Numerical Determination of Stresses in Earth Masses

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General numerical techniques are developed for the purpose of analyzing stress distributions in commonly encountered foundation conditions. These conditions include the effect of rigid boundaries, mixed boundary conditions, nonhomogeneous materials, and time-dependent material characteristics. The numerical development is oriented toward computer utilization. Typical results for the various problem categories are included.

•THE IMPACT of the computer revolution on contemporary society is self-evident to most observers. In the area of research, new disciplines have emerged in the computer environment which already offer great promise. The more traditional disciplines have been able to develop refined techniques of solution for classical problems. The latter topic is pursued in this presentation.

The calculation of stresses in earth masses is beset with numerous difficulties. It would do well to elucidate these difficulties to appraise the potential contribution of numerical methods in conjunction with the computer.

Of fundamental importance to any stress analysis problem is the description of the mechanical behavior of the material involved, together with a complete description of the shape of, and conditions at, the boundaries, and any spatial or time variations in material characteristics which may occur in the regions under consideration.

With respect to an adequate description of the mechanical behavior of soil, the computer has little to offer except in a supplementary role. Only patient work in the soil mechanics laboratory will eventually unwind the mysterious nonlinearities of soil. Until this is accomplished, the traditional assumptions concerning elastic, viscoelastic or plastic behavior will have to be judiciously utilized.

The incorporation of environmental realities into the stress analysis of a particular earth mass problem can, however, be greatly enhanced by the use of the computer. Nonhomogeneities, so prevalent in nature, and irregularly shaped boundaries subjected to mixed stress and displacement conditions are typical of the types of problems which can now be analyzed with relative ease. The relative importance of these effects is discussed before the suggested techniques of solution.

A final comment for the benefit of the reader is that this paper is presented in the spirit of "how to" rather than the "results of."

ENVIRONMENTAL BOUNDARY EFFECTS

Rigid Base Beneath a Layer of Soil

Figure 1 (Terzaghi, 1) demonstrates the effect on the vertical stress distribution of a rigid base beneath a layer of elastic soil subjected to a surface load. The significant

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feature is that the departure from the semi-infinite case is a function of the depth or position of the rigid base with respect to the applied load.

Rigid Vertical Boundary

Figure 2 shows the difference between measured values of horizontal stress along a rigid vertical boundary and the calculated values, assuming an elastic half space as given by Spangler (2). The departure is again a function of the relative position of the applied load with respect to the vertical constraint.

Friction Along a Rigid Boundary

The increase in vertical stress along a rigid horizontal boundary due to lateral restraint provided by friction along that boundary is shown in Figure 3 (1). Thus, resist-

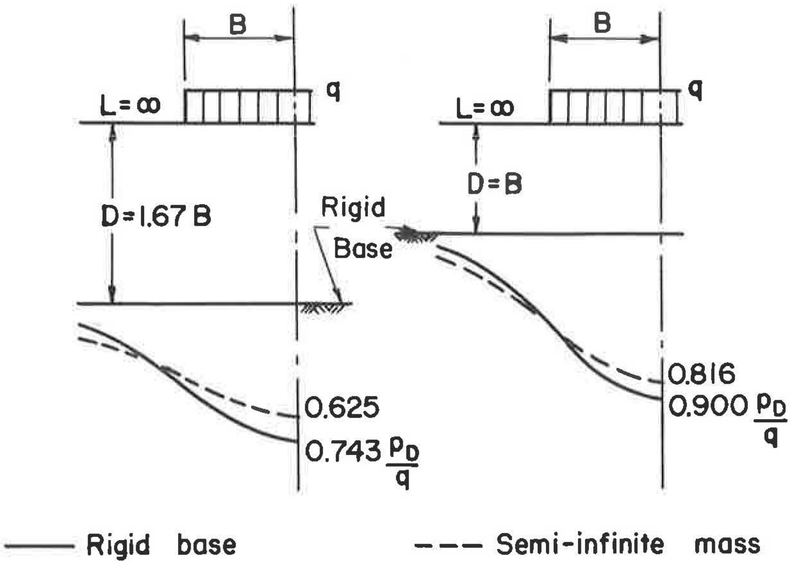


Figure 1. Pressure distributions at indicated depths.

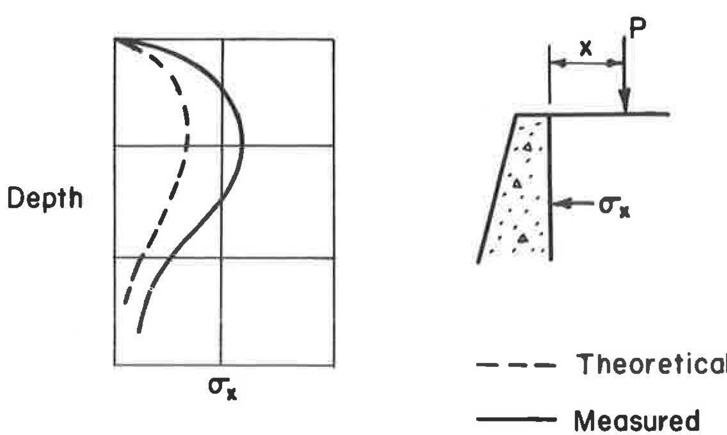


Figure 2. Horizontal pressures on retaining walls due to concentrated surface loads.

