An Analytical-Experimental Method for Determining Interface Tractions for Buried Structures Subjected to Static Loads

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The outer boundary tractions of normal pressure and shear may be determined for a rigid circular culvert by studying the displacement field of the inner boundary. Given the inner boundary displacements, the elastic response of the structure will yield, on analysis, that unique loading responsible for the displacements. This paper develops the equations of the elastic analysis of the culvert and plots charts for ease in application of the parameters.

•THE DESIGN engineer responsible for the structural design of culverts is faced with the unenviable task of having to design the structure without any clear knowledge of the loadings on it. This deficiency inhibits his direct determination, in a design sense, of the state of stress throughout the structure.

This paper shows that the displacement response of the inner boundary of a buried rigid circular culvert furnishes sufficient information to determine that unique soil loading causing this response. Knowledge of this soil loading (surface tractions on the buried structure) would permit a quantitative estimate of the phenomenon of soil arching. Knowledge of these soil-structure interface tractions would also permit a critical evaluation of the effectiveness of some current design assumptions of loads, bedding conditions, and backfill techniques. This paper is only concerned with the demonstration of the fact that inner boundary displacements are sufficient to determine the soil loadings. In short, the buried structure is used as a transducer.

ASSUMPTIONS

1. Symmetry of loading about a vertical axis exists and variations from this assumption are small, normally distributed (in a statistical sense), and self-compensating.
2. The culvert responds in a linear elastic manner. Any variations from this idealization are assumed to be small enough to be neglected at the usual levels of working stress for concrete.
3. No assumption is made concerning the response of the soil to self-induced loading. No claim is made that the measured displacements are historically independent of the character and placement of the backfill. Only the response of the structure is of interest.
4. Deflections of the culvert are small enough so that the principle of superposition may be invoked for the stresses and displacements of the culvert.
5. In the central interior portion of the buried culvert, far from its ends, conditions are such that the problem may reasonably be studied in the domain of plane strain.

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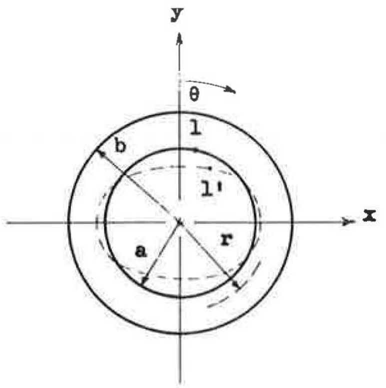


Figure 1. Inner boundary displacements.

DESCRIPTION OF INNER BOUNDARY DISPLACEMENTS

In Figure 1, point 1 on the original circular boundary moves to point 1′ on the distorted boundary. The displacement in the radial direction is denoted by u, and the displacement in the tangential direction by v. When it is radially outward, u is considered positive; v is positive when in the direction of increasing θ.

With the displacements of sufficiently many points (such as point 1 in Fig. 1) measured experimentally, it is possible to deduce, in a statistical sense, a contour which best describes the movement of all points of the inner boundary due to the movement of the measured points. It can be shown mathematically that of all trigonometric polynomials, those of the Fourier series type provide the best approximation (in the sense of least squares) to a contour described by discrete points (1).

Figure 1 also shows that initial polar symmetry of the geometry coupled with symmetry of displacements about the y-axis ($\theta = 0$) implies that the shape of the inner boundary to the left of the vertical axis is the mirror image of that to the right of the vertical axis. The radial displacements on the left are equal to the corresponding radial displacements on the right. Likewise, the tangential displacements on the left are the negatives of the tangential displacements on the right. As a consequence, the radial displacement, u, may be described only by even functions, and the tangential displacement, v, may be described only by odd functions.

Therefore, the displacement field of the structure has been described in general as:

$u = u(r, \theta) \equiv$ radial displacement $=$ even function of θ, and
$v = v(r, \theta) \equiv$ tangential displacement $=$ odd function of θ.

