement slabs. Corrections to this theory can be obtained from "thick plate" theory and introduced into any solution where needed (33).

Summary of Elastic Theory

The theory described herein is helpful to the development of any method for analyzing plate bending. Closed-form solutions of the problems, however, become more difficult as complexities increase. Hand solutions of isotropic plates are readily accomplished, but for solutions of homogeneous orthotropic plates one must usually resort to computers. The addition of elastic support or finite cracks forces the use of approximate methods and limiting assumptions. Furthermore, each solution represents a special case, and a multitude of special-case solutions are required for the problems of interest. A more general, more rapid method would be of great advantage to the engineer. It would also be helpful if these solutions could be accomplished without resorting to higher order functions such as Bessel and Hankel functions. Such a general theory is the object of the research described herein.

FINITE-ELEMENT THEORY

The theories discussed in the preceding section are based on infinitesimal calculus. There are many rules governing the use of such calculus. In general, the functions must be continuous, and fourth order systems must have two continuous derivatives. Many complex engineering problems do not properly fulfill these conditions and cannot, therefore, be solved by resorting to the calculus. Furthermore, many such classical or analytical methods may not be well adapted for use on high-speed digital computers. As a consequence, approximate, or so-called "numerical," methods have been developed. Hardy Cross (8, p. 1) pioneered the use of such methods in civil engineering with moment distribution methods. Newmark (8, p. 138) and Southwell (8, p. 66) have also been instrumental in these developments. In such numerical methods, the differential equation concerned is replaced by its finite difference equivalent. The problem then reduces to solving a large number of simultaneous algebraic equations instead of one complex differential equation.

The method described herein is slightly different and involves breaking a plate or slab into a system of finite elements, each consisting of rigid bars connected by elastic blocks. The algebraic equations describing the system are derived by free-body anaysis of the finite model.

Assumptions

It is impossible to develop a completely general theory describing the behavior of any structure. It is often difficult to find solutions for the mathematical equations describing even limited theories; therefore, additional conditions and assumptions are often imposed to permit solution. While many of these assumptions are known, it seems worthwhile to restate the assumptions and conditions relative to the finite-element model describing slab behavior.

1. Planes of the plate lying initially normal-to-the-middle surface of the plate remain on the normal-to-the-middle surface of the plate after bending.

2. Normal stresses in the direction perpendicular to the plate surface can be disregarded for the bending solution.

3. There is no axial deformation in the middle plane of the plate and, thus, this plane remains "neutral" during bending.

4. All deformations are small with regard to dimensions of the plate.

5. The bar elements of the model are infinitely stiff and weightless.

6. Each joint in the model is composed of an elastic homogenous and orthotropic material which can be described by four independent elastic constants.

7. Loads, masses, and bending strains occur at the joint.

8. Torsional stiffness of the plate element can be invested in torsion bars.

9. The neutral axis lies in the same plane for all elements even for nonuniform cross-sections. (Violation of this assumption, as stated by Ang and Newmark (2), has been shown to cause little error.)

10. The spacing of the beam elements, designated by h_x and h_y, need not be equal but must be constant for all parallel beams.

11. The number of increments into which each beam is divided is equal to the length of the beam divided by the increment length.

The Physical Model

Numerical methods are most often used as mathematical approximations of a governing differential equation by the substitution of finite-difference forms for derivatives, or by the approximation of a continuum problem with a discrete nodal system. A second and perhaps preferable method is to model the plate or slab physically by a system of finite elements whose behavior can properly be described with algebraic equations. Newmark (8) pioneered such models for articulated beams and plates. He states". . . the use of the model (finite-element model) offers certain advantages; there is no ambiguity concerning the boundary conditions; statical checks on the results have a physical meaning and can be made more accurately; variations in dimensions and physical properties can be more easily treated." For many problems, the finite-difference equations developed by direct substitution for the differential equation and the finite-element model equations developed from a free-body analysis of the model are equivalent. This, however, is not always the case. The physical model seems preferable because it facilitates visualization of the problem and formulation of proper boundary and loading conditions. It is useful, however, to use difference equations to describe the bending moments in the finite-element beams.

Model of a Beam-Column

The basic element in the plate model developed here is the model of a beam subjected to transverse and axial loads (termed a beam-column and developed by Matlock, et al.) (18, 19). Figure 2 shows the development of this model. Figure 2a illustrates a beam element deformed by the action of pure bending and subjected to the assumptions of conventional beam theory. For linearly elastic stress and strain, the stresses acting on the beam element are shown in Figure 2b. If these distributed stresses are to be replaced by concentrated forces as shown in Figure 2c, as they often are for design purposes, it seems reasonable to develop the mechanical model, Figure 2d. Here the deformed beam element is replaced by a pair of hinged plates with linear springs containing the elastic flexural stiffness of the beam restraining movement of the plates, top and bottom. Thus, a beam could be represented by a series of such beam element models (Fig. 2e).

If the thickness of the plates between hinged joints is increased, a cruder representation results (Fig. 2f). It has been shown, however, that representation of real beams by models containing as few as six elements or increments (as they will be called hereafter) give satisfactory approximation of real beams. As a specific example of modeling, Figure 3 indicates a beam-on-foundation subjected to both lateral and axial loads. Supports may be linearly-elastic, non-linear, or fixed. Figure 4 shows these loads and supports depicted in the finite-element model. Many other loads and load com-

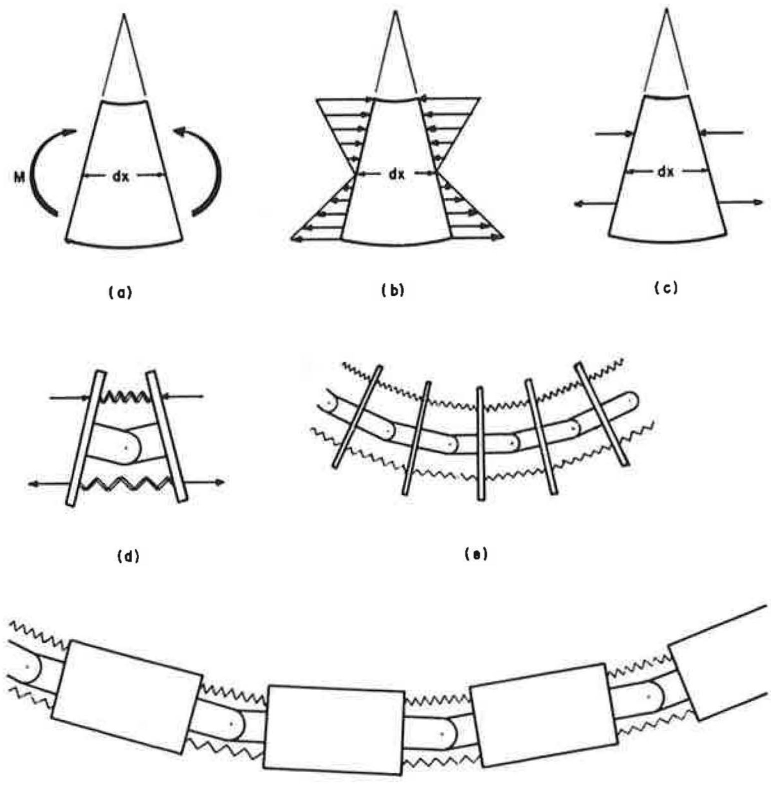


Figure 2. Finite mechanical representation of a conventional beam.

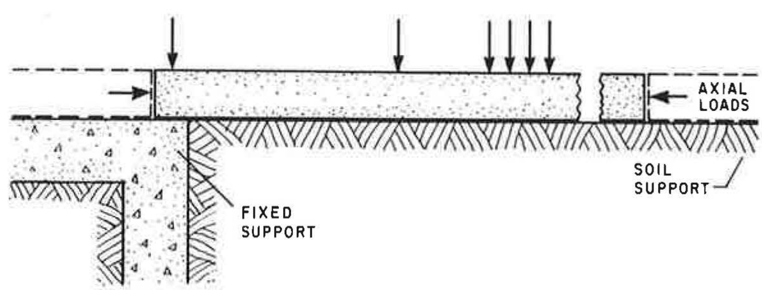


Figure 3. Example beam on foundation subjected to lateral and axial loads.

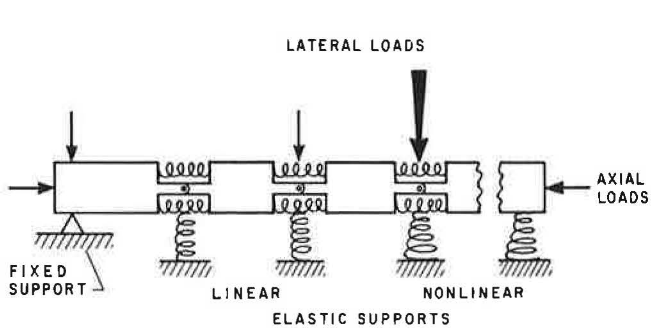


Figure 4. Finite-element model of Figure 3.

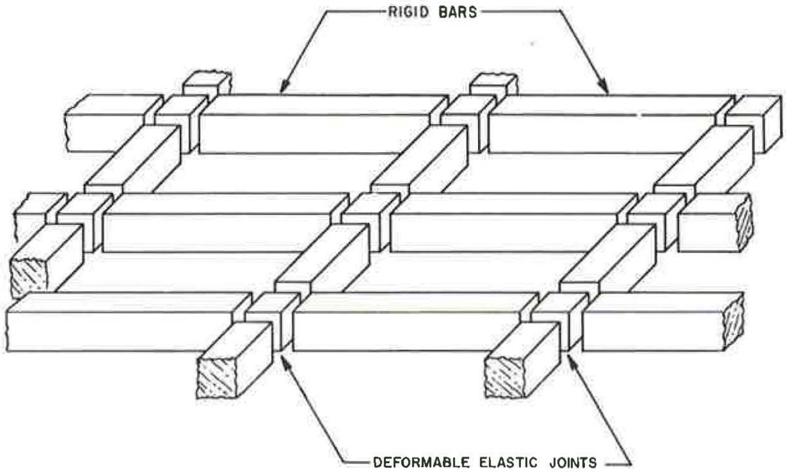


Figure 5. Finite-element model of grid-beam system.

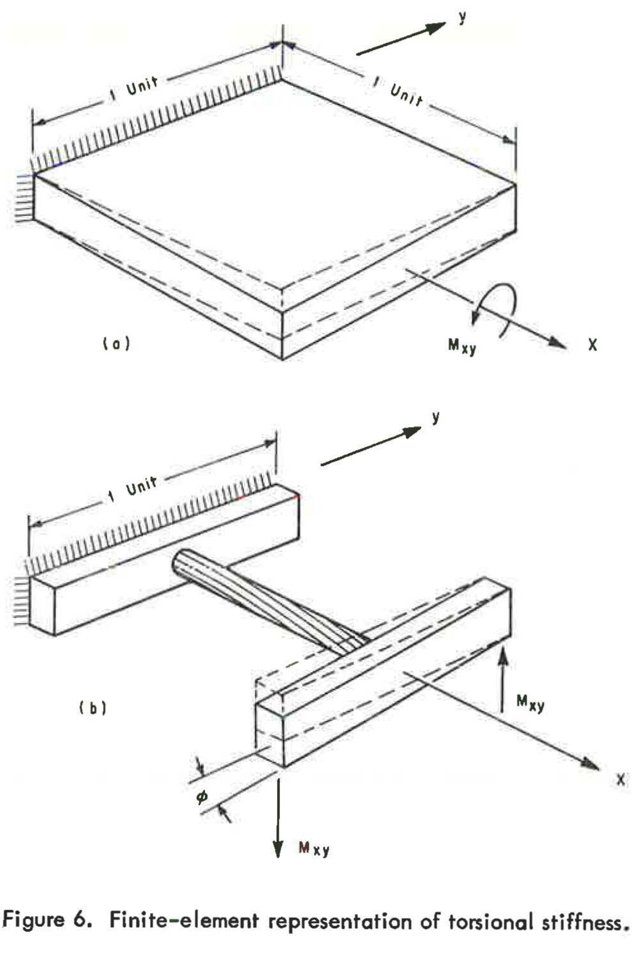


Figure 6. Finite-element representation of torsional stiffness.

binations are possible. These include distributed or concentrated, transverse loads, transverse couples, axial loads, and bending moments. Elastic restraints are included as linear or nonlinear supports, or, as distributed or concentrated rotational restraints. In short, almost any physical combination of loads or restraints can be applied to a beam-column with this method.

Simple Two-Dimensional Systems

If one or more of these beams in each horizontal orthogonal direction are combined, they form a grid-beam system similar to the girder and stiffener system of a bridge deck or similar to the beam system of a waffle floor (Fig. 5). Tucker and Matlock (29) extended the use of the beam-column model to such systems. Each of the beams in this grid-beam system can be solved by the beam-column method as a line member. However, the effect of one beam on the next beam is important if the beams act as a monolithic system.

Such systems account for pure bending only. No torsion or Poisson's ratio effect is considered. In a true grid-beam system, these effects are small and do not affect the solution significantly.

Plates and Slabs

For the plate solution, however, the effects of torsion in particular are of significant importance, and the Poisson's ratio effects are more important than for grid-beam problems. Tucker (30) has worked on this problem as have Ang and Newmark (2). The next step was to determine some method for including these two factors in the model.