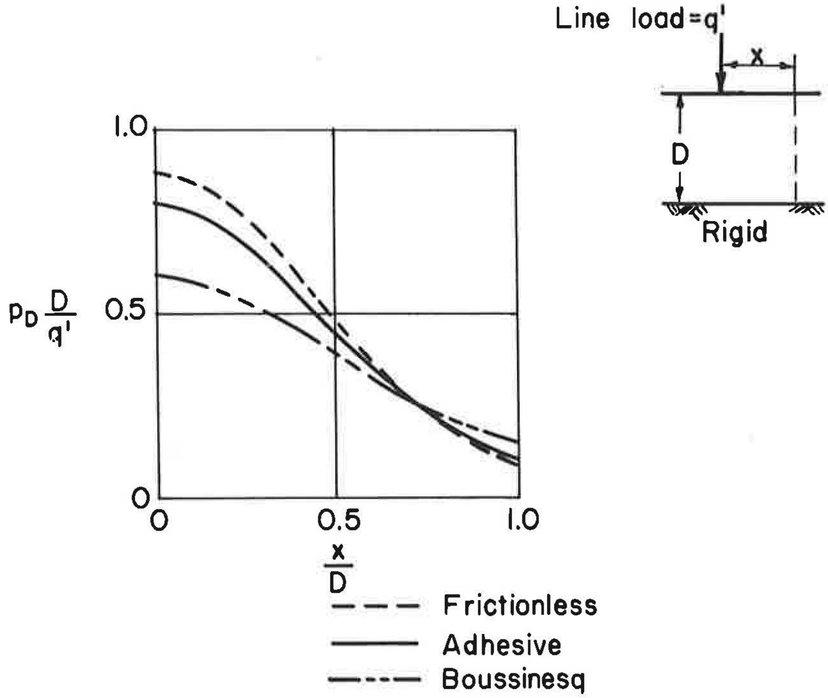


Figure 3. Distribution of normal pressure on rigid base of elastic layer acted on by line load.

ance to displacement in a direction parallel as well as perpendicular to a rigid boundary may have a significant effect on stress distribution.

VARIATIONS IN MATERIAL PROPERTIES

Spatial Variation in Elastic Modulus

The effect of spatial variations in mechanical properties as typified by an elastic modulus which is a function of position, E(r, z), is shown in Figure 4 (3). The stress distributions are forced in the direction of the stiffer material, resulting in a considerable modification of the Boussinesq solution.

Anisotropy

Directional variations in elastic properties as well as spatial variations may alter stress distributions, as shown in Figure 5 (1). The extent of the modification is, of course, dependent on the magnitude of the ratio of the moduli in the principal directions.

Time-Dependent Material Characteristics

Rather radical redistributions of stress as a function of time may occur in time-dependent or viscoelastic materials. Figure 6 (4) shows such a redistribution for a Maxwell type material.

METHODOLOGY

Many numerical techniques exist for obtaining approximate solutions to differential systems. Probably the most widely accepted approach involves the use of finite differences. Basically, a finite difference approximation simply does not allow a derivative to be taken to the limit. Thus a difference is "discrete" and therefore compatible with the digital computer.

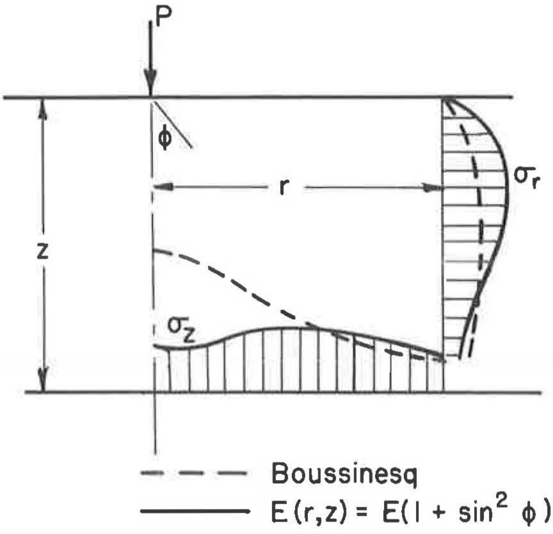


Figure 4. Stress distributions with and without spatial variations in elastic modulus (E).

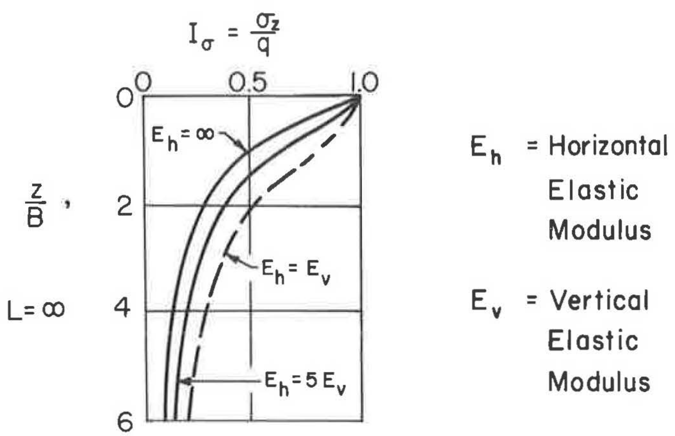


Figure 5. Relation between depth and vertical unit pressure beneath center line of flexible strip load with $E_h \geq E_v$.

The common finite difference approximations (Table 1) have been formulated according to the grid notation shown in Figure 7. The value of w_n is the magnitude of the function at the n^{th} grid point.