In particular on the inner boundary, where $r = a$, we have

$$u(a, \theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta$$

$$v(a, \theta) = \sum_{n=1}^{\infty} B_n \sin n\theta$$

where the coefficients A_0, A_n, and B_n are readily calculated from the experimentally determined inner boundary displacements by means of a regression analysis using trigonometric functions of sines and cosines. Computer programs are available for these computations.

DESCRIPTION OF OUTER BOUNDARY TRACTIONS

Let the stresses in the plane of the culvert cross section be designated by the following:

$\tau_{rr} = \tau_{rr}(r, \theta) \equiv$ normal radial stress,
$\tau_{\theta\theta} = \tau_{\theta\theta}(r, \theta) \equiv$ normal tangential stress, and
$\tau_{r\theta} = \tau_{r\theta}(r, \theta) \equiv$ shear stress.

The traction on the boundary of the soil-structure interface can be arbitrarily described by the two following components:

$p(\theta) = \tau_{rr}(b, \theta) \equiv$ normal component of surface traction (pressure), and
$q(\theta) = \tau_{r\theta}(b, \theta) \equiv$ shear component of surface traction.

Using the same argument of symmetry as was used in the case of the inner boundary displacements, the pressure, p, may be described by even functions only and the shear at the interface by odd functions only (Fig. 2).

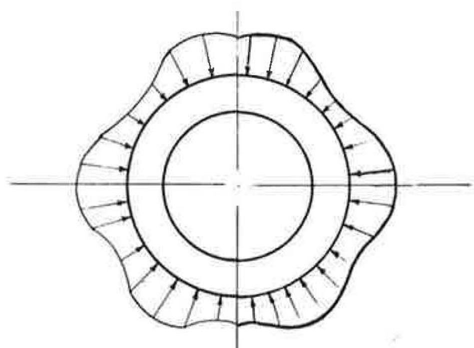
If it is the case that the true form of p and q can be described by a function that is at least piecewise continuous, then these functions can always be represented by a Fourier series; consequently,

$$p(\theta) = \tau_{rr}(b, \theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$$

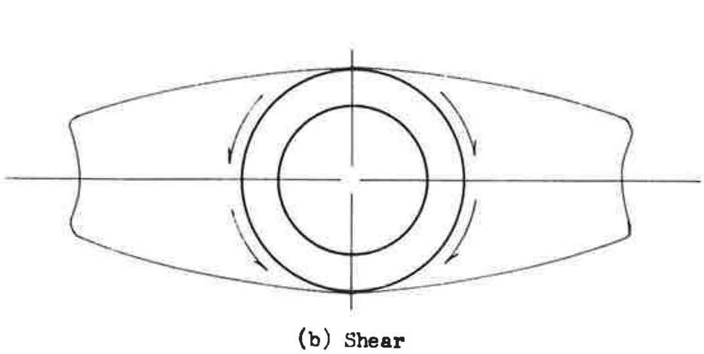
$$q(\theta) = \tau_{r\theta}(b, \theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$$

where a_0, a_n, and b_n are the as yet undetermined coefficients which will be found in the elastic analysis that follows.

The sign convention adopted is as follows: (a) p is positive when its effect is to produce tension, and p is negative when its effect is to produce compression; and (b) q is positive when its effect is to produce clockwise shear, and q is negative when its effect is to produce counterclockwise shear.



(a) Normal Pressure



(b) Shear

Figure 2. Normal pressure and shear.

ELASTIC ANALYSIS OF THE RIGID CIRCULAR CULVERT

The elastic analysis begins with the displacements of the inner boundary which are considered known; i. e., obtainable by experiment.