First, consider torsion in Figure 6. If a unit element is removed from the slab, a twisting moment M_{xy} can be applied about the x-axis. The torsional stiffness C of the slab is defined as the applied twisting moment divided by the resulting angular rotation, φ, across the element. Then

$$C = \frac{M_{xy}}{\varphi} \tag{40}$$

Considering this element as two beams connected by a torsion bar, the bar modulus can be chosen equal to C so that an applied twisting moment will produce the same rela-

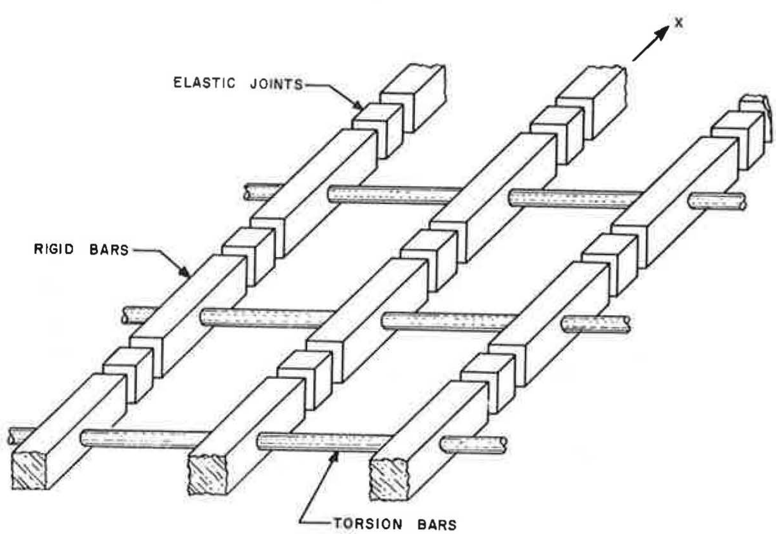
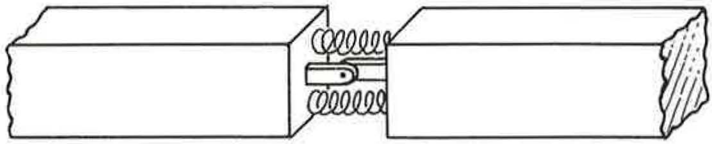
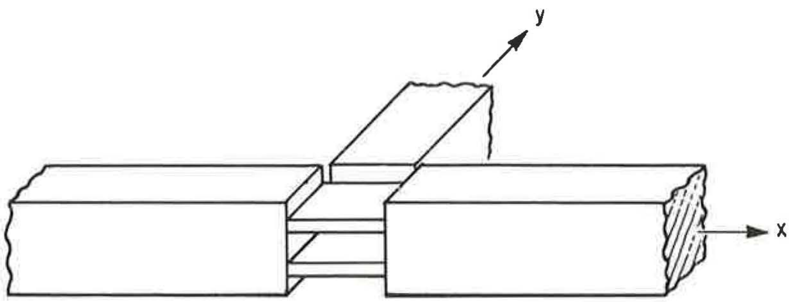


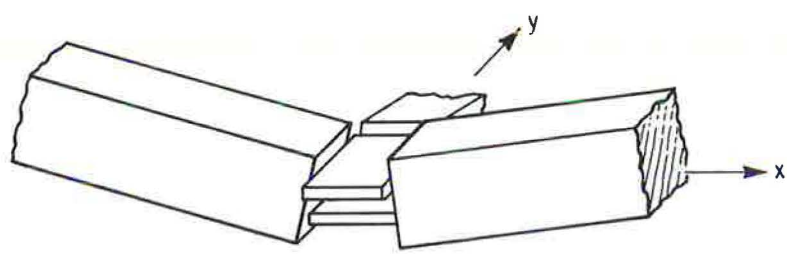
Figure 7. Finite-element x-beam system with torsion bars acting between segments.



(a) Typical joint from the beam-column model.



(b) Typical joint from the plate model (partial cutaway).



(c) Deformed joint from the plate model.

Figure 8. Action of Poisson's ratio at a finite joint.

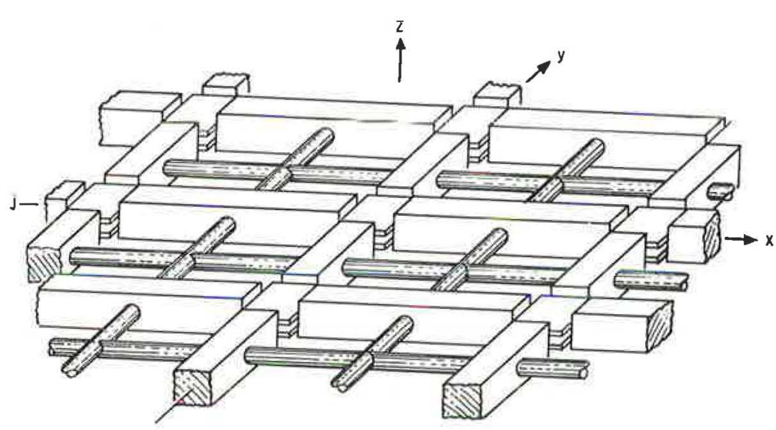


Figure 9. Finite-element model of a plate or slab.

tive angle change φ as in the real unit element of the plate. Using this technique, torsion bars can be inserted between the adjacent bar elements of all the beams in the y-direction of the grid-beam system as shown in Figure 7. These torsion bars are also inserted, of course, between the beams in the orthogonal x direction as shown in Figure 9. It is convenient to think of one set of torsion bars acting with each set of beams since the solution proceeds in this manner. It should be emphasized here that these torsion bars represent the real torsional stiffness of the slab and are always active in the system.

The effect of Poisson's ratio is easier to handle than torsion. Remember that the bending stiffness EI of a beam-column is vested in linear springs restraining the movement of the finite-elements at each joint. The analogous bending stiffness of a plate

$$D = \frac{Et^3}{12(1 - \nu^2)} \tag{41}$$

replaces the EI of the beam and must also be concentrated. A pair of linear springs, however, is not satisfactory for this purpose since they can transfer no Poisson's ratio load. These stiffness springs are, therefore, replaced in the plate model by elastic blocks whose stress-strain relationship is equivalent to that of the real plate and which have Poisson's ratio equal that of the plate. Figure 8 illustrates the action of these elastic blocks. The blocks in Figure 8b replace the springs in Figure 8a. If the beams in the x-direction are bent up, the beams in the orthogonal y-direction bend down due to Poisson's ratio (unless they are restrained). The force required to restrain them results in an additional bending moment which equals

$$\Delta M_y = \nu D \left(\frac{\partial^2 w}{\partial x^2}\right) \tag{42}$$

This is likewise true for the action of the y-beams on the x-beams. As a result, the bending moment in an x-beam becomes

$$M_x = D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \tag{43}$$

Figure 9 shows the assembled slab model. The torsion bars in Figure 9 are considered to resist only torsion.

Input Values for Model

Having developed a model, it is necessary to relate it to a real plate or slab. The plate is divided into increments in the x- and y-directions with increment length h_x and h_y, respectively. These "beam" increments are designated with i in the x-direction and j in the y-direction. The mesh point or joint on the positive end of each increment is arbitrarily numbered the same as that increment. This numbering system then gives the i, j grid indicated in Figure 10. It is also convenient to denote segments of the plate bounded by increments in both the x- and y-directions, because these segments correspond to the torsion bars in Figure 10.

Stiffness and Lateral Load.—To describe the real plate with the model, it is appropriate to look at the j^{th} x-beam. Figure 11 shows a side view of this beam which may be irregular in profile and may be loaded by a varying distributed load q. One increment width of the load, centered on the i^{th} mesh point, is assigned to Station i on the j^{th} x-beam (Station i, j on the model). $Q_{i,j}$ is the lumped load applied at Station i, j. $D_{i,j}$ represents the bending stiffness of the plate segment of which mesh point i, j is the center. The sketch is intended to illustrate that the stiffness may vary. In this case, it decreases from Station i-1 toward Station i+1. The load also varies but increases from Station i-1 toward Station i+1. $Q_{i,j}$ can be expressed by the equation

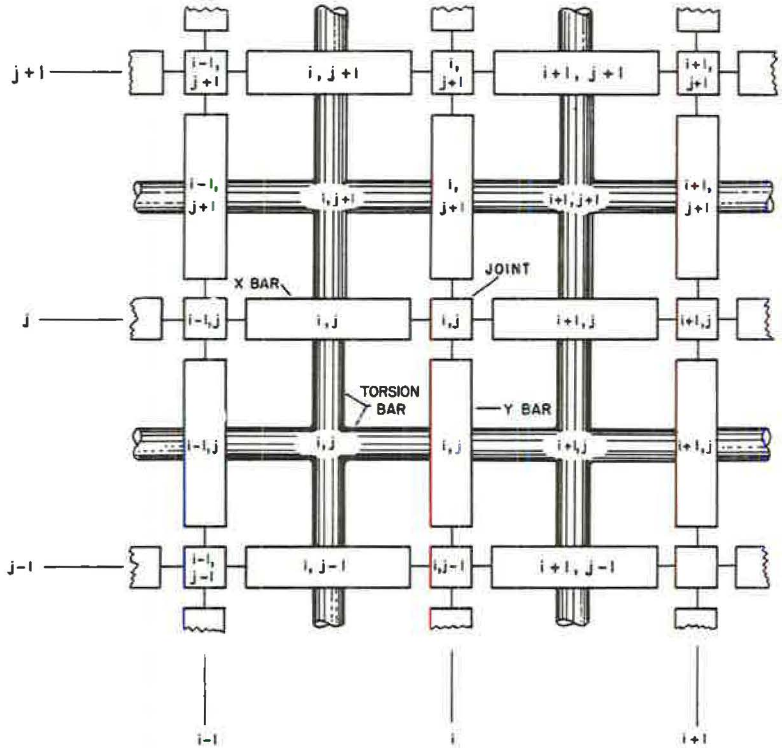


Figure 10. Plan view of the slab model showing all parts with generalized numbering system.

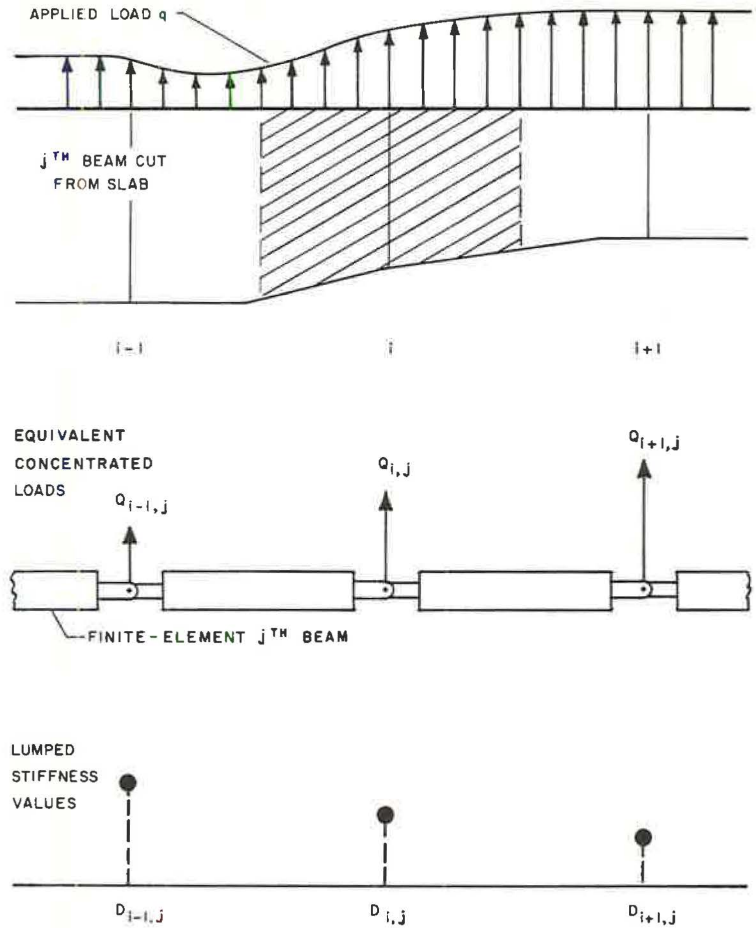


Figure 11. Finite-element representation of a beam cut from slab with finite-element loads and stiffness.

$$Q_{i,j} = \sum_{i-\frac{1}{2}}^{i+\frac{1}{2}} \sum_{j-\frac{1}{2}}^{j+\frac{1}{2}} q + Q_c \tag{44}$$

where Q_c is any concentrated load which may be present at the station in addition to the distributed load.

The stiffness $D_{i,j}$ for a plate is a unit value per inch of width. It is convenient for use in computations to input average values over a full increment width. If $D^x_{i,j}$ represents the average stiffness in the x-direction, it can be calculated

$$D^x_{i,j} = \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \frac{D^x_{i,j}}{h_x h_y} \tag{45}$$

that is, the average bending stiffness of the plate over an area one increment wide and one increment long, centered at Station i, j. Full development of all input for the model is provided in Ref. 11.

Other Input Values.—It is convenient to represent the torsional stiffness of plate segment i, j as torsion bars i, j acting at the midpoint of the model elements (Fig. 10). It is also helpful for external couples or torques applied to the plate to be input into the stiff beam elements. This is properly shown on the free-body in the next section. Axial loads, P, are also input into the bars with the changes, ΔP, considered to occur at mesh points.

Summary of Finite-Element Theory

A physical model has been chosen to represent the plate or slab for solution by numerical methods in preference to expressing the differential equation governing slab behavior in finite-difference form. The model is straightforward and assists visualization of the problem. Discontinuities and freely discontinuous changes in load, bending stiffness, torsional stiffness, and other parameters are easily understood with the use of a physical model, but limitations on continuity of the differential equation make direct difference approximations suspect.

The greater the number of increments used to model a particular problem, the greater the accuracy of the solution. All exact solutions are based on infinitesimal changes in the real structure. Experience with this model indicates that reasonable results can be obtained with most problems using 8 to 20 increments in each direction, although the number of increments to be used will certainly depend on the dimensions of the problem as well as the accuracy required and the local complexity to be resolved. This is discussed in the section on "Example Problems and Verification of the Method."