Utilizing these algebraic approximations to derivatives, it is possible to replace a differential field equation describing the stress distribution in a region by a set of algebraic equations. The solution of this set of equations yields numerical values for the dependent variable at the nodes (intersection of the mesh lines).

The choice of the dependent variable (stress or displacement) cannot be divorced from the circumstances surrounding a particular problem. For example, if all the boundary conditions are of the stress variety, then stress would be a probable solution variable. If the boundary conditions are mixed, displacement may be a more convenient choice. This situation is analogous to the force and displacement alternatives which occur in the matrix analysis of structures.

Each approach is presented along with some applications.

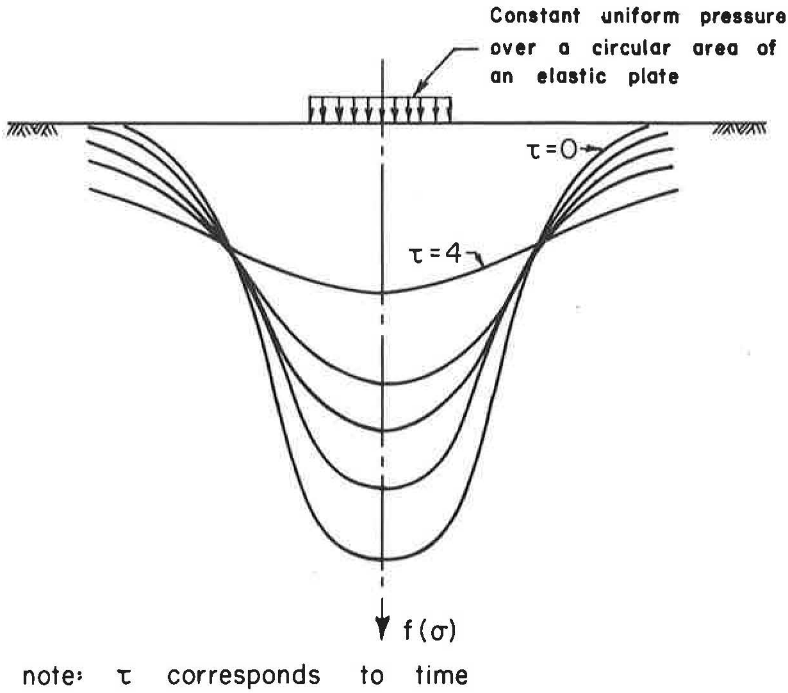


Figure 6. Reactive pressure distribution for plate on Maxwell foundation at various times τ.

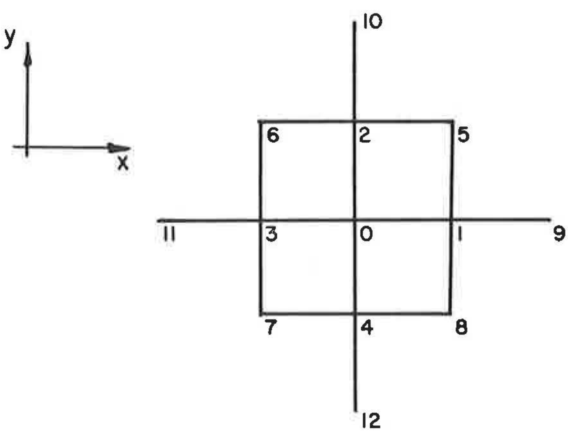


Figure 7. Finite difference grid notation.

Force Method

The governing field equation describing the stress distribution in a linearly elastic body is the biharmonic

$$\nabla^4 \phi = 0 \tag{1}$$

TABLE 1

$$\left[\frac{\partial w}{\partial x}\right]_0 = \frac{w_1 - w_3}{2h}$$

$$\left[\frac{\partial w}{\partial y}\right]_0 = \frac{w_2 - w_4}{2h}$$

$$\left[\frac{\partial^2 w}{\partial x^2}\right]_0 = \frac{w_1 + w_3 - 2w_0}{h^2}$$

$$\left[\frac{\partial^2 w}{\partial y^2}\right]_0 = \frac{w_2 + w_4 - 2w_0}{h^2}$$

$$\left[\frac{\partial^2 w}{\partial x\,\partial y}\right]_0 = \frac{w_5 - w_6 + w_7 - w_8}{4h^2}$$

$$\left[\frac{\partial^3 w}{\partial x^3}\right]_0 = \frac{w_9 - 2w_1 + 2w_3 - w_{11}}{2h^3}$$

$$\left[\frac{\partial^3 w}{\partial y^3}\right]_0 = \frac{w_{10} - 2w_2 + 2w_4 - w_{12}}{2h^3}$$

$$\left[\frac{\partial^4 w}{\partial x^4}\right]_0 = \frac{6w_0 - 4w_1 - 4w_3 + w_9 + w_{11}}{h^4}$$

$$\left[\frac{\partial^4 w}{\partial y^4}\right]_0 = \frac{6w_0 - 4w_2 - 4w_4 + w_{10} + w_{12}}{h^4}$$

$$\left[\frac{\partial^4 w}{\partial x^2\,\partial y^2}\right]_0 = \frac{4w_0 - 2\,\Sigma\, w_1 + \Sigma\, w_5}{h^4}$$

where ϕ is the Airy stress function describing all stresses in either a plane strain or plane stress situation. The extension of the biharmonic to include spatial variations in material properties is (5):