$$u(a, \theta) = A_o + \sum_{n=1}^{\infty} A_n \cos n\theta \tag{1a}$$

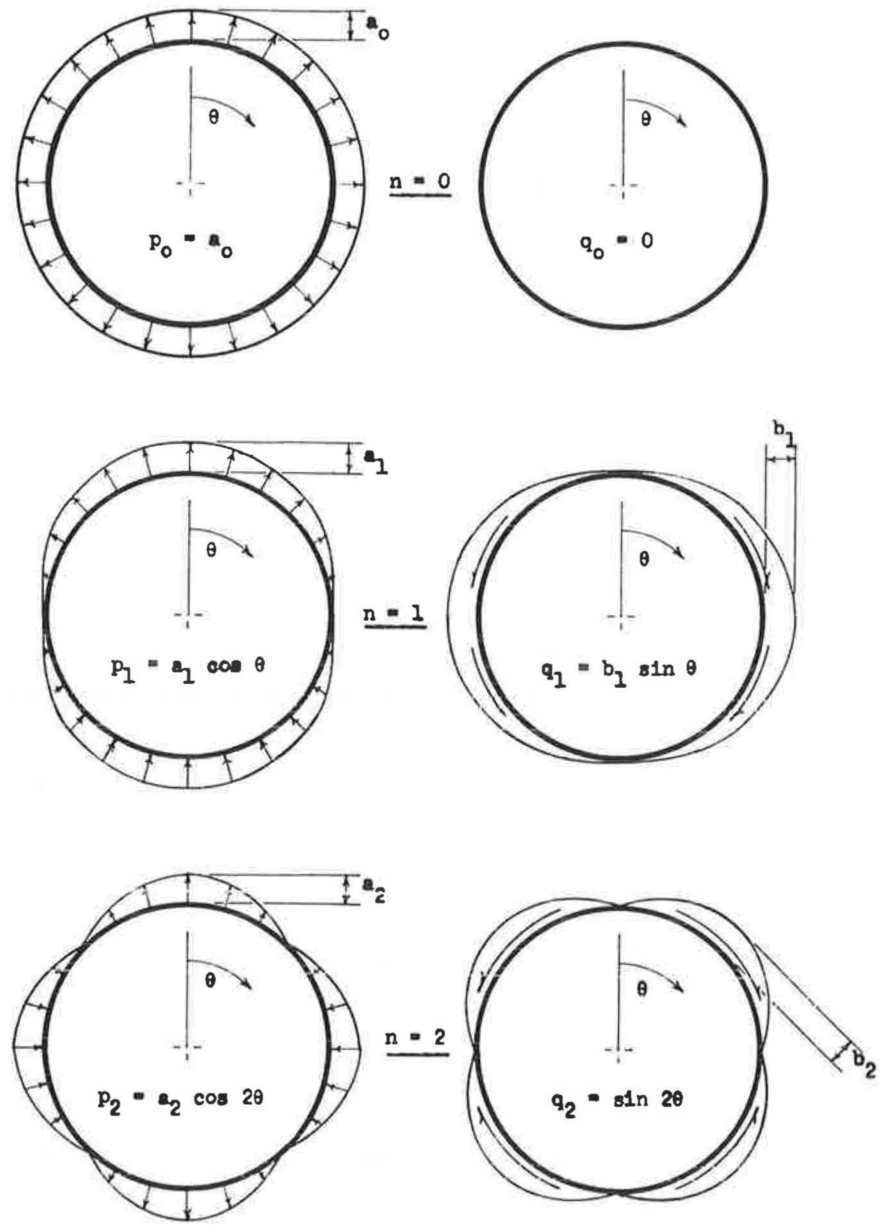


Figure 3. Soil structure interface loading components, Fourier harmonics, n = 0, 1, 2, 3, 4, 5.