FORMULATION OF EQUATIONS

The purpose of this section is to formulate from a free-body analysis the equations necessary to solve for the bending of a slab. It is intended here to give a readable and concise account of these developments rather than a complete mathematical treatment.

Free-Body Analysis

To derive the equations for solution of the bending of a plate or slab, it is helpful to refer to a free-body of the model. Consider first a section of the assembled slab model centered at any mesh point i, j (Fig. 12). For the present, the x-bar to the left of point i, j is called Bar a, and the x-bar to the right of point i, j is called Bar b.

Figure 13 shows these same bars as a free-body with other members of the model fixed and replaced by a system of equivalent forces. $Q^y_{i,j}$ represents the load carried

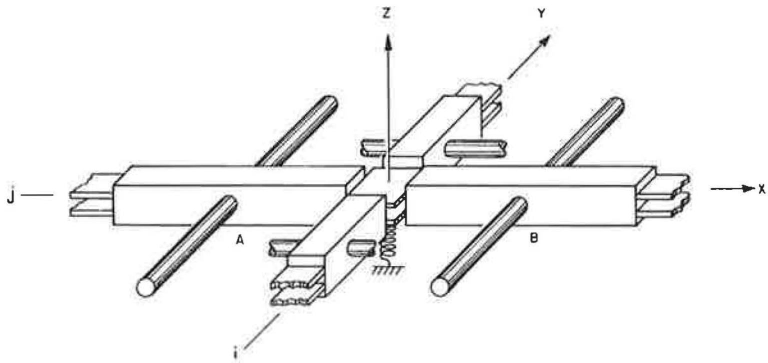


Figure 12. Typical joint i,j taken from finite-element slab model.

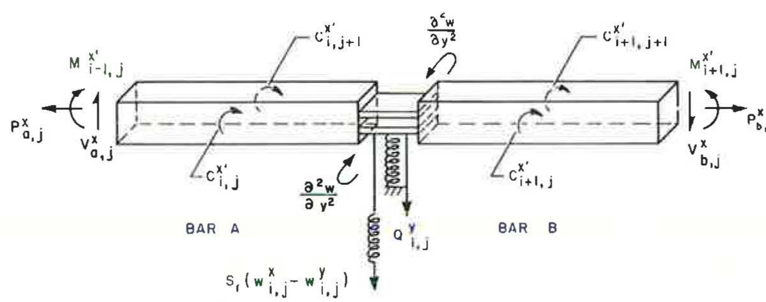


Figure 13. Free-body of joint i,j with other members of the model replaced by an equivalent force system.

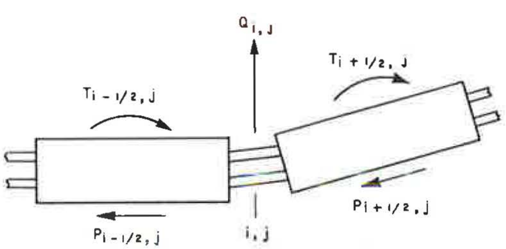


Figure 14. Typical joint i,j with force and restraint inputs shown.

by the y-beam at this intersection and the term $\partial^2 w/\partial y^2$ represents the restraint of the y-beam which provides the Poisson's ratio effect in the x-beam moment. The term $S_f\left(w^x_{i,j} - w^y_{i,j}\right)$ represents the load stored in the fictitious spring closure parameter. These closure springs will be discussed fully in the next two sections. Figure 14 shows the external forces which can be applied to these same two bars. Any of these forces may be zero but are considered to be present for generality. Combining the system of equivalent forces and external loads gives the general free-body of the slab model in Figure 15. This free-body is for a section of an x-beam. A similar free-body can be developed for the y-beam by changing all x's for y's, and all y's for x's.

Summing vertical forces in Figure 15 at joint i, j with up taken as positive gives

$$\Sigma F^x_{v_{i,j}} = Q^y_{i,j} + V^x_{a,j} - V^x_{b,j} - S_{i,j}\left(w^x_{i,j}\right) - Q_{i,j} - S_f\left(w^x_{i,j} - w^y_{i,j}\right) = 0 \qquad (46)$$

To evaluate the shear $V^x_{a,j}$, sum moments acting on Bar a about the center of the bar (clockwise rotations are positive). For equilibrium

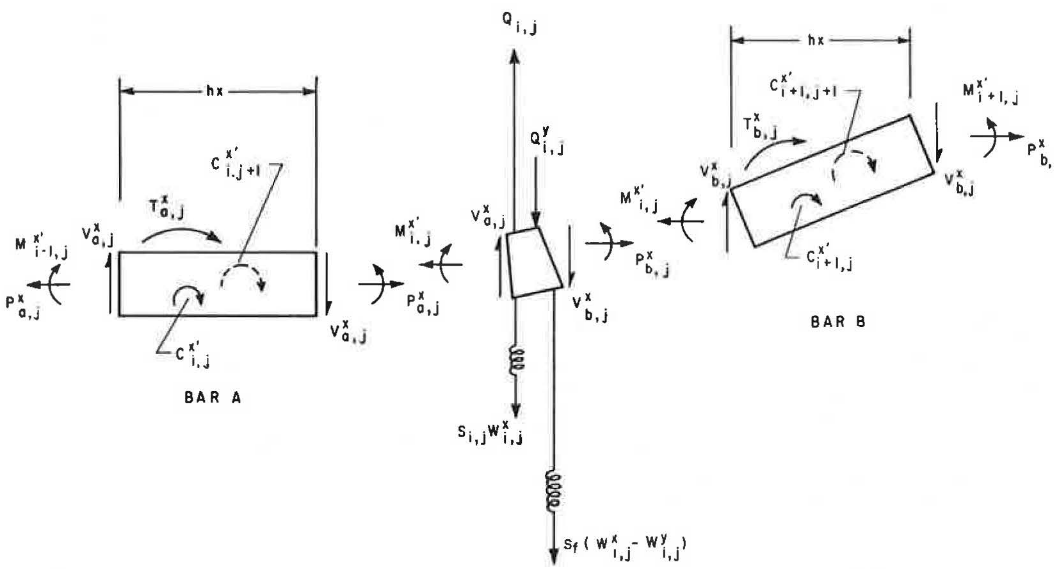


Figure 15. Generalized free-body of joint i,j with all forces and restraints shown.

$$\sum M_a = 0 = M^{x'}_{i-1,j} - M^{x'}_{i,j} + T^{x}_{a,j} + C^{x'}_{i,j} + C^{x'}_{i,j+1} + V_{a,j} h_x$$

$$+ 2P^{x}_{a,j} \left(\frac{-w^{x}_{i-1,j} + w^{x}_{i,j}}{2} \right) = 0 \tag{47}$$

Multiplying through by h_x and clearing obtains

$$-h_x V^{x}_{a,j} = M^{x'}_{i-1,j} - M^{x'}_{i,j} + T^{x}_{a,j} + C^{x'}_{i,j} + C^{x'}_{i,j+1} + P^{x}_{a,j} \left(- w^{x}_{i-1,j} + w^{x}_{i,j} \right) \tag{48}$$

Likewise summing moments about Bar b and multiplying through by h_x obtains an expression for the shear $V^{x}_{b,j}$ as follows:

$$- h_x V^{x}_{b,j} = M^{x'}_{i,j} - M^{x'}_{i+1,j} + T^{x}_{b,j} + C^{x'}_{i+1,j} + C^{x'}_{i+1,j+1}$$

$$+ P^{x}_{b,j} \left(- w^{x}_{i,j} + w^{x}_{i+1,j} \right) \tag{49}$$

Multiplying Eq. 46 through by h_x and substituting Eqs. 48 and 49 for the shears obtains the equation of interest. After convenient grouping of terms and transfer of all known values to the right-hand side of the equation, with a sign change, it becomes

$$\left(M^{x'}_{i-1,j} - 2M^{x'}_{i,j} + M^{x'}_{i+1,j}\right) - \left(- C^{x'}_{i,j} - C^{x'}_{i,j+1} + C^{x'}_{i+1,j} + C^{x'}_{i+1,j+1}\right)$$

$$+ P^{x}_{a,j}\left(- w^{x}_{i-1,j} + w^{x}_{i,j}\right) - P^{x}_{b,j}\left(- w^{x}_{i,j} + w^{x}_{i+1,j}\right) + S_{i,j}h_{x}w^{x}_{i,j}$$

$$= h_{x}\left[Q_{i,j} - Q^{y}_{i,j} - S_{f}\left(w^{x}_{i,j} - w^{y}_{i,j}\right)\right] - T^{x}_{a,j} + T^{x}_{b,j} \tag{50}$$

This equation relates forces and deflections at point i, j, but all of the prime terms must be evaluated further before the required mathematical manipulations can be performed. It is necessary at this point to substitute the central difference formulations of moment. Accordingly, they are substituted at Stations i-1, j; i, j; and i+1, j.

The term $C'_{i,j}$ represents the force exerted on the x-beam due to the relative rotation between this beam and its neighbors. These expressions must be written for $C^{x'}$ at Stations i, j; i, j+1; and i+1, j+1.

After making these substitutions, Eq. 50 becomes

$$h_{y}D^{x}_{i-1,j}\left[\left(\frac{w^{x}_{i-2,j} - 2w^{x}_{i-1,j} + w^{x}_{i,j}}{h^{2}_{x}}\right) + \nu_{yx}\left(\frac{w^{y}_{i-1,j-1} - 2w^{y}_{i-1,j} + w^{y}_{i-1,j+1}}{h^{2}_{y}}\right)\right]$$

$$- 2h_{x}D^{x}_{i,j}\left[\left(\frac{w^{x}_{i-1,j} - 2w^{x}_{i,j} + w^{x}_{i,j}}{h^{2}_{x}}\right) + \nu_{yx}\left(\frac{w^{y}_{i,j-1} - 2w^{y}_{i,j} + w^{y}_{i,j+1}}{h^{2}_{y}}\right)\right]$$

$$+ h_{y}D^{x}_{i+1,j}\left[\left(\frac{w^{x}_{i,j} - 2w^{x}_{i+1,j} + w^{x}_{i+2,j}}{h^{2}_{x}}\right) + \nu_{yx}\left(\frac{w^{y}_{i+1,j-1} - 2w^{y}_{i+1,j} + w^{y}_{i+1,j+1}}{h^{2}_{y}}\right)\right]$$

$$+ \frac{C^{x}_{i,j}}{h_{y}}\left(w^{x}_{i-1,j-1} - w^{x}_{i-1,j} - w^{x}_{i,j-1} + w^{x}_{i,j}\right)$$

$$+ \frac{C^{x}_{i,j+1}}{h_{y}}\left(- w^{x}_{i-1,j} + w^{x}_{i,j} + w^{x}_{i-1,j+1} - w^{x}_{i,j+1}\right)$$

$$- \frac{C^{x}_{i+1,j}}{h_{y}}\left(- w^{x}_{i,j} + w^{x}_{i+1,j} + w^{x}_{i,j-1} - w^{x}_{i+1,j-1}\right)$$

$$- \frac{C^{x}_{i+1,j+1}}{h_{y}}\left(- w^{x}_{i,j} + w^{x}_{i+1,j} + w^{x}_{i,j+1} - w^{x}_{i+1,j+1}\right)$$

$$+ P^x_{a,j}\left(-w^x_{i-1,j} + w^x_{i,j}\right) - P^x_{b,j}\left(- w^x_{i,j} + w^x_{i+1,j}\right) + h_x\left(S_{i,j} + S_f\right)w^x_{i,j}$$

$$= h_x\left(Q_{i,j} - Q^y_{i,j} + S_f w^y_{i,j}\right) - T^x_{a,j} + T^x_{b,j} \tag{51}$$

It is convenient in computation to use the same numbering system for bars, torsion bars, and joints. So far in these developments bars have been referred to as a and b. Referring to the numbering system shown in Figure 10, it will be recognized that in reality a becomes i and b becomes i+1. Therefore, for example, $T^x_{a,j}$ becomes $T^x_{i,j}$, $P^x_{b,j}$ becomes $P^x_{i+1,j}$, etc.