$$\frac{\partial^2}{\partial x^2}\left[A\frac{\partial^2\phi}{\partial x^2}\right] + \frac{\partial^2}{\partial y^2}\left[A\frac{\partial^2\phi}{\partial y^2}\right] + \frac{\partial^2}{\partial x\,\partial y}\left[2A(1+B)\frac{\partial^2\phi}{\partial x\,\partial y}\right]$$

$$-\frac{\partial^2}{\partial x^2}\left[AB\frac{\partial^2\phi}{\partial y^2}\right] - \frac{\partial^2}{\partial y^2}\left[AB\frac{\partial^2\phi}{\partial x^2}\right] \tag{2}$$

$$-\nabla^2\left[A(1-B)V\right] + \nabla^2(C\alpha T) = 0$$

where: $A = \dfrac{1-\nu^2}{E}$

$B = \dfrac{\nu}{1-\nu}$

$C = 1+\nu$

V = potential of body forces

αT = thermal expansion term

For the elastic case with $\nu = \frac{1}{2}$, B = 1, and neglecting the thermal expansion term, Eq. 2 reduces to the relationship indicated in Eq. 3.

$$A(\nabla^4\phi) + 2\frac{\partial A}{\partial x}\left[\frac{\partial^3\phi}{\partial x^3} + \frac{\partial^3\phi}{\partial x\,\partial y^2}\right] + 2\frac{\partial A}{\partial y}\left[\frac{\partial^3\phi}{\partial y^3} + \frac{\partial^3\phi}{\partial x^2\partial y}\right]$$

$$+\left[\frac{\partial^2 A}{\partial y^2} - \frac{\partial^2 A}{\partial x^2}\right]\left[\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial x^2}\right] + 4\left[\frac{\partial^2 A}{\partial x\partial y}\cdot\frac{\partial^2\phi}{\partial x\partial y}\right] = 0 \tag{3}$$

The value of the modulus at the point as well as its first, second, and cross derivatives appear as coefficients of the unknown stress function. Thus, these quantities must be initially calculated numerically from the known variations in modulus. They can then be substituted as numerical coefficients into the following finite difference approximation to Eq. 3.

$$\phi_0(20A) + \phi_1(-8A - 4A_x - A_{yy} + A_{xx}) +$$

$$\phi_2(-8A - 4A_y + A_{yy} - A_{xx}) + \phi_3(-8A + 4A_x - A_{yy} + A_{xx}) +$$

$$\phi_4(-8A + 4A_y + A_{yy} - A_{xx}) + \phi_5(2A + A_x + A_y + A_{xy}) +$$

$$\phi_6(2A - A_x + A_y - A_{xy}) + \phi_7(2A - A_x - A_y + A_{xy}) +$$

$$\phi_8(2A + A_x - A_y - A_{xy}) + \phi_9(A + A_x) + \phi_{10}(A + A_y) +$$

$$\phi_{11}(A - A_x) + \phi_{12}(A - A_y) = 0$$

where: $A_x = \dfrac{\partial A}{\partial x} \cdot h$; $A_y = \dfrac{\partial A}{\partial y} \cdot h$; $A_{xx} = \dfrac{\partial^2 A}{\partial x^2} \cdot h^2$;

$$A_{yy} = \frac{\partial^2 A}{\partial y^2} \cdot h^2 \quad \text{and} \quad A_{xy} = \frac{\partial^2 A}{\partial x \, \partial y} \cdot h^2$$

This difference approximation is formulated at every interior grid point of the region. The boundary conditions are introduced into those equations which contain points on or outside of the boundary (6). A typical statement of boundary conditions is shown in Figure 8. The homogeneous solution refers to the known stress function solution for a homogeneous half space.

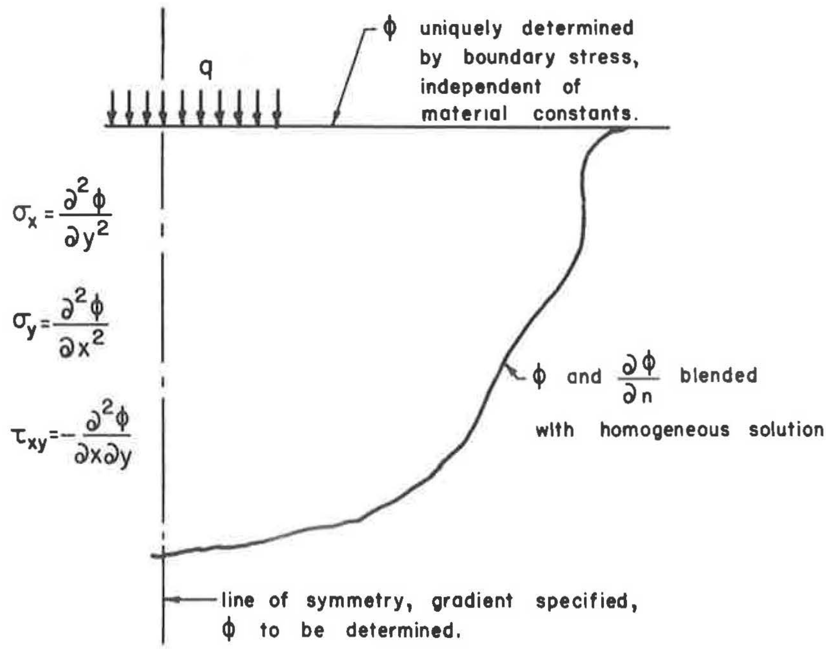
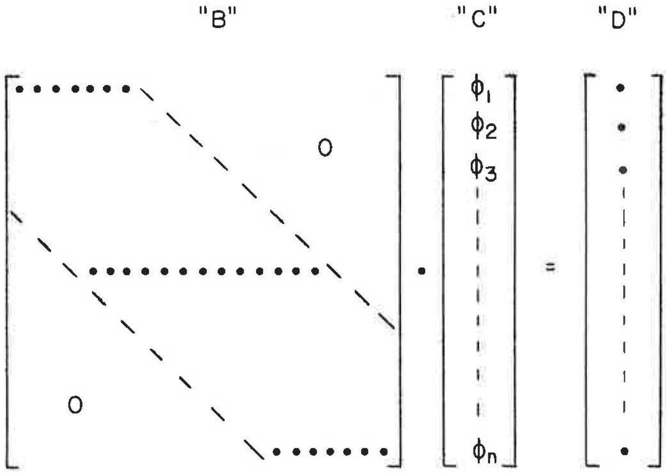