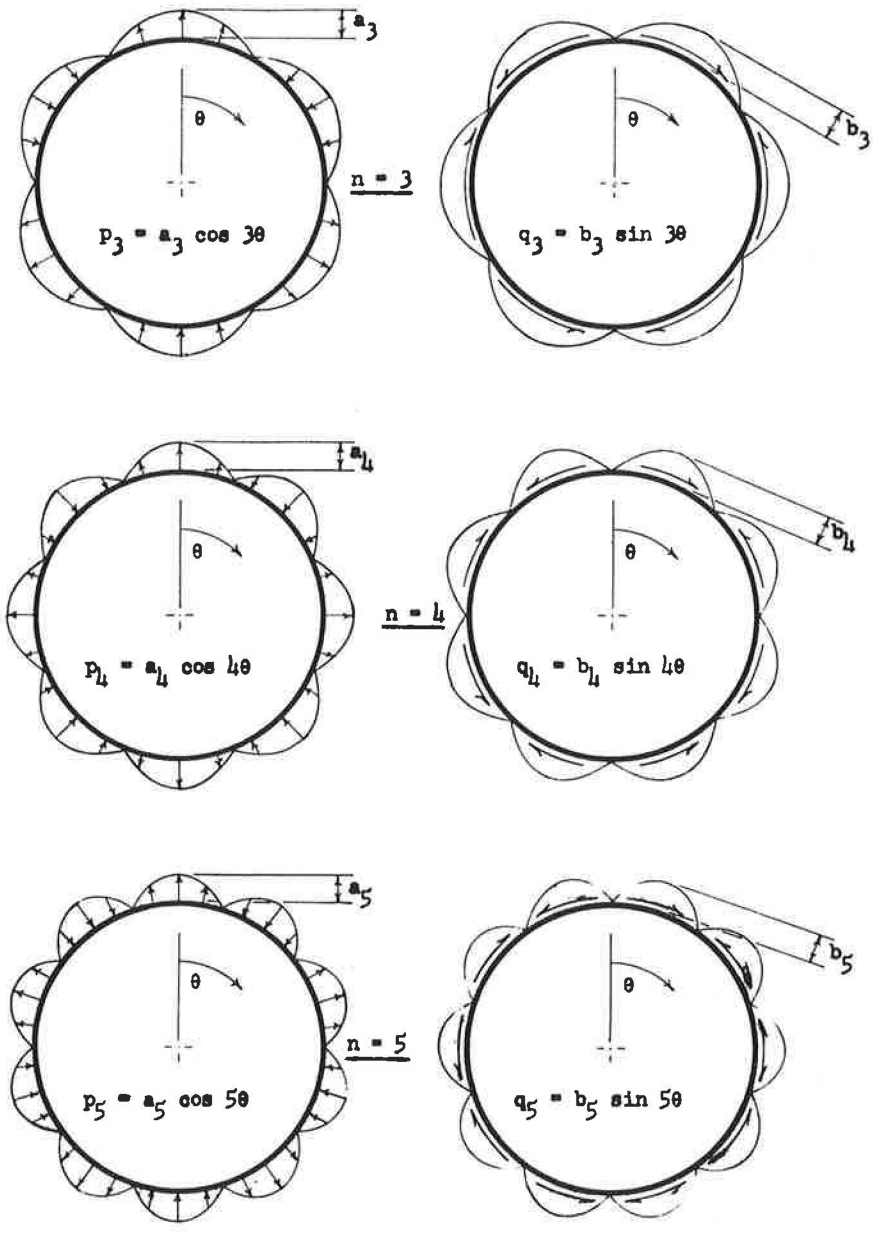


Figure 3. (Continued)

$$v(a, \theta) = \sum_{n=1}^{\infty} B_n \sin n\theta \tag{1b}$$

Eqs. 1a and 1b are a consequence of the experimentally determined inner boundary displacements. (Of course, for numerical work the series shown will be truncated at some finite number of terms which will, however, be large enough to assure accuracy as well as consistency with the experimental data.)

The tractions on the outer boundary are unknown, but as previously stated, if they can be described by functions that are at least piecewise continuous, then they may be described by a Fourier series as follows:

$$\tau_{rr}(b,\theta) = p(\theta) = \sum_{n=0}^{\infty} p_n = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta \tag{2a}$$

$$\tau_{r\theta}(b,\theta) = q(\theta) = \sum_{n=0}^{\infty} q_n = \sum_{n=1}^{\infty} b_n \sin n\theta \tag{2b}$$

Invoking the principle of superposition (previously discussed), we will now calculate the response displacement of the structure to each harmonic of loading.