This will be an implicit solution for $w^x_{i,j}$, the deflection of the j^th^ x-beam at Station i. It is convenient for solution, however, to utilize the last estimated values for all deflections, w^x, not falling on the j^th^ beam for a particular iteration, and transfer them to the right-hand side of the equation. Furthermore, all of the y-beam deflections $w^y_{i,j}$ will be assumed known from a previous iteration and will also appear on the right-hand side of the equation. After making the notation change of a to i and transferring known values to the right-hand side, it is helpful to clear fractions and rearrange terms. The resulting equation is the equation we seek and is most conveniently written in terms of five unknown deflections, i.e.,

$$a_x w^x_{i-2,j} + b_x w^x_{i-1,j} + c_x w^x_{i,j} + d_x w^x_{i+1,j} + e_x w^x_{i+2,j} = f_x \tag{52}$$

where

$$a_x = \frac{h_y^2}{h_x^2} D^x_{i-1,j} \tag{53}$$

$$b_x = -2.0\frac{h_y^2}{h_x^2}\left(D^x_{i-1,j} + D^x_{i,j}\right) - C^x_{i,j} - C^x_{i,j+1} - h_y P^x_{i,j} \tag{54}$$

$$c_x = \frac{h_y^2}{h_x^2}\left(D^x_{i-1,j} + 4D^x_{i,j} + D^x_{i+1,j}\right) + C^x_{i,j} + C^x_{i+1,j} + C^x_{i,j+1}$$

$$+ C^x_{i+1,j+1} + h_x h_y\left(S_{i,j} + S_f\right) + h_y\left(P^x_{i,j} + P^x_{i+1,j}\right) \tag{55}$$

$$d_x = -2.0\frac{h_y^2}{h_x^2}\left(D^x_{i,j} + D^x_{i+1,j}\right) - C^x_{i+1,j} - C^x_{i+1,j+1} - h_y P^x_{i+1,j} \tag{56}$$

$$e_x = \frac{h_y^2}{h_x^2} D^x_{i+1,j} \tag{57}$$

$$f_x = h_x h_y \left(Q_{i,j} - Q^y_{i,j} + S_f w^y_{i,j}\right) + h_y \left(T^x_{i,j} + T^x_{i+1,j}\right)$$

$$- \nu_{yx} \Bigg[D^x_{i-1,j} \left(w^y_{i-1,j-1} - 2w^y_{i-1,j} + w^y_{i-1,j+1}\right)$$

$$- 2D^x_{i,j} \left(w^y_{i,j-1} - 2w^y_{i,j} + w^y_{i,j+1}\right) + D^x_{i+1,j} \Big(w^y_{i+1,j-1} - 2w^y_{i+1,j}$$

$$+ w^y_{i+1,j+1}\Big)\Bigg] - C^x_{i,j} \left(w^x_{i-1,j-1} - w^x_{i,j-1}\right) - C^x_{i,j+1} \left(w^x_{i-1,j+1} - w^x_{i,j+1}\right)$$

$$+ C^x_{i+1,j} \left(w^x_{i,j-1} - w^x_{i+1,j-1}\right) + C^x_{i+1,j+1} \left(w^x_{i,j+1} - w^x_{i+1,j+1}\right) \qquad (58)$$

One term remains to be evaluated, $Q^y_{i,j}$, the load absorbed by the y-beams at any time. This load can be evaluated by numerical differentiation of the deflected pattern of the y-beam system, but it can also be done from the free-body analysis by summing vertical forces in terms of load absorbed by both sets of beams, $Q^x_{i,j}$ and $Q^y_{i,j}$. This summation on the free-body in Figure 15 gives

$$Q_{i,j} - Q^y_{i,j} - Q^x_{i,j} - S_{i,j} w^x_{i,j} + S_f \left(w^x_{i,j} - w^y_{i,j}\right) = 0 \qquad (59)$$

After necessary algebraic manipulations, the appropriate equation for evaluating $Q^y_{i,j}$ is seen to be as follows

$$Q^y_{i,j} = \overline{QBMY}_{i,j} + \overline{QTMY}_{i,j} + \overline{QPY}_{i,j} + \frac{T^y_{i,j} - T^y_{i,j+1}}{h_y} \qquad (60)$$

If this process is repeated for a segment of y-beam, equations comparable to Eqs. 52 through 58 can be developed for the y-beams.

Summary

Eqs. 52 through 58 conveniently describe the model at Station i, j and are statically correct since the summation of forces at any time during the solution will equal zero. There are two such sets of equations, one for the x-system and one for the y-system at each mesh point i, j. The number of stations in each direction is equal to the number of increments plus 4. As an example, a problem divided into eight increments in the x-direction and eight increments in the y-direction would require equations at 12 stations in each direction. Thus the number of equations required to describe the system would be 288; 144 for the y-beams and 144 for the x-beams. This readily explains the need to resort to digital computers to perform the mathematical manipulations.

SOLUTION OF EQUATIONS

The equations derived in the preceding section are formidable. Two sets of such equations are required to describe each mesh point in the system, one for the x-beams and one for the y-beams. To make these equations useful, some general technique for solving them rapidly is necessary. Although some hand methods have been developed for small mesh systems, the high-speed digital computer offers the desirable approach. This section presents several methods available for solution of these equations. A general description of the method chosen for use in this work is included.

Current Methods for Solution of Simultaneous Equations

The methods developed to solve systems of equations like Eq. 52 fall into about five major categories: (a) simple direct-elimination methods, (b) methods involving iterative techniques similar to moment distribution, (c) general relaxation techniques, (d) successive over-relaxation, and (e) alternating-direction-implicit methods. Actually, there are many other methods and many variations of the major methods listed above. The purpose of this writing, however, is not to survey the field of numerical analysis, but to apply a useful method to the solution of plates and slabs.

White and Cottingham (35) found a simple elimination method to be useful in their solution of plate buckling problems. Such elimination methods, however, are time consuming, requiring time in proportion to the cube of the number of equations involved. Another major drawback of this method is storage space since every term in the matrix must be stored even though many are zero.

Newmark (20) discusses several methods for solving simultaneous equations including successive approximation and step-by-step methods, as well as the distribution method. Distribution methods are somewhat more formal than relaxation methods and are organized for hand computations by technicians. Such methods are too cumbersome for efficient use in a digital computer.

The relaxation methods, or more specifically the method of "successive relaxation of constraints," is based on the concept that the structure is maintained in a continuous state, but has acting on it residual loads which are not statically consistent with the correct loading. The "residuals" are reduced by introducing arbitrary changes in displacement until convergence or statical balance is obtained. Southwell (8, p. 66) pioneered such methods. These were also originally developed for hand computation but are flexible enough for use in computers. Liebmann (8, p. 147) coded relaxation techniques for use on digital computers and speeded the process up considerably. Even so, he states, "The disadvantages of this procedure are the slow rate of convergence in many cases and the possible lack of convergence." Other work on this technique includes that by Jacobi, Gauss and Seidel, Richardson, and Frankel.

The SOR method, successive over-relaxation, provides still faster and better trial-and-error solutions by applying a complex relaxation factor which over-relaxes or over-compensates the adjustment of the existing data on any given trial. Otherwise, the method is basically that of relaxation.

The alternating-direction method presented by Conte and Dames (5) appears to offer by far the best techniques for solving the plate equation. Others who have used this method include Griffin and Varga (7) and Tucker (30). Because of its applicability, a more complete discussion of this method is warranted.

Alternating-Direction Implicit Solution

Conte and Dames (5) present an implicit alternating-direction iterative scheme which appears to be more efficient than any of the relaxation methods. Their procedure is an extension of methods developed by Douglas and Rachford. Conte and Dames (5) present a solution of the partial differential equation.which governs slab behavior. In simplest terms, the method divides the partial differential equation into two ordinary differential equations and couples their solution by trial and error in a methodical fashion, proceeding first in the x-Cartesian direction, then in the y-direction, thus the name alternating-direction. The most difficult part of using this method is the selection of proper iteration parameters. Proof of convergence exists for certain parameter selection for regular, well-conditioned systems. For the diverse systems described herein, however, much remains to be done.

Experimentation by Matlock, Tucker, Ingram, Salani, and Haliburton (18, 30, 12, 24) with these methods has led to the use of the alternating-direction iterative method in the solutions in this report. This technique has many favorable characteristics which warrant its use, as follows:

1. The method is rapid and well adapted for computer use.
2. The method fits well with the mechanical model used to describe the system.
3. The process can be easily visualized as a trial-and-error solution of the model.
4. The method is logical and can be understood by practicing engineers.

The concepts developed herein are general in nature. They do not emphasize mathematical rigor and completeness, but are shown to be applicable to many engineering problems. No attempt will be made to prove mathematically absolute convergence, although such proof is available for special-case uniform, homogeneous, isotropic systems. Rather, the advantages and capabilities of the method will be demonstrated by examples later in this report. The validity of these diverse examples and their exhibition of closure or convergence to acceptable tolerance (10^{-6} in. for deflection or 1.0 lb for load) is offered as adequate proof of satisfactory closure.

Use of the alternating-direction iterative method is greatly enhanced by judicious choice of closure parameters. They have been shown to be related to the limiting eigenvalues or characteristic values of the set of equations involved. Many mathematicians maintain that closure parameter values selected for square systems must be used for both halves of any iterative cycle. Ingram (12) has demonstrated a method, however, which is not troubled by this restriction. Furthermore, diverse problems which prove troublesome to solve with the classical single-iteration control methods are readily solved using the Ingram dual-control techniques.

Details of Solutions

For solving the large number of simultaneous equations which result in each half-cycle of the alternating-direction iterative method, Matlock and Haliburton (18) used an efficient two-pass method to solve linearly elastic beam-columns. The method involves the elimination of four unknowns, two each in two passes. The first pass from top to bottom eliminates deflections w^x_{i-2} and w^x_{i-1} from each equation (see Eq. 52). The second pass, in reverse order, eliminates deflections w^x_{i+2} and w^x_{i+1} from each equation, and thus results in the solution for the desired deflection w^x_i. Those readers not familiar with this technique are invited to read Ref. 18.

One of the valuable assets of this method is that boundary conditions as normally discussed are automatically provided with two dummy stations specified at each end of each beam in the system. These dummy stations in reality have no bending stiffness; therefore, a bending stiffness equal zero is input for them. Equation 52 is then formulated for every station in the beam plus two dummy stations on each end.

To solve for $w^x_{i,j}$ then, the plate is considered to be two systems of orthogonal beams interconnected at Station i, j by S_f, the fictitious closure spring constant. Figure 16 shows the slab model with closure springs acting during closure.

With the beam-column as a basic tool, the solution of the system of equations for plates and slabs proceeds as follows:

1. Solve each x-beam successively through the system considering all the y-beams to be held fixed in space. At any particular solution of any x-beam then, the fictitious closure spring acts as restraint on the x-beam of interest.
2. After all x-beams are solved and their new deflection pattern is known, alternate or change directions and fix the x-beams in this new pattern.
3. Solve for the deflected shape of each y-beam in turn. The fictitious springs now act as loads or restraints on the y-beam, serving to transfer the load which has been stored in them from the deflected x-beams.

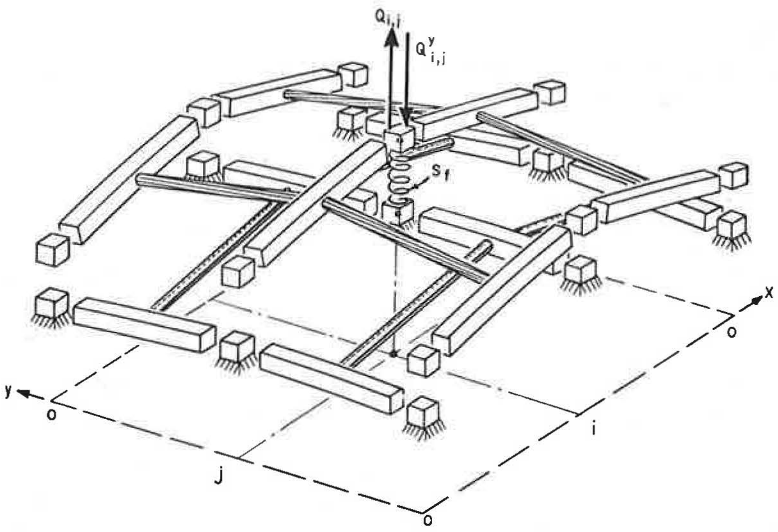


Figure 16. Plate represented in the closure process as two orthogonal systems with closure spring acting between them at Station i,j.

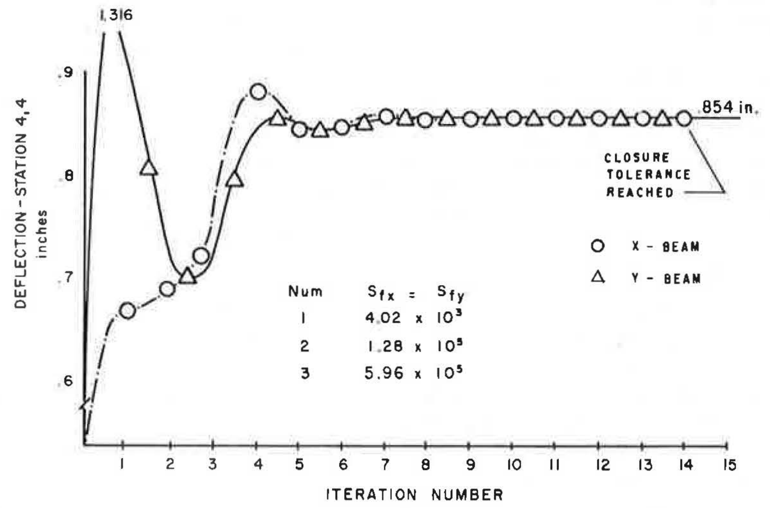


Figure 17. Plot of closure for Example Problem 102.

4. This procedure is repeated alternately until all of the load is properly distributed throughout the system. At this point the summation of static forces at each joint in the system will equal zero within the specified tolerance and the deflection of the x-beam system $w^x_{i,j}$, at any point will equal the deflection of the y-beam system, $w^y_{i,j}$, at the same point within the specified tolerance so that the term $S_f\left(w^x_{i,j} - w^y_{i,j}\right)$ vanishes.

The process described is a rapid one requiring from 5 to 25 iterations for most simple problems with closure to six significant digits.

Closure Process

Figure 17 illustrates the closure process for an 8- × 8-increment square plate. The parameters were chosen in accordance with the rules set forth in Ref. 11, p. 59.

THE COMPUTER PROGRAM

The equations derived in the section on "Formulation of Equations" are not useful for hand calculations, but they are extremely well adapted for digital computer methods. During the 18 months of this investigation, 12 programs have been developed which are useful for solving slab and plate problems of various types. The earlier programs are simple in format and application. The most general program is known as SLAB 17. The number 17 signifies that this is the seventeenth version in the chronological sequence of development of SLAB Programs.