Figure 8. Statement at boundary conditions.

The resulting set of algebraic equations can be stated in the following matrix form.



The B matrix is a diagonal matrix whose coefficients are determined by the spatial variations in material properties; C is the unknown stress function column vector, and D is a known column vector dictated by the values of stress at the boundaries. Performing a matrix inversion yields values for the stress function and thus stresses at the interior grid points.

The results of a rather unusual application of this technique are shown in Figure 9.

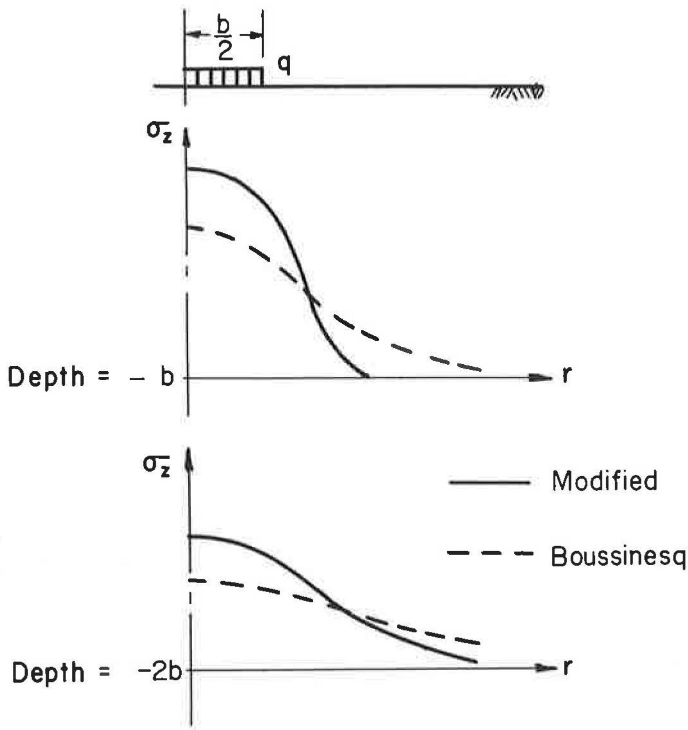


Figure 9. Comparison of homogeneous and nonhomogeneous cases.

In this case the modulus was assumed to vary as the first stress invarient $\left[\frac{\sigma_x + \sigma_y + \sigma_z}{3}\right]$ thus imposing a nonlinearity which required iteration of the previous technique.

For the particular case of a discontinuity in modulus, such as a layered system, the biharmonic would apply in each of the homogeneous layers; however, compatibility of stress and strain would have to be insured (5) according to the following relations.

$$\left.\begin{aligned} \phi &= \Phi \\ \frac{\partial \phi}{\partial y} &= \frac{\partial \Phi}{\partial y} \end{aligned}\right\} \text{stress requirements}$$

$$\left.\begin{aligned} A_1\left[\frac{\partial^2 \phi}{\partial y^2} - B_1 \frac{\partial^2 \phi}{\partial x^2}\right] &= A_2\left[\frac{\partial^2 \Phi}{\partial y^2} - B_2 \frac{\partial^2 \Phi}{\partial x^2}\right] \\ -A_1\left[\frac{\partial^3 \phi}{\partial y^3} + (2 + B_1)\frac{\partial^3 \phi}{\partial x^2 \partial y}\right] &= -A_2\left[\frac{\partial^3 \Phi}{\partial y^3} + (2 + B_2)\frac{\partial^3 \Phi}{\partial x^2 \partial y}\right] \end{aligned}\right\} \text{strain requirements}$$

$$\nabla^4 \phi = 0$$

$$\nabla^4 \Phi = 0$$

where: A and B are material constants

These equations expressed in finite difference form allow the elimination of points outside each of the respective regions as indicated in Figure 10 whereupon ϕ and Φ can be determined as usual.