Zeroth Harmonic

Considering only the $n = 0$ term of the summations in Eqs. 2a and 2b, we obtain the following (Fig. 3):

$$p_0 = a_0$$
$$q_0 = 0$$

In Appendix A the relations between stresses and displacements are derived, and the following is obtained for the zeroth harmonic of inner boundary displacement:

$$u(a,\theta) = A_0$$
$$v(a,\theta) = 0$$

The corresponding outer boundary tractions are

$$p_0 = \frac{A_0}{2a}\left(1 - \frac{a^2}{b^2}\right) E_1 = \frac{E}{2a(1-\nu^2)}\left(1 - \frac{a^2}{b^2}\right) A_0$$

$$q_0 = 0$$

Here E_1 denotes the "plane strain" modulus of elasticity.

$$E_1 = \frac{E}{(1-\nu^2)}$$

First Harmonic

For the $n = 1$ terms in Eqs. 2a and 2b, we obtain the tractions of the first harmonic loading (Fig. 3):

$$p_1 = a_1 \cos\theta$$
$$q_1 = b_1 \sin\theta$$

Reference is again made to Appendix A where the derivation of the relations between stresses and displacements is shown.

For the first harmonic of inner boundary displacement,

$$u(a,\theta) = A_1 \cos\theta$$
$$v(a,\theta) = B_1 \sin\theta$$

the corresponding outer boundary tractions are

$$p_1 = \frac{b}{a^2}\left(1 - \frac{a^4}{b^4}\right)\left(\frac{E_1}{1-\nu_1}\right) A_1 \cos\theta = \frac{b}{a^2}\left(1 - \frac{a^4}{b^4}\right)\frac{E}{(1+\nu)(1-2\nu)} A_1 \cos\theta$$

$$q_1 = \frac{b}{a^2}\left(1 - \frac{a^4}{b^4}\right)\left(\frac{E_1}{3+2\nu_1}\right) B_1 \sin\theta = \frac{b}{a^2}\left(1 - \frac{a^4}{b^4}\right)\frac{E}{(1+\nu)(3-\nu)} B_1 \sin\theta$$

Here again, E_1 denotes the "plane strain" modulus of elasticity and $\nu_1 = \nu/(1 - \nu)$ denotes the "plane strain" Poisson's ratio.

nth Harmonic, $n \geq 2$

In Eqs. 2a and 2b the tractions of the nth harmonic of loading are described as follows (Fig. 3):

$$p_n = a_n \cos n\theta$$
$$q_n = b_n \sin n\theta$$

Again, the analysis of Appendix A results in the following: for the nth harmonic of inner boundary displacement,

$$u(a, \theta) = A_n \cos n\theta$$
$$v(a, \theta) = B_n \sin n\theta$$

The corresponding outer boundary tractions are

$$p_n = \frac{E}{2a(1 - \nu^2)} \left[\frac{(C_{1n})^3 A_n + (C_{2n})^3 B_n}{(C_{3n})^3 A_n + (C_{4n})^3 B_n}\right] A_n \cos n\theta$$

$$q_n = \frac{E}{2a(1 - \nu^2)} \left[\frac{(C_{5n})^3 A_n + (C_{6n})^3 B_n}{(C_{7n})^3 A_n + (C_{8n})^3 B_n}\right] B_n \sin n\theta$$

In the foregoing equations the coefficients C_{1n}, C_{2n}, ..., C_{8n} are quantities functionally dependent on α, the ratio of inner boundary diameter to outer boundary diameter. For ease in future computations and applications, these coefficients were precalculated by digital computer for a range of values of α and are charted in Appendix B.