These programs are written in FORTRAN computer language for the Control Data Corporation 1604 Digital Computer which has a 48 bit word length and is operated with a FORTRAN-63 monitor system. The compile time for the basic program is less than two minutes; however, normal operating decks may be compiled on binary cards, thus reducing compile time in the computer to about 15 seconds. The exact storage requirements of the program as presently dimensioned are undetermined. In general, however, the dimension statements are such that the program will handle as large a problem as practical with present storage capacity. This program can be modified for use with the IBM 7090 computer by the modification of about 12 input-output cards.

The time required to run problems varies, of course, with the complexity and size of the system, i.e., the number of increments involved, and the number of iterations required to obtain the desired accuracy. To give a general idea of operating time, eight-by-eight problems close to a tolerance of 10^{-6} inches in 10 to 60 iterations, and require 30 to 100 seconds for solution. An increase in size to 16 × 16 with fairly uniform stiffnesses in both directions can be closed to similar tolerances in about 100- to 200-sec computer time. While this may seem high when compared to solution time for simpler problems, the cost of three minutes of computer time ($15 to $30, depending on rental rates) is small compared to three to four days of laborious computation time required to do the problem by hand. More important, perhaps, is the fact that this computer program provides a useful way of making some solutions for the first time.

The FORTRAN Program

A summary flow diagram for the SLAB Programs is given in Figure 18. This flow diagram describes the program tasks briefly. A detailed flow diagram and listing of the program SLAB 17 is provided in Ref. 11.

The format used for inputing data into the program is arranged as conveniently as possible. No effort is made to be frugal with the number of cards required to input one problem. Instead, every effort is made to organize the program input logically and concisely. The problem input deck starts with two cover cards used to identify the program and the particular run being made. After these come necessary data cards.

Output Information

The program output is arranged to be useful to the user. A format which can be trimmed to standard 8½- by 11-in. size is provided. For convenience and help in identifying problems, the program prints out all original input data at the beginning of each problem. These values are tabulated and labeled just as they were input. The first output computed by the program itself is Table 4, "Monitor Deflection." (See Appendix.) This table prints out deflections for both the x-beams and y-beams at the four pre-selected monitor stations specified in Table 1. (See Appendix.) This data can be plotted using other versions of the SLAB Program.

The results desired from the program are printed out in Table 5. (See Appendix.) This table prints in two parts in keeping with the 8½- by 11-in. format. The first half prints external station numbers, x- and y-deflections, bending moments in the x- and y-directions, the external reaction of the slab and the true error in statics as determined

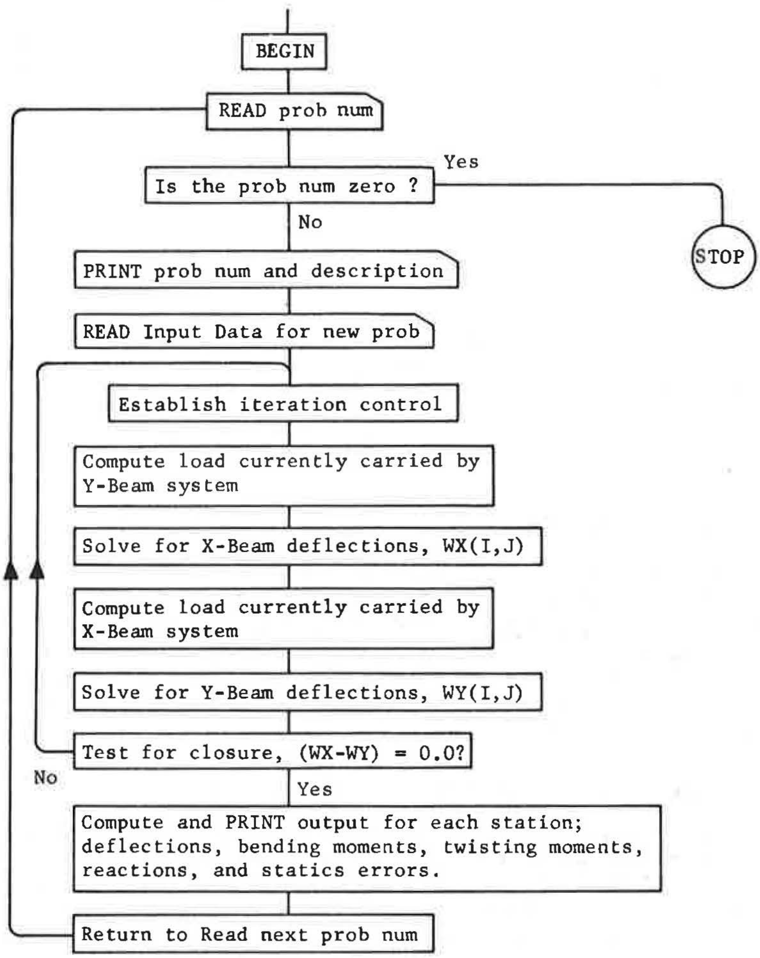


Figure 18. Summary flow chart, slab program.

by summation of vertical forces at each station. Part 2 of Table 5 prints out station numbers and twisting moments in the x- and y-directions at each station. Four additional spaces are provided for printing out stresses and direction of principal stress in later programs to be equipped with stress calculating options.

An automatic plot routine can be coupled with SLAB 17 and used to plot any of the variables available at mesh points in the system, although its major use is normally plotting deflection contours.

The bending and twisting moment outputs are calculated by numerical differentiation of the deflected shape. In both cases central differences are used to provide moments at each mesh point in order that these moments may be available for calculation of principal stresses.

Summary of Program Details

SLAB 17 is the most general and useful program available at the present time for solving the equations developed herein. The program is written in FORTRAN-63 language for the CDC 1604 computer and solves slab and plate problems very rapidly. Ref. 11 contains all information about the program from flow diagrams to output information, and can be extracted as an operating manual for use with the program. Twelve other programs are available for solving various types of problems. Several of these will be developed to provide special-case solutions which solve more rapidly than the general method.

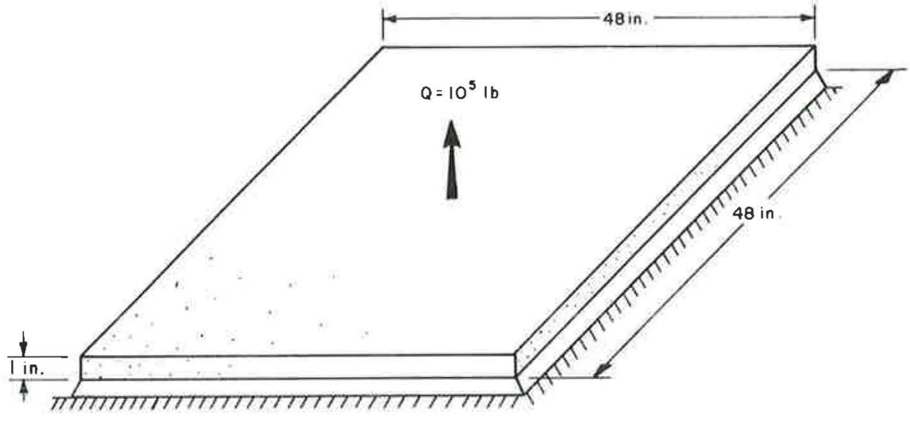


Figure 19. Square steel plate simply supported at all edges (Example Problem 101).

EXAMPLE PROBLEMS AND VERIFICATION OF THE METHOD

Developments of equations and discussions of techniques are important in analytical work of this kind; however, application of the method and demonstration of technique in solving actual problems is equally important. This section provides the solution to several example problems to demonstrate Program SLAB 17 and its use in engineering calculations. Closed-form solutions for some of the problems are provided as a mathematical check to the computed solutions. Sample computer output for example Problem 201 is provided in the Appendix.

Problem Series 100—Simply-Supported Plate With Variations

As a first example, a series of problems illustrating many of the variations possible in the program are applied to a 48-in.-sq simply supported steel plate 1 in. thick (Fig. 19). The modulus of elasticity is 30,000,000 psi. Poisson's ratio is 0.25. Loading variations will be discussed with the individual cases. Once the reader acquaints himself with the physical properties of this plate, it will be possible to evaluate very rapidly five separate cases of load and variations of parameters.

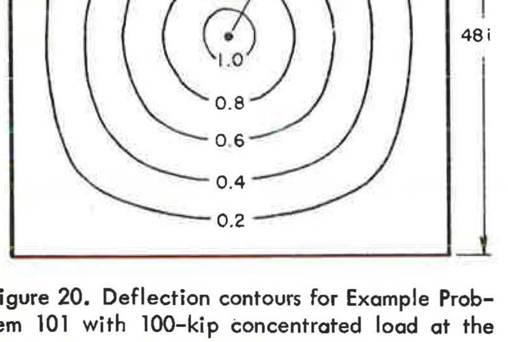


Figure 20. Deflection contours for Example Problem 101 with 100-kip concentrated load at the center.

Problem 101—Concentrated Load.—The problem of a simply supported square or rectangular plate with concentrated load is considered by Timoshenko for various load conditions. Several equations for solving this problem are presented using single and double trigonometric series. A consensus value of solutions for maximum deflection, which occurs under the load, is 1.07 inches. Figure 20 is a plot obtained automatically from SLAB 17 coupled with a plot routine for the complete deflected shape of the plate when it is divided into eight 6-in. increments in each direction. The maximum deflection w_{max} is noted to be 1.138 inches. This differs 0.07 inch, or 6 percent from the closed-form solution. If the number of increments is increased to 16 in each direction, a maximum deflection of 1.08 inches results. Thus, the error is reduced to 1 percent, probably as good as the accuracy of the closed-form solution using a truncated double trigonometric series. Contours of maximum bending moment or twisting moment could have been plotted, if desired, just as easily as the deflections.

Problem 102—In-Plane Forces.—In addition to the concentrated load at the center, add a uniform in-plane force in the y-direction of 16,667 pounds per inch of plate width. In the closed-form solution this term appears in the denominator of the series solution and does not have as much effect as might be expected. The maximum deflection occurs under the load and is 0.787 inch. The computed solution for an 8 × 8 grid is 0.854 inch. The difference of 0.067 inch is almost identical with that in Problem 101. Increasing the number of increments would reduce the difference accordingly.

Problem 103—Two-Way In-Plane Forces.—Add to Problem 102 an equal in-plane tensile force in the x-direction. The computed solution for maximum deflection reduces to 0.661 inch. If the force in the x-direction is tensile or positive but the force in the y-direction is compressive or negative, the effects on maximum deflection offset each other as would be expected. This solution gives a maximum deflection of 1.14 inches, the same as Problem 101.

Problem 106—End Supports With Line Loads.—Modify the basic problem slightly by removing the simple supports under two edges of the plate. This leaves the plate supported as a wide-beam on simple supports (Fig. 21). Unlike a beam, however, the plate should exhibit Poisson's ratio effects. Poisson's ratio manifests itself in such a structure by anticlastic bending. This may be explained in the following way. If moments are applied to the plate at opposite ends of the x-axis, a simple analysis would indicate that a uniform moment in the x-direction, M_x, would be present throughout the plate. Two conditions are known from physical equations governing plate behavior. First, the bending moment in the y-direction at the free y-edges must be zero. And, second, the bending moment in the y-direction may be stated as follows:

$$M_y = D_y\left(\frac{\partial^2 w}{\partial y^2} + \nu_{xy}\frac{\partial^2 w}{\partial x^2}\right) \tag{61}$$

The first stated condition requires that the second condition, Eq. 61, be identically zero. Note that the bending in the x-direction is not zero, thus the differential $\partial^2 w/\partial x^2$ cannot be equal to zero. Then for Eq. 61 to be identically zero

$$\frac{\partial^2 w}{\partial y^2} = -\nu\frac{\partial^2 w}{\partial x^2} \tag{62}$$

Thus, bending in the y-direction will be present at the two edges with a sense opposite that in the x-direction. This is illustrated in Figure 22. Figure 23 illustrates the same plate when Poisson's ratio equals zero. This can be recognized as bending equivalent to that of a beam in which Poisson's ratio can be neglected. Brief reference to Eq. 61 indicates that if Poisson's ratio equals zero, the bending in the y-direction is unaffected by bending in the x-direction since they are related only through Poisson's ratio.

Two solutions were run, one with Poisson's ratio $\nu = 0.0$, the second with Poisson's ratio $\nu = 0.25$. The hand solution as a beam gives w_{max} at the center of the beam or plate of -0.566 inch. The SLAB 17 solution for eight increments $\nu = 0.0$ gives w_{max} = -0.576 inch, a difference of 2 percent. A 16 × 16 solution reduces this difference to less than one percent. For Poisson's ratio of 0.25 a center deflection of w = -0.575 inch results. This increases to -0.640 inch at the two edges due to anticlastic bending.

Problem 107—End Supports with Applied Torques.—This problem is the equivalent of Problem 106 except the moment due to the applied line loads acting at 6-in. distance from the two simple supports is converted to a uniform moment applied near the ends.

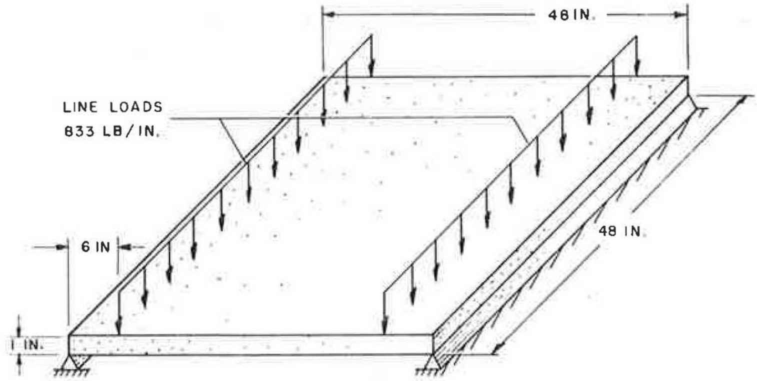


Figure 21. Plate simply supported on two edges with line loads (Example Problem 106).