Displacement Method

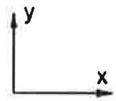
In contrast to the single fourth order field equation in the force method, the displacement method requires the simultaneous satisfaction of two second order differential equations (6) which are

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 v}{\partial x \partial y} = 0$$

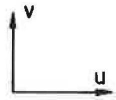
(4)

$$\mu\frac{\partial^2 v}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u}{\partial x \partial y} = 0$$

where: $\lambda = \dfrac{E\nu}{(1 + \nu)(1 - 2\nu)}$



$\mu = \dfrac{E}{2(1 + \nu)}$



Discretizing (finite difference conversion) these equations causes them to assume the following form:

$$\frac{(\lambda + 2\mu)}{h^2}(u_3 - 2u_0 + u_1) + \frac{\mu}{h^2}(u_2 - 2u_0 + u_4)$$

(5)

$$+ \frac{(\lambda + \mu)}{4h^2}(v_5 - v_6 + v_7 - v_8) = 0$$

$$\frac{\mu}{h^2}(v_3 - 2v_0 + v_1) + \frac{(\lambda + 2\mu)}{h^2}(v_2 - 2v_0 + v_4)$$

(6)

$$+ \frac{(\lambda + \mu)}{4h^2}(u_5 - u_6 + u_7 - u_8) = 0$$

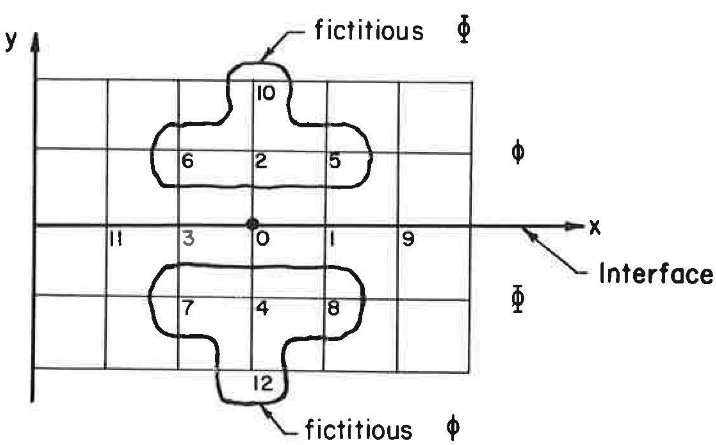
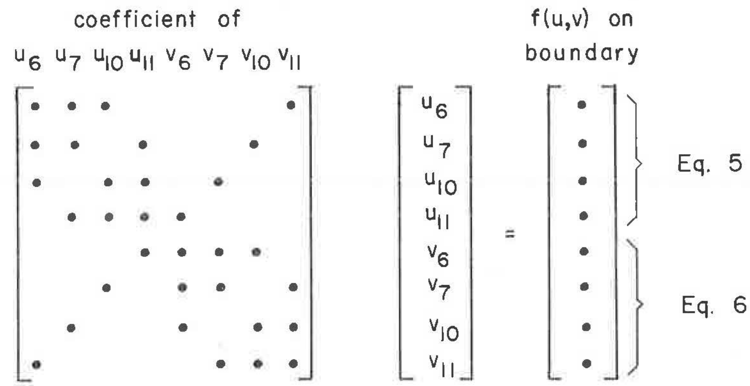


Figure 10. Finite difference notation at interface.

The application of these difference equations is probably best explained with a specific example. Consider the region EFGH in Figure 11 and assume the displacements u and v are completely specified on the boundary. Thus eight unknowns must be determined, the horizontal and vertical displacements at nodes 6, 7, 10 and 11. Application of Eqs. 5 and 6 to these nodes yields the following set of equations in matrix notation.



Again, matrix inversion gives the solution for the unknown displacements.

When stress conditions as well as displacements are imposed on the boundary, the use of the displacement approach requires the conversion of the stress condition to a displacement condition via the stress strain relation for the material. For an elastic material, the following equations provide the conversion.

$$\sigma_x = \lambda\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_y = \lambda\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{xy} = \mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$$

The displacements at a stressed boundary as well as interior displacements must be determined. Application of the field equations at the boundary involves mesh points outside the region as shown in Figure 12. However, these fictitious points are eliminated using the known values of stress at the boundary and the stress strain relation. The results for a typical mixed boundary condition problem of this type are given in Figure 13.

This problem can be considered as a fill in a rigid (rock) valley subjected to a uniform load. The increase in vertical stress near the rigid boundary at the lower section is quite evident.

Viscoelasticity and the Displacement Method

The incorporation of a viscoelastic assumption in a two-dimensional stress analysis increases the number of independent variables from two to three. Thus a solution must be obtained at a given time, t, as a function of the spatial coordinates x and y and then "marched" to a new t as dictated by the governing equations.

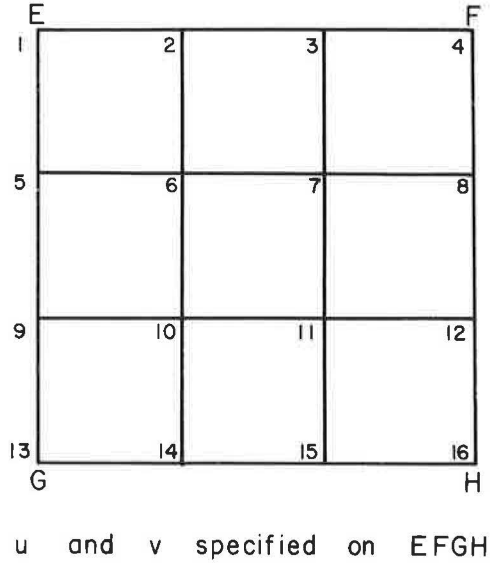


Figure 11. Region considered in displacement method example.

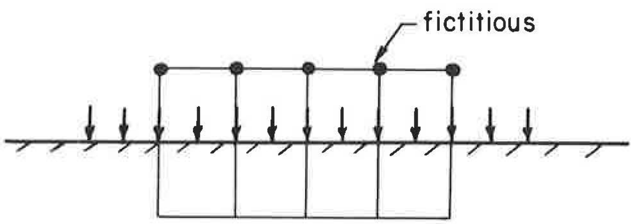


Figure 12. Location of fictitious grid points at boundary.