CONCLUSION

It is possible to use a round circular culvert under fill directly as a transducer for the determination of the unique outer boundary loading causing the experimentally determined displacements. The displacements of the inner boundary are represented in the form of a Fourier series:

$$\text{radial displacement} = u(a, \theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta$$

$$\text{tangential displacement} = v(a, \theta) = \sum_{n=1}^{\infty} B_n \sin n\theta$$

The A_0, A_n, and B_n which are determined by a regression analysis of the experimental data are now used as inputs in the solutions for the outer boundary tractions as follows:

$$p(b, \theta) = \frac{E}{2a(1 - \nu^2)} \left[1 - \left(\frac{a}{b}\right)^2\right] A_0 + \frac{E}{(1 - \nu)(1 - 2\nu)} \frac{b}{a^2} \left[1 - \left(\frac{a}{b}\right)^4\right] A_1 \cos \theta$$

$$+ \sum_{n=2}^{\infty} \frac{E}{2a(1 - \nu^2)} \left[\frac{(C_{1n})^3 A_n + (C_{2n})^3 B_n}{(C_{3n})^3 A_n + (C_{4n})^3 B_n}\right] A_n \cos n\theta$$

$$q(b, \theta) = \frac{E}{(1 + \nu)(3 - \nu)} \frac{b}{a^2} \left[1 - \left(\frac{a}{b}\right)^4\right] B_1 \sin \theta$$

$$+ \sum_{n=2}^{\infty} \frac{E}{2a(1 - \nu)^2} \left[\frac{(C_{5n})^3 A_n + (C_{6n})^3 B_n}{(C_{7n})^3 A_n + (C_{8n})^3 B_n}\right] B_n \sin n\theta$$

The n may be chosen so as to give any predetermined degree of precision to these converging series. With the A_0, A_1, ..., A_n and B_1, ..., B_n known, and with the $C1n$, $C2n$, ..., $C8n$ calculated or taken from the charts in Appendix B, the normal traction, p, and the tangential traction, q, may be calculated.

REFERENCE

1. Gaskell, Robert E. Engineering Mathematics. Dryden Press, p. 220, 1958.

Appendix A

DERIVATION OF RELATIONS BETWEEN STRESSES AND DISPLACEMENTS

Given:

1. Displacements of inner boundary (r = a)

$$u(a, \theta) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta) \tag{3a}$$

$$v(a, \theta) = \sum_{n=1}^{\infty} (B_n \sin n\theta) \tag{3b}$$

2. Tractions on outer boundary (r = b)

$$p = \tau_{rr}(b, \theta) = a_0 + \sum_{m=1}^{\infty} (a_m \cos m\theta)$$

$$q = \tau_{r\theta}(b, \theta) = \sum_{m=1}^{\infty} (b_m \sin m\theta)$$

denote the radial tension and the tangential traction, respectively. To show:

1. $a_0 = f_1(A_0)$
   $a_0 = 0$ when $n \geq 1$
2. $a_m = f_2(A_n, B_n)$ when $m = n \geq 1$
   $a_m = 0$ when $m \neq n \geq 1$
3. $b_m = f_3(A_n, B_n)$ when $m = n \geq 1$
   $b_m = 0$ when $m \neq n \geq 1$

In addition, the tractions on the outer boundary will be evaluated as functions of the displacements of the inner boundary.

Zeroth Harmonic

Tractions of the zeroth harmonic of loading are described as:

$$p_0 = a_0 \tag{4a}$$

$$q_0 = 0 \tag{4b}$$

The general solution (2, pp. 58-59) of the problem of a stress distribution symmetrical about a longitudinal axis (the Lamé problem) is obtained as follows:

$$\text{Stress function } \varphi = A \log r + Br^2 \log r + Cr^2 + D$$

The stresses are given by:

$$\tau_{rr} = \frac{1}{r}\frac{\partial \varphi_0}{\partial r} = \frac{A}{r^2} + B(1 + 2\log r) + 2C$$

$$\tau_{\theta\theta} = \frac{\partial^2 \varphi_0}{\partial r^2} = -\frac{A}{r^2} + B(3 + 2\log r) + 2C$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi_0}{\partial \theta}\right) = 0$$