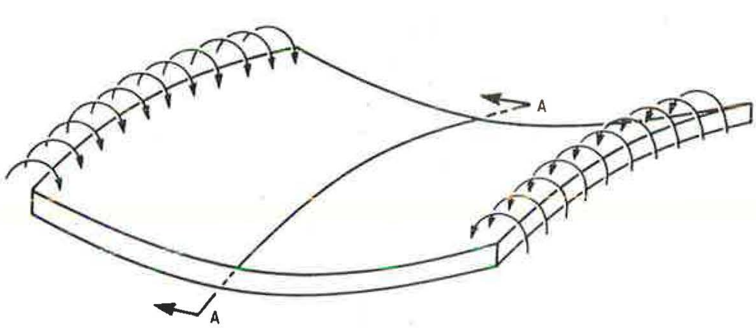


Figure 22. Anticlastic bending of plate subjected to uniform bending moment at opposite edges (Example Problem 107).

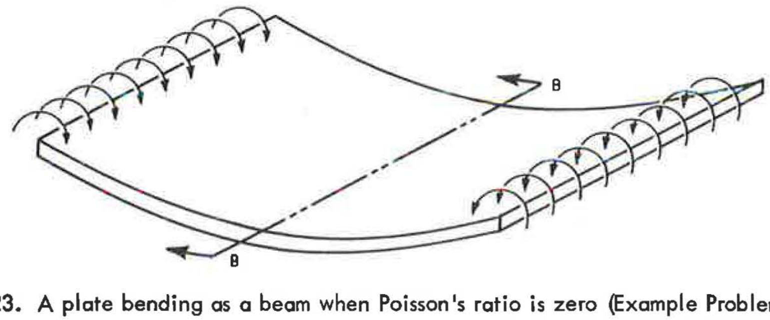


Figure 23. A plate bending as a beam when Poisson's ratio is zero (Example Problem 106).

It is illustrated in Figure 24. The results are exactly comparable to those of Problem 106 as was expected. This indicates that Program SLAB 17 handles applied torques satisfactorily.

Slabs-on-Foundation—Westergaard Cases, Problem Series 200

For slab-on-foundation problems, the matter of checking theory becomes more complicated because of the lack of closed-form solutions. Three example problems related to the three Westergaard cases are presented here since these solutions are well known

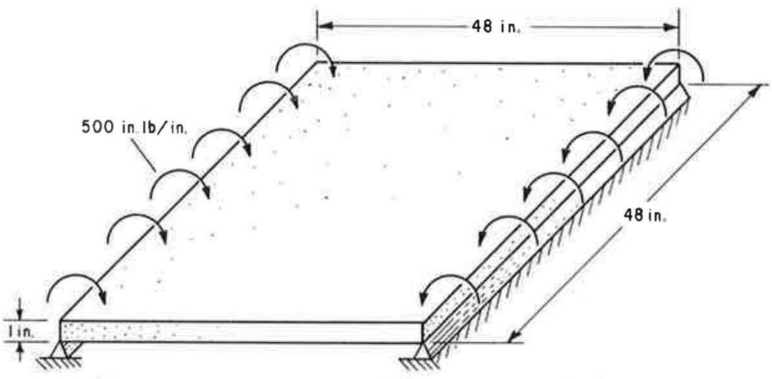


Figure 24. Simply supported plate with a bending moment applied at opposite ends (Example Problem 107).

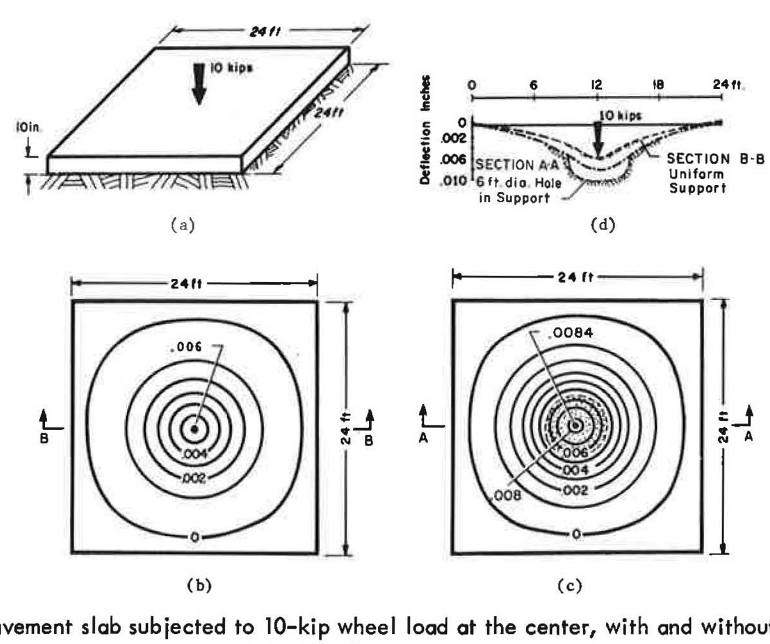


Figure 25. Pavement slab subjected to 10-kip wheel load at the center, with and without uniform subgrade support.

and are currently used as a basis for most rigid pavement design. A single pavement slab was chosen for comparison and examined separately for the three Westergaard cases. The closed-form solutions come from Westergaard, (32, p. 102). A standard slab example is used for the computed deflection as shown in Figure 25a.

The examples all involve a 10-in. slab thickness, 24 ft square in plan dimension with a modulus of elasticity of 3,000,000 psi and $\nu = 0.20$. The subgrade modulus was assumed to be 200 pounds per square inch per inch of deflection and a single concentrated load of 10 kips was applied in each case.

Problem 201—Center Load.—With these physical constants the Westergaard solution gives the deflection under a load applied at the center of an infinite slab to be -0.0057 inch. The computed results are -0.0060 inch. In addition, one can see from Figure 25b that the computed solution gives the complete deflection contours of the slab, whereas the Westergaard equation gives the deflection only under the load. This solution

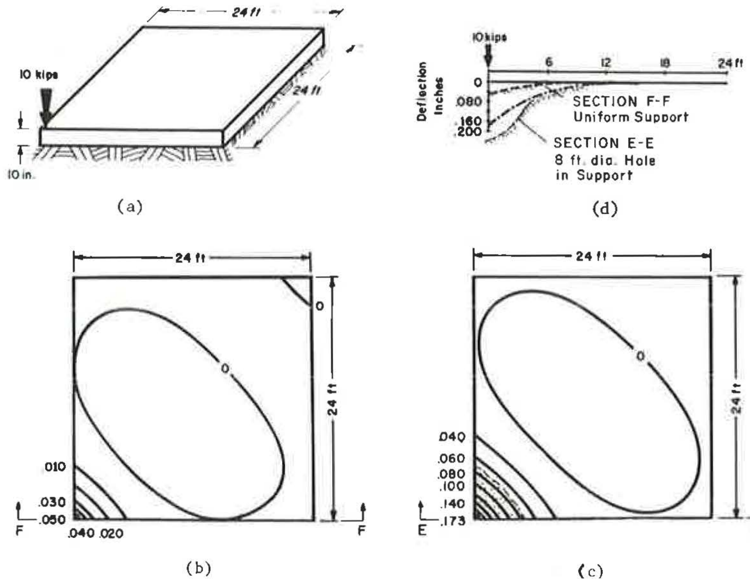


Figure 26. Pavement slab subjected to 10-kip wheel load at the corner, with and without uniform subgrade support.

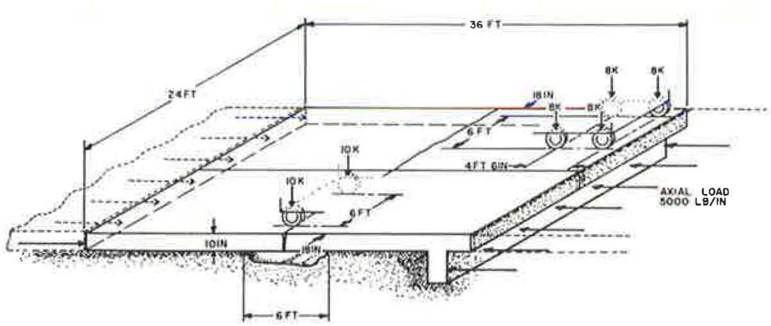


Figure 27. Lug anchor example problem.

involves 8 increments. A solution using 12 or 16 increments gives deflection results closer to that of Westergaard.

Problem 202—Edge Load.—For the case of edge loading, Westergaard gives -0.019-inch deflection for a point under the load at the edge of a slab and infinitely far from any other boundaries. These, of course, are not realistic boundaries since pavements certainly have finite limits. In reality, because of cracking or jointing, the load is nearly always relatively close to some boundary in any direction. The finite-element solution based again on the 24-ft-square slab, but with the load centered along one edge, gives a deflection of -0.018 inch. These results compare within 4 percent. Exact comparison need not be expected since one solution is for a real slab and the other for an infinite slab.

Problem 203—Corner Load.—The third Westergaard case is the load applied at a rectangular corner, infinitely far from any other discontinuity. The comparable real slab is shown in Figure 26a. The Westergaard solution gives -0.049 inch of deflection under the load. The finite-element solution shown in Figure 26b is -0.050 inch. The deflection contours are also of interest and are not easily obtained from the Westergaard solution.

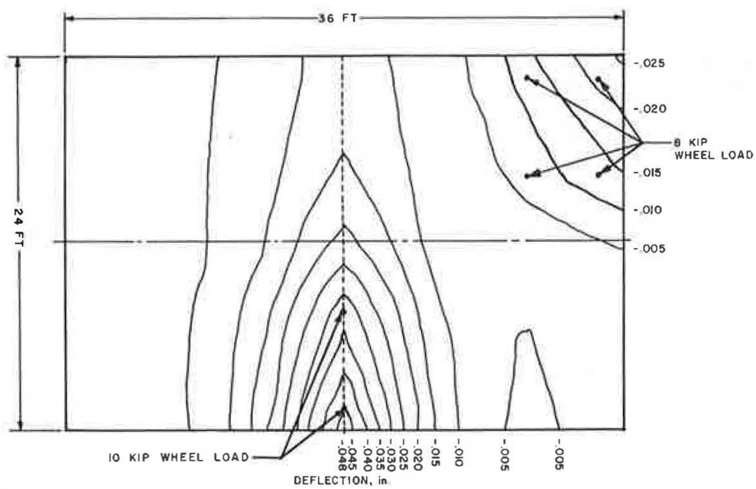


Figure 28. Deflection contours of lug anchor example problem in Figure 27.

To summarize these comparisons it has been shown that the finite-element method described herein agrees within 2 to 5 percent with the Westergaard slab-on-foundation solutions which are currently used for pavement design. In addition, the new method readily provides deflection contours. The same is not true with the Westergaard solution although computer programs do exist to solve those equations explicitly.

Nonuniform Subgrade Support.—To illustrate nonuniform subgrade support, the three cases previously described for center, edge, and corner loadings were rerun with a hole cut in the subgrade centered under the load. For the center load case, the hole 6 feet in diameter cut in the subgrade results in an increased deflection of 40 percent to -0.0084 inch. For the nonuniformly supported edge load, a hole 6 feet in diameter is centered under the load. The resulting deflection under the load is -0.35 inch (Fig. 25c) or nearly double that of the slab with uniform support. Figure 25d compares an edge view with and without uniform support. The corner load case has a hole 8 feet in diameter cut in the subgrade centered at the corner. The resulting deflection increased to -0.173 inch or about 3½ times that of the uniform case as shown in Figures 26c and 26d. It is not intended to draw conclusions at this time concerning these relative increases in deflection nor their effect on pavement performance. It is merely desired to indicate that the method is easily adaptable to solutions for such nonuniform cases which probably represent a majority of pavement actually in service in the United States.

Lug Anchor Example, Problem 602

One of the strengths of the method described herein is its ability to handle complex problems with combination loads and variations in flexure stiffness and support conditions. Figure 27 illustrates such a problem. A 10-in. thick reinforced concrete slab was used. Near one end the slab has a 24-in. deep lug anchor extending into the subgrade. The slab has a centerline joint and a crack which developed from a combination of shrinkage and previous overstress. The joint and crack develop only 20 percent of the stiffness of the surrounding slab. The soil has settled under the crack in the slab and left a section unsupported. For any nonuniformly supported slab such as this, the dead weight of the slab must be considered when evaluating moments and stresses. This weight acts as a uniform load of 300 lb per station. Two 10-kip and four 8-kip wheel loads are considered in this example. An axial load of 5,000 lb/in. is being resisted by the lug anchor. The resulting deflected shape is shown in Figure 28. The maximum deflection of 0.048 inch occurs at the edge of the slab at the transverse crack. It is virtually impossible to obtain this general solution by any other existing methods of analysis.

Summary of Example Problems

The foregoing example problems have been solved to indicate the broad capability of the new method. Two points are worth noting. For those cases having closed-form solutions, the finite-element solution with 8 to 10 increments produced results within 2 to 5 percent of the closed-form solution. If the number of increments was increased to 16, the error comparison reduced to 1 to 3 percent. Perhaps more important are those cases for which no closed-form solution exists. The finite-element method permits, for the first time, the evaluation of such cases. It will be helpful if solutions of these new cases can be compared with experimental data obtained from field or model studies.

SUMMARY

This report examines the analysis of plates and pavement slabs. A study of the technical literature resulted in the selection of some sixty helpful references. Many of these papers contain solutions for special-case plates with simple supports and simple load patterns. These solutions are mathematically complex and are often shrouded in jargon not always relatable to real problems. A particular void is noted in the analysis of pavement slabs. The best work available (Westergaard's) is limited by special-case loads and severe assumptions including infinite or semi-infinite plan dimensions and uniform support conditions.