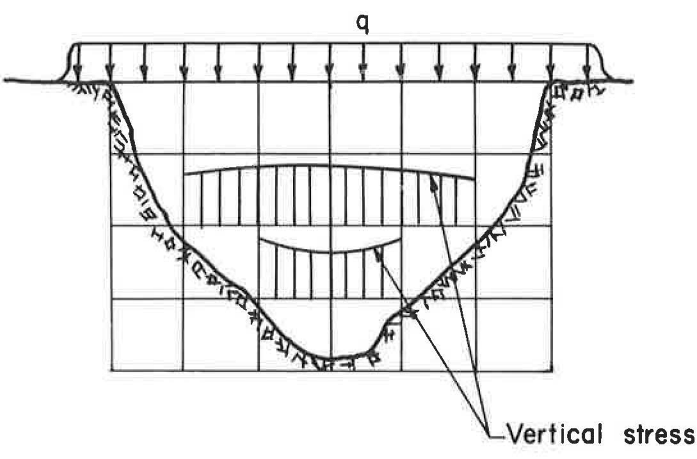


Figure 13. Plot of vertical stress distribution in irregular cross-section.

Again, a particular case may best serve to demonstrate the technique. The displacement field equations for a Voigt solid (7) are

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + (\lambda' + 2\mu')\frac{\partial}{\partial t}\left[\frac{\partial^2 u}{\partial x^2}\right] + \mu\frac{\partial^2 u}{\partial y^2} + \mu'\frac{\partial}{\partial t}\left[\frac{\partial^2 u}{\partial y^2}\right] + (\lambda + \mu)\frac{\partial^2 v}{\partial x \partial y} + (\lambda' + \mu')\frac{\partial}{\partial t}\left[\frac{\partial^2 v}{\partial x \partial y}\right] = 0 \tag{7}$$

$$\mu\frac{\partial^2 v}{\partial x^2} + \mu'\frac{\partial}{\partial t}\left[\frac{\partial^2 v}{\partial x^2}\right] + (\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + (\lambda' + 2\mu')\frac{\partial}{\partial t}\left[\frac{\partial^2 v}{\partial y^2}\right] + (\lambda + \mu)\frac{\partial^2 u}{\partial x \partial y} + (\lambda' + \mu')\frac{\partial}{\partial t}\left[\frac{\partial^2 u}{\partial x \partial y}\right] = 0 \tag{8}$$

where λ' and μ' are the viscous components of the material behavior.

With the grid notation shown in Figure 14 the algebraic expression of these equations takes the indicated form where the Crank-Nicolson technique of averaging the space derivatives at either end of the time increment is employed for reasons of mathematical stability.

$$A_1(u_3 - 2u_0 + u_1)_{t_0+\Delta t} + B_1(u_2 - 2u_0 + u_4)_{t_0+\Delta t} + A_2(v_5 - v_6 + v_7 - v_8)_{t_0+\Delta t} = A_3(u_3 - 2u_0 + u_1)_{t_0} + B_2(u_2 - 2u_0 + u_4)_{t_0} + A_4(v_5 - v_6 + v_7 - v_8)_{t_0}$$

where:

$$A_1 = (\lambda\Delta t + 2\mu\Delta t + 2\lambda' + 4\mu')$$

$$A_3 = (-\lambda\Delta t - 2\mu\Delta t + 2\lambda' + 4\mu')$$

$$A_2 = (1/4)(\lambda\Delta t + \mu\Delta t + 2\lambda' + 2\mu')$$

$$A_4 = (1/4)(-\lambda\Delta t - \mu\Delta t + 2\lambda' + 2\mu')$$

and $B_1 = \mu\Delta t + 2\mu'$ $\quad B_2 = -\mu\Delta t + 2\mu'$

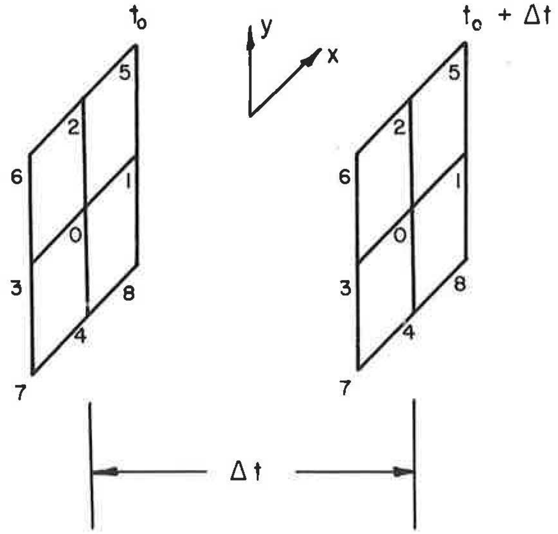


Figure 14. Grid notation for spatial and time coordinates.

The viscoelastic solution is similar to a repeated elastic solution in that a set of simultaneous equations must be solved at the end of each finite time increment.

A typical set of results using this approach is shown in Figure 15, which essentially describes the time-dependent displacement of the surface of a Voigt foundation material subjected to a uniform line load.

CRITICAL EQUILIBRIUM OF EARTH MASSES

The previous discussion has been confined to continuous behavior assuming small strains satisfying equilibrium and compatibility requirements.