A method has been presented which is not limited by the simplifying assumptions needed for closed-form solutions. The technique is based on a physical model of the problem which is described mathematically. The principal features of the method are as follows:

1. Representation of the plate or slab by a finite-element model of beam-column elements with freely discontinuous stiffness and load. These line elements are grouped into two systems of orthogonal beams or beam-columns.
2. A rapid, direct solution of individual beams using recursive techniques.
3. An alternating-direction iteration method for combining the solutions of the individual beams into a coordinated slab solution.

The finite-element model is helpful in visualizing the problem and forming the solution. The model consists of:

1. Infinitely stiff and weightless bar elements to connect the joints.
2. Elastic joints where bending occurs, made of an elastic, homogeneous, and orthotropic material which can be described by four independent elastic constants.
3. Torsion bars which represent the torsional stiffness of the plate.
4. Elastic support springs which provide foundation support.

All properties and loads can be freely variable from point to point. Concentrated or distributed loads can be handled including transverse loads, in-plane forces, and external couples. Elastic restraints are provided by vertical support springs.

The alternating-direction iteration method is used to solve the equations describing the behavior of the model because it is well adapted and easy to visualize. The model and method are too complex for hand calculations. A computer program which solves the equations implicitly for the deflection patterns has been developed. The program is written in FORTRAN-63 for the CDC 1604 computer. Minor changes of input formats are required to convert it for use on an IBM 7090. Compile time is 90 to 100 seconds but binary decks are available which compile in about 15 seconds. Automatic plot routines are available for use with the program.

This method has application to a broad variety of complex plate and slab problems which cannot be solved by any other existing method. Applications to complex pavement design problems are of particular interest. The use of the method as a tool in stochastic modeling of pavement life and performance studies is of particular interest. Immediate use of the method in developing new pavement design information is suggested.

NEEDED RESEARCH

The finite-element model described herein is a useful tool. The development of such a method opens the door to studies which heretofore could not be conducted. Such applications of this method are discussed in this section. A look into the method pinpoints several areas of study which could lead to improvement.

Study of Material Properties

It would be helpful if the orthotropic properties of materials used in slabs could be determined for exact input into this program. In particular, information is needed on the relationship of Poisson's ratio and Young's modulus for orthotropic materials, and on torsional rigidity for "torsionally stiff" and "torsionally soft" rib reinforced orthotropic plates.

Comparison With Field Measurements

"The proof of any pudding is in the eating." It is desirable that studies of pavement structures be made in the laboratory and in the field, and that corresponding mathematical analyses be made with the finite-element model. Correlation of these data will help in the evaluation of theory and will lead to improved methods for determining some of the unknowns in the field studies. Deflection measurements will be helpful in this regard as will curvature measurements.

Evaluation of Support Characteristics

Current methods of measuring and specifying pavement support are probably unsatisfactory. It is not adequate to describe a constant k-value for a subgrade or a subbase to be used under a pavement slab. This value is not a linear quantity but is highly dependent on the deflection of the slab lying immediately above. It is also related to overall slab deflections. The true support value is dependent on many things. The first step in such evaluation is the study of nonlinear support conditions for the finite-element model.

ACKNOWLEDGMENTS

This investigation was conducted under the auspices of the Center for Highway Research, The University of Texas. The project of which it is a part was sponsored by the Texas Highway Department and the U. S. Bureau of Public Roads. This support is gratefully acknowledged as is that of the Computation Center, The University of Texas, for the use of its computer.

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Appendix

SAMPLE COMPUTER SOLUTIONS OF EXAMPLE PROBLEMS

EXAMPLE DATA INPUT

```
201      8X8 SOF ,  PR=.20  CENTER LOAD, HWY REPORT 6          WRH      29AP5
  4    6    0   24    8    8 3.600E 01 3.600E 01 1.000E-05 0.200E 00
  4    4    0    0    0    4    4    0
4.920E 03 2.460E 04 1.230E 05 6.160E 05
4.920E 03 2.460E 04 1.230E 05 6.160E 05
  0    0    8    8 0.652E 08 0.652E+08           0.650E 05
  0    1    8    7 0.652E 08 0.652E+08           0.650E 05
  1    0    7    8 0.652E 08 0.652E+08           0.650E 05
  1    1    7    7 0.652E 08 0.652E+08           0.650E 05
  1    1    8    8                                         2.080E+08 2.080E+08
  4    4    4    4                     1.000E 04
```

```
PROGRAM SLAB 17 - MASTER DECK - WR HUDSON, H MATLOCK     REVISION DATE 26 JUL 65
CE051022   HWY SLAB PROJECT        SLAB 17     W R HUDSON
RUN EXAMPLE PROBLEMS FOR FINAL CHECK DECK 3 SLAB 17


PROB
  201      8X8 SOF ,   PR=.20  CENTER LOAD, HWY REPORT-6          WRH      29AP5


TABLE 1.   CONTROL DATA

     NUM VALUES TABLE 2                                                        4
     NUM CARDS TABLE 3A                                                        6
     NUM CARDS TABLE 3B                                                        0
     MAX NUM ITERATIONS                                                       24
     NUM INCREMENTS  MX                                                        8
     NUM INCREMENTS  MY                                                        8
     INCR LENGTH  HX                                                   3.600E 01
     INCR LENGTH  HY                                                   3.600E 01
     CLOSURE TOLERANCE                                                 1.000E-05
     POISSONS RATIO                                                    2.000E-01

     MONITOR STAS  I,J              4  4      0  0      0  4      4  0

TABLE 2A.  ITERATION CONTROL DATA

     F. SPRING REPRESENTING X BEAM

      4.920E 03
      2.460E 04
      1.230E 05
      6.160E 05


TABLE 2B.  ITERATION CONTROL DATA

     F. SPRING REPRESENTING Y BEAM

      4.920E 03
      2.460E 04
      1.230E 05
      6.160E 05

TABLE 3A. STIFFNESS AND LOAD DATA, FULL VALUES ADDED AT ALL STAS I,J IN RECT.

   FROM    THRU      DX          DY           Q           S           CX          CY

   0  0   8  8  6.520E 07  6.520E 07           0  6.500E 04           0           0
   0  1   8  7  6.520E 07  6.520E 07           0  6.500E 04           0           0
   1  0   7  8  6.520E 07  6.520E 07           0  6.500E 04           0           0
   1  1   7  7  6.520E 07  6.520E 07           0  6.500E 04           0           0
   1  1   8  8          0          0           0           0  2.080E 08  2.080E 08
   4  4   4  4          0          0   1.000E 04           0           0           0
```

TABLE 3B. STIFFNESS AND LOAD DATA, FULL VALUES ADDED AT ALL STAS I,J IN RECT.

FROM	THRU	TX	TY	PX	PY

TABLE 4. MONITOR TALLY AND DEFLS AT 4 STAS

	ITR NUM	FICT SPRING	CYC NUM	NOT STAB	NOT CLOS	I,J 4 4	0 0	0 4	4 0
X	1	4.920E 03	1	43		1.195E-02	0	6.298E-04	0
Y		4.920E 03		78	76	5.451E-03	-5.746E-05	-3.113E-04	3.841E-04
X	2	2.460E 04	2	81		8.522E-03	-6.072E-05	-5.283E-04	-2.819E-04
Y		2.460E 04		76	76	5.789E-03	-1.189E-04	-3.142E-04	-3.246E-04
X	3	1.230E 05	3	74		7.334E-03	-1.426E-04	-3.809E-04	-2.665E-04
Y		1.230E 05		68	62	6.118E-03	-1.743E-04	-2.991E-04	-3.159E-04
X	4	6.160E 05	4	65		6.739E-03	-1.837E-04	-2.734E-04	-2.833E-04
Y		6.160E 05		46	39	6.414E-03	-1.921E-04	-2.795E-04	-2.628E-04
X	5	4.920E 03	1	44		6.576E-03	-2.457E-04	-2.788E-04	-2.747E-04
Y		4.920E 03		16	37	6.444E-03	-2.251E-04	-2.696E-04	-2.618E-04
X	6	2.460E 04	2	9		6.562E-03	-2.460E-04	-2.798E-04	-2.711E-04
Y		2.460E 04		4	28	6.457E-03	-2.351E-04	-2.669E-04	-2.651E-04
X	7	1.230E 05	3	8		6.548E-03	-2.410E-04	-2.739E-04	-2.695E-04
Y		1.230E 05		4	17	6.473E-03	-2.398E-04	-2.680E-04	-2.678E-04
X	8	6.160E 05	4	9		6.529E-03	-2.403E-04	-2.694E-04	-2.687E-04
Y		6.160E 05		5	11	6.496E-03	-2.404E-04	-2.690E-04	-2.687E-04
X	9	4.920E 03	1	3		6.515E-03	-2.403E-04	-2.686E-04	-2.686E-04
Y		4.920E 03		0	10	6.500E-03	-2.408E-04	-2.689E-04	-2.694E-04
X	10	2.460E 04	2	0		6.513E-03	-2.405E-04	-2.686E-04	-2.686E-04
Y		2.460E 04		0	9	6.501E-03	-2.407E-04	-2.688E-04	-2.691E-04
X	11	1.230E 05	3	0		6.512E-03	-2.406E-04	-2.687E-04	-2.687E-04
Y		1.230E 05		0	4	6.503E-03	-2.406E-04	-2.688E-04	-2.688E-04
X	12	6.160E 05	4	0		6.510E-03	-2.406E-04	-2.688E-04	-2.688E-04
Y		6.160E 05		0	0	6.505E-03	-2.406E-04	-2.688E-04	-2.688E-04

PROGRAM SLAB 17 - MASTER DECK - WR HUDSON, H MATLOCK REVISION DATE 26 JUL 65
0E051022 HWY SLAB PROJECT SLAB 17 W R HUDSON
RUN EXAMPLE PROBLEMS FOR FINAL CHECK DECK 3 SLAB 17

PROB (CONTD)
201 8X8 SOF , PR=.20 CENTER LOAD, HWY REPORT-6 WRH 29AP5

TABLE 5. RESULTS -- ITERATION 12

I,J		X-DEFL	Y-DEFL	BMX	BMY	REACT	TRERR
-1	-1	-1.943E-04	-1.943E-04	0	0	0	0
0	-1	-2.506E-04	-2.505E-04	0	0	0	0
1	-1	-3.067E-04	-3.067E-04	0	0	0	0
2	-1	-4.812E-04	-4.813E-04	0	0	0	0
3	-1	-6.743E-04	-6.743E-04	0	0	0	0
4	-1	-7.640E-04	-7.639E-04	0	0	0	0
5	-1	-6.742E-04	-6.742E-04	0	0	0	0
6	-1	-4.809E-04	-4.812E-04	0	0	0	0
7	-1	-3.072E-04	-3.068E-04	0	0	0	0
8	-1	-2.504E-04	-2.505E-04	0	0	0	0
9	-1	-1.947E-04	-1.947E-04	0	0	0	0
-1	0	-2.506E-04	-2.506E-04	0	0	0	0
0	0	-2.406E-04	-2.406E-04	0	0	1.564E 01	2.021E-03
1	0	-2.306E-04	-2.306E-04	-4.197E 00	3.056E-10	2.999E 01	-9.718E-03
2	0	-2.641E-04	-2.641E-04	1.638E 00	-2.219E-09	3.434E 01	-6.739E-03
3	0	-2.806E-04	-2.806E-04	2.740E 00	-7.276E-10	3.646E 01	1.745E-02
4	0	-2.688E-04	-2.688E-04	-2.284E 00	-6.985E-10	3.492E 01	1.445E-02
5	0	-2.806E-04	-2.806E-04	2.727E 00	-1.295E-09	3.655E 01	-7.672E-02
6	0	-2.641E-04	-2.641E-04	1.655E 00	-1.397E-09	3.424E 01	9.601E-02
7	0	-2.306E-04	-2.306E-04	-4.214E 00	-1.310E-10	3.005E 01	-7.103E-02
8	0	-2.406E-04	-2.406E-04	0	0	1.562E 01	2.274E-02
9	0	-2.507E-04	-2.507E-04	0	0	0	0
-1	1	-3.068E-04	-3.068E-04	0	0	0	0
0	1	-2.306E-04	-2.306E-04	7.235E-06	-4.195E 00	2.998E 01	1.905E-03
1	1	-1.458E-04	-1.458E-04	2.580E 00	2.580E 00	3.790E 01	4.549E-03
2	1	-5.029E-05	-5.030E-05	1.692E 01	2.456E 01	1.308E 01	-2.777E-03
3	1	1.074E-04	1.074E-04	5.935E 00	6.263E 01	-2.789E 01	-2.968E-02
4	1	2.310E-04	2.311E-04	-2.841E 01	9.665E 01	-6.012E 01	5.312E-02
5	1	1.075E-04	1.074E-04	5.912E 00	6.268E 01	-2.793E 01	3.091E-03
6	1	-5.033E-05	-5.034E-05	1.693E 01	2.450E 01	1.320E 01	-1.133E-01
7	1	-1.459E-04	-1.457E-04	2.603E 00	2.594E 00	3.776E 01	1.512E-01
8	1	-2.306E-04	-2.307E-04	-3.446E-03	-4.193E 00	3.004E 01	-5.606E-02
9	1	-3.067E-04	-3.067E-04	0	0	0	0
-1	2	-4.813E-04	-4.813E-04	0	0	0	0
0	2	-2.641E-04	-2.641E-04	9.725E-04	1.640E 00	3.433E 01	5.795E-03
1	2	-5.030E-05	-5.031E-05	2.456E 01	1.693E 01	1.307E 01	5.382E-03
2	2	2.731E-04	2.731E-04	5.236E 01	5.238E 01	-7.096E 01	-4.441E-02
3	2	8.133E-04	8.134E-04	6.068E 00	1.195E 02	-2.118E 02	3.012E-01
4	2	1.261E-03	1.261E-03	-1.330E 02	2.008E 02	-3.269E 02	-9.502E-01
5	2	8.125E-04	8.137E-04	6.472E 00	1.193E 02	-2.129E 02	1.489E 00
6	2	2.739E-04	2.727E-04	5.191E 01	5.270E 01	-6.991E 01	-1.152E 00
7	2	-5.059E-05	-5.010E-05	2.478E 01	1.675E 01	1.277E 01	3.214E-01
8	2	-2.641E-04	-2.642E-04	8.784E-03	1.667E 00	3.434E 01	4.296E-03

9	2	-4.811E-04	-4.811E-04	0	0	0	0
-1	3	-6.742E-04	-6.742E-04	0	0	0	0
0	3	-2.806E-04	-2.806E-04	-3.976E-04	2.738E 00	3.649E 01	-1.310E-02
1	3	1.074E-04	1.074E-04	6.262E 01	5.947E 00	-2.792E 01	4.367E-03
2	3	8.133E-04	8.134E-04	1.196E 02	5.954E 00	-2.117E 02	2.715E-01
3	3	2.132E-03	2.132E-03	3.410E 00	3.926E 00	-5.529E 02	-1.456E 00
4	3	3.465E-03	3.467E-03	-5.024E 02	6.019E 01	-9.049E 02	3.721E 00
5	3	2.135E-03	2.131E-03	2.127E 00	4.579E 00	-5.493E 02	-5.178E 00
6	3	8.109E-04	8.148E-04	1.210E 02	5.136E 00	-2.150E 02	3.683E 00
7	3	1.080E-04	1.066E-04	6.203E 01	6.392E 00	-2.705E 01	-8.539E-01
8	3	-2.804E-04	-2.804E-04	-3.260E-02	2.683E 00	3.654E 01	-8.526E-02
9	3	-6.747E-04	-6.747E-04	0	0	0	0
-1	4	-7.639E-04	-7.639E-04	0	0	0	0
0	4	-2.688E-04	-2.688E-04	1.538E-04	-2.288E 00	3.491E 01	2.787E-02
1	4	2.311E-04	2.310E-04	9.667E 01	-2.844E 01	-6.006E 01	-1.778E-02
2	4	1.261E-03	1.261E-03	2.005E 02	-1.327E 02	-3.271E 02	-6.930E-01
3	4	3.466E-03	3.467E-03	6.145E 01	-5.037E 02	-9.045E 02	3.309E 00
4	4	6.510E-03	6.505E-03	-1.471E 03	-1.468E 03	8.315E 03	-7.357E 00
5	4	3.461E-03	3.469E-03	6.358E 01	-5.048E 02	-9.099E 02	8.974E 00
6	4	1.264E-03	1.258E-03	1.984E 02	-1.315E 02	-3.226E 02	-5.333E 00
7	4	2.308E-04	2.321E-04	9.726E 01	-2.903E 01	-6.052E 01	3.339E-01
8	4	-2.694E-04	-2.688E-04	7.027E-02	-2.271E 00	3.446E 01	5.236E-01
9	4	-7.641E-04	-7.641E-04	0	0	0	0
-1	5	-6.742E-04	-6.742E-04	0	0	0	0
0	5	-2.806E-04	-2.806E-04	-1.825E-03	2.740E 00	3.654E 01	-6.301E-02
1	5	1.073E-04	1.074E-04	6.266E 01	5.937E 00	-2.796E 01	5.036E-02
2	5	8.134E-04	8.135E-04	1.197E 02	5.924E 00	-2.125E 02	9.802E-01
3	5	2.133E-03	2.131E-03	2.697E 00	4.160E 00	-5.501E 02	-4.287E 00
4	5	3.463E-03	3.469E-03	-5.009E 02	5.960E 01	-9.096E 02	8.546E 00
5	5	2.138E-03	2.129E-03	6.181E-01	5.342E 00	-5.455E 02	-9.088E 00
6	5	8.101E-04	8.158E-04	1.215E 02	4.739E 00	-2.155E 02	4.100E 00
7	5	1.066E-04	1.068E-04	6.256E 01	6.363E 00	-2.885E 01	1.108E 00
8	5	-2.796E-04	-2.809E-04	-8.486E-02	2.813E 00	3.745E 01	-1.012E 00
9	5	-6.726E-04	-6.726E-04	0	0	0	0
-1	6	-4.814E-04	-4.814E-04	0	0	0	0
0	6	-2.641E-04	-2.641E-04	2.617E-03	1.636E 00	3.425E 01	8.581E-02
1	6	-5.019E-05	-5.035E-05	2.450E 01	1.694E 01	1.318E 01	-1.123E-01
2	6	2.730E-04	2.730E-04	5.224E 01	5.243E 01	-7.026E 01	-7.218E-01
3	6	8.124E-04	8.141E-04	6.857E 00	1.192E 02	-2.146E 02	3.179E 00
4	6	1.264E-03	1.259E-03	-1.346E 02	2.015E 02	-3.223E 02	-5.674E 00
5	6	8.099E-04	8.155E-04	7.919E 00	1.184E 02	-2.162E 02	4.905E 00
6	6	2.743E-04	2.720E-04	5.161E 01	5.304E 01	-6.997E 01	-1.050E 00
7	6	-4.905E-05	-5.045E-05	2.416E 01	1.689E 01	1.467E 01	-1.734E 00
8	6	-2.648E-04	-2.636E-04	4.675E-02	1.514E 00	3.346E 01	8.902E-01
9	6	-4.832E-04	-4.832E-04	0	0	0	0
-1	7	-3.067E-04	-3.067E-04	0	0	0	0
0	7	-2.306E-04	-2.306E-04	-1.171E-03	-4.191E 00	3.005E 01	-6.400E-02
1	7	-1.459E-04	-1.458E-04	2.625E 00	2.565E 00	3.778E 01	1.390E-01
2	7	-5.023E-05	-5.026E-05	1.697E 01	2.453E 01	1.289E 01	1.722E-01
3	7	1.079E-04	1.070E-04	5.578E 00	6.286E 01	-2.689E 01	-1.040E 00
4	7	2.298E-04	2.318E-04	-2.778E 01	9.617E 01	-6.148E 01	1.473E 00
5	7	1.083E-04	1.069E-04	5.509E 00	6.310E 01	-2.749E 01	-4.893E-01
6	7	-4.998E-05	-5.057E-05	1.683E 01	2.449E 01	1.386E 01	-7.844E-01
7	7	-1.465E-04	-1.453E-04	2.914E 00	2.370E 00	3.702E 01	9.054E-01
8	7	-2.304E-04	-2.308E-04	8.369E-03	-4.108E 00	3.027E 01	-2.882E-01

9	7	-3.057E-04	-3.057E-04	0	0	0	0
-1	8	-2.506E-04	-2.506E-04	0	0	0	0
0	8	-2.406E-04	-2.406E-04	7.149E-11	3.575E-10	1.562E 01	1.737E-02
1	8	-2.306E-04	-2.306E-04	-4.206E 00	1.455E-11	3.003E 01	-5.303E-02
2	8	-2.641E-04	-2.641E-04	1.636E 00	4.293E-10	3.430E 01	2.947E-02
3	8	-2.807E-04	-2.805E-04	2.780E 00	1.135E-09	3.646E 01	2.511E-02
4	8	-2.685E-04	-2.689E-04	-2.337E 00	2.874E-09	3.485E 01	8.014E-02
5	8	-2.806E-04	-2.807E-04	2.708E 00	7.130E-10	3.684E 01	-3.586E-01
6	8	-2.645E-04	-2.639E-04	1.742E 00	-7.130E-10	3.395E 01	3.983E-01
7	8	-2.303E-04	-2.308E-04	-4.267E 00	7.713E-10	3.009E 01	-1.098E-01
8	8	-2.406E-04	-2.406E-04	-7.149E-11	-3.575E-10	1.563E 01	4.378E-03
9	8	-2.507E-04	-2.507E-04	0	0	0	0
-1	9	-1.945E-04	-1.945E-04	0	0	0	0
0	9	-2.505E-04	-2.505E-04	0	0	0	0
1	9	-3.069E-04	-3.068E-04	0	0	0	0
2	9	-4.811E-04	-4.813E-04	0	0	0	0
3	9	-6.739E-04	-6.738E-04	0	0	0	0
4	9	-7.637E-04	-7.647E-04	0	0	0	C
5	9	-6.763E-04	-6.740E-04	0	0	0	0
6	9	-4.791E-04	-4.807E-04	0	0	0	0
7	9	-3.069E-04	-3.075E-04	0	0	0	0
8	9	-2.515E-04	-2.503E-04	0	0	0	0
9	9	-1.957E-04	-1.957E-04	0	0	0	0

PROB (CONTD)
201 8X8 SOF , PR=.20 CENTER LOAD, HWY REPORT-6 WRH 29AP5

TABLE 5. RESULTS(CONTD) -- ITERATION 12

I,J		TMX	TMY
-1	-1	0	0
0	-1	0	0
1	-1	0	0
2	-1	0	0
3	-1	0	0
4	-1	0	0
5	-1	0	0
6	-1	0	0
7	-1	0	0
8	-1	0	C
9	-1	0	0
-1	0	0	0
0	0	2.742E 00	-2.742E 00
1	0	8.245E 00	-8.247E 00
2	0	1.245E 01	-1.245E 01
3	0	1.132E 01	-1.131E 01
4	0	8.279E-04	1.639E-03
5	0	-1.132E 01	1.132E 01
6	0	-1.245E 01	1.245E 01
7	0	-8.241E 00	8.247E 00
8	C	-2.742E 00	2.740E 00
9	C	0	0

1	1	0	0
0	1	8.246E 00	-8.246E 00
1	1	2.250E 01	-2.250E 01
2	1	3.666E 01	-3.666E 01
3	1	3.983E 01	-3.981E 01
4	1	-3.227E-02	-1.063E-02
5	1	-3.980E 01	3.982E 01
6	1	-3.664E 01	3.666E 01
7	1	-2.253E 01	2.248E 01
8	1	-8.233E 00	8.244E 00
9	1	0	0
1	2	0	0
0	2	1.245E 01	-1.245E 01
1	2	3.666E 01	-3.666E 01
2	2	7.109E 01	-7.107E 01
3	2	9.511E 01	-9.519E 01
4	2	9.510E-02	4.493E-02
5	2	-9.520E 01	9.513E 01
6	2	-7.116E 01	7.106E 01
7	2	-3.655E 01	3.671E 01
8	2	-1.248E 01	1.244E 01
9	2	0	0
-1	3	0	0
0	3	1.131E 01	-1.131E 01
1	3	3.982E 01	-3.981E 01
2	3	9.513E 01	-9.518E 01
3	3	1.710E 02	-1.708E 02
4	3	-1.386E-01	-7.556E-02
5	3	-1.709E 02	1.709E 02
6	3	-9.499E 01	9.522E 01
7	3	-3.996E 01	3.973E 01
8	3	-1.132E 01	1.134E 01
9	3	0	0
1	4	0	0
0	4	-2.189E-03	2.596E-04
1	4	2.239E-03	-3.631E-03
2	4	3.485E-02	2.586E-02
3	4	-9.773E-02	-5.706E-02
4	4	7.729E-02	5.058E-02
5	4	6.506E-02	2.284E-02
6	4	-1.648E-01	-8.230E-02
7	4	6.059E-02	5.622E-02
8	4	6.911E-02	-3.794E-02
9	4	0	0
-1	5	0	0
0	5	-1.131E 01	1.131E 01
1	5	-3.982E 01	3.982E 01
2	5	-9.517E 01	9.515E 01
3	5	-1.709E 02	1.709E 02
4	5	6.813E-02	3.388E-02
5	5	1.708E 02	-1.709E 02
6	5	9.516E 01	-9.514E 01
7	5	3.991E 01	-3.978E 01
8	5	1.125E 01	-1.131E 01
9	5	0	0

-1	6	0	0
0	6	-1.245E 01	1.245E 01
1	6	-3.666E 01	3.666E 01
2	6	-7.110E 01	7.106E 01
3	6	-9.506E 01	9.522E 01
4	6	-1.582E-01	-9.235E-02
5	6	9.520E 01	-9.511E 01
6	6	7.126E 01	-7.101E 01
7	6	3.649E 01	-3.677E 01
8	6	1.244E 01	-1.242E 01
9	6	0	0
-1	7	0	0
0	7	-8.249E 00	8.247E 00
1	7	-2.250E 01	2.249E 01
2	7	-3.662E 01	3.669E 01
3	7	-3.993E 01	3.976E 01
4	7	1.080E-01	6.090E-02
5	7	3.987E 01	-3.980E 01
6	7	3.647E 01	-3.675E 01
7	7	2.259E 01	-2.243E 01
8	7	8.305E 00	-8.283E 00
9	7	0	0
-1	8	0	0
0	8	-2.740E 00	2.740E 00
1	8	-8.245E 00	8.249E 00
2	8	-1.246E 01	1.243E 01
3	8	-1.129E 01	1.134E 01
4	8	-5.604E-02	1.309E-03
5	8	1.132E 01	-1.136E 01
6	8	1.252E 01	-1.241E 01
7	8	8.187E 00	-8.239E 00
8	8	2.712E 00	-2.731E 00
9	8	0	0
-1	9	0	0
0	9	0	0
1	9	0	0
2	9	0	0
3	9	0	0
4	9	0	0
5	9	0	0
6	9	0	0
7	9	0	0
8	9	0	0
9	9	0	0

TIME = 1 MINUTES, 40 AND 43/60 SECONDS