In the case of discontinuous behavior which occurs when failure or slip planes develop in an earth mass, equilibrium must still be satisfied at impending failure; however, the yield or failure criterion for the material replaces the compatibility require-

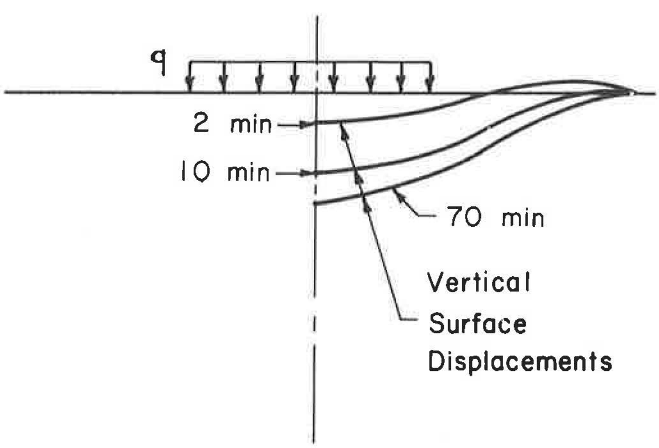


Figure 15. Typical viscoelastic surface displacements.

ment. The failure criterion for a material with cohesion and internal friction as given by Scott (8) is

$$\left.\begin{matrix}\sigma_x \\ \sigma_z\end{matrix}\right\} = \sigma\,(1 \mp \sin\phi \cos 2\theta) - H$$

$$\tau_{xz} = \sigma \sin\phi \sin 2\theta$$

where: ϕ = angle of internal friction,

θ = defined in Figure 16,

$H = c \cdot \cot\phi$, and

$\sigma = 1/2\,(\sigma_x + \sigma_z + 2H) = 1/2(\sigma_1 + \sigma_3 + 2H)$

with the various angles shown in Figure 16. The failure criterion can be combined with the equations of equilibrium,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} = \gamma$$

to form a set of nonlinear equations which by the following transformations,

$$\chi = \frac{\cot\phi}{2} \log_e \frac{\sigma}{\sigma_o}$$

$$\xi = \chi + \theta$$

$$\eta = \chi - \theta$$

yield the ordinary differential equations,

$$\frac{d\xi}{dz} = -\frac{\gamma \sin(\theta - \alpha)}{2\sigma \sin\phi \cos(\theta + \alpha)}$$

$$\frac{d\eta}{dz} = \frac{\gamma \sin(\theta + \alpha)}{2\sigma \sin\phi \cos(\theta - \alpha)} \qquad (9)$$

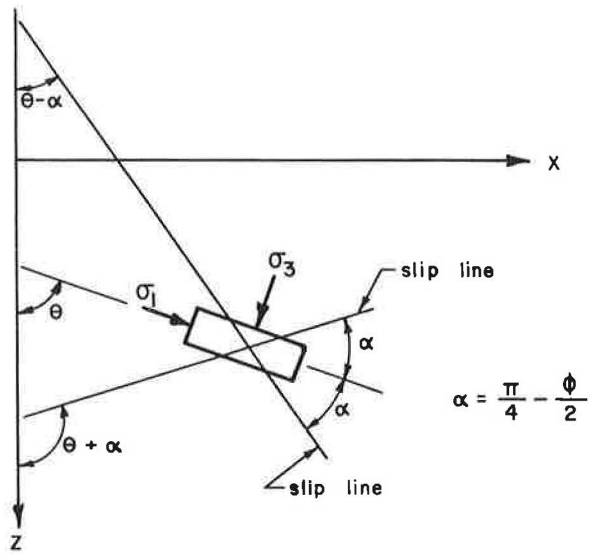


Figure 16. Slip line formation in real space.

as valid relationships along the "slip" lines. Thus starting at a boundary where ξ and η are defined, the family of slip lines within the earth mass can be developed by integrating Eq. 9.

Again utilizing difference techniques the expression for ξ typically becomes

$$\xi_C = \xi_A - \frac{(z_C - z_A)\,\gamma\,\sin(\theta - \alpha)}{2\sigma_A\,\sin\phi\,\cos(\theta + \alpha)}$$

which, in conjunction with the assumption of a straight-line approximation to short segments of a slip line, allows the discrete development of the slip line grid. The repetitive nature of this process is particularly well adapted to the computer.

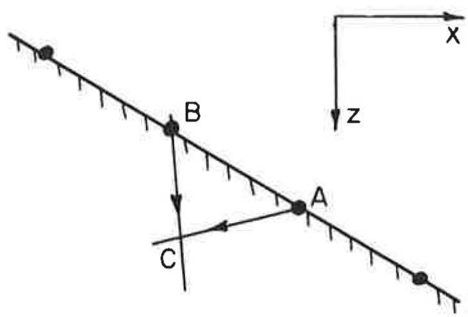


Figure 17. Reference diagram for beginning solution at a boundary.

Sokolovski (9) has improved the numerical process by developing the following expressions for the coordinates of the intersection of slip lines (Fig. 17).

$$x_C = x_B + (z_C - z_B)\tan(\theta_B - \alpha)$$

$$z_C = \frac{z_B \tan(\theta_B - \alpha) - x_B - z_A \tan(\theta_A + \alpha) + x_A}{\tan(\theta_B - \alpha) - \tan(\theta_A + \alpha)}$$

CONCLUSIONS

The purpose of this somewhat cursory excursion into the realm of numerical techniques as related to foundation problems has been to stimulate awareness of the current availability of tools that may allow the job to be done better.

Independent of this presentation, the future will record that its purpose has been realized.

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