Single valued displacements require B = 0 (2, p. 68). Therefore,

$$\tau_{rr} = \frac{A}{r^2} + 2C \tag{5a}$$

$$\tau_{\theta\theta} = -\frac{A}{r^2} + 2C \tag{5b}$$

$$\tau_{r\theta} = 0 \tag{5c}$$

Substituting the boundary conditions:

$$p_0 = \tau_{rr}(b, \theta) = a_0 = \frac{A}{b^2} + 2C$$

$$q_0 = \tau_{rr}(a, \theta) = 0 = \frac{A}{a^2} + 2C$$

Solving for A and C and substituting in Eqs. 5a, 5b, and 5c, we obtain the following:

$$\tau_{rr} = \frac{a_0 b^2}{(b^2 - a^2)}\left(1 - \frac{a^2}{r^2}\right) = K_0\left(1 - \frac{a^2}{r^2}\right) \tag{6a}$$

$$\tau_{\theta\theta} = \frac{a_0 b^2}{(b^2 - a^2)}\left(1 + \frac{a^2}{r^2}\right) = K_0\left(1 + \frac{a^2}{r^2}\right) \tag{6b}$$

$$\tau_{r\theta} = 0 \tag{6c}$$

where

$$K_0 = \frac{a_0 b^2}{(b^2 - a^2)}$$

For the two-dimensional system of polar coordinates employed, the strain-displacement relations are (2, pp. 65-66):

$$\epsilon_{rr} = \frac{\partial u}{\partial r} \tag{7a}$$

$$\epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{7b}$$

$$\gamma_{r\theta} = 2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{7c}$$

The stress-strain relations for an elastic body in a two-dimensional polar coordinate system are (2, pp. 65-66).

$$\epsilon_{rr} = \frac{1}{E_1}(\tau_{rr} - \nu_1 \tau_{\theta\theta}) \tag{8a}$$

$$\epsilon_{\theta\theta} = \frac{1}{E_1}(\tau_{\theta\theta} - \nu_1 \tau_{\theta\theta}) \tag{8b}$$

$$\gamma_{r\theta} = \frac{1}{G_1}\tau_{r\theta} \tag{8c}$$

Here E_1 and ν_1 are the "plane strain" elastic constants where

$$E_1 = \frac{E}{1-\nu^2}, \quad \nu_1 = \frac{\nu}{1-\nu}, \quad G_1 = G = \frac{E_1}{2(1+\nu_1)}$$

Eliminating the strains between the strain-displacement relations and the stress-strain relations (Eqs. 7a, 7b, 7c and 8a, 8b and 8c) we obtain:

$$\frac{\partial u}{\partial r} = \frac{1}{E_1}(\tau_{rr} - \nu_1 \tau_{\theta\theta}) \tag{9a}$$

$$\frac{\partial v}{\partial \theta} = \frac{1}{E_1}(\tau_{\theta\theta} - \nu_1 \tau_{rr}) - u \tag{9b}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \frac{1}{G_1}\tau_{r\theta} \tag{9c}$$

Substituting the equation for stress (Eq. 6a) in Eq. 9a:

$$\frac{\partial u}{\partial r} = \frac{K_0}{E_1}\left[\left(1 - \frac{a^2}{r^2}\right) - \nu_1\left(1 + \frac{a^2}{r^2}\right)\right] \tag{10}$$

Integrating with respect to r:

$$u(r, \theta) = \frac{K_0 r}{E_1}\left[\left(1 + \frac{a^2}{r^2}\right) - \nu_1\left(1 - \frac{a^2}{r^2}\right)\right] + g_1(\theta) \tag{11}$$

Substituting the equation for stress (Eq. 6b) and the equation for displacement $u(r, \theta)$, Eq. 11, into the stress-displacement relation Eq. 9b:

$$\frac{\partial v}{\partial \theta} = - g_1(\theta)$$

Integrating with respect to θ:

$$v(r, \theta) = -\int g_1(\theta)\, d\theta + g_2(r) \tag{12}$$

To evaluate the functions $g_1(\theta)$ and $g_2(r)$, Eqs. 11, 12 and 6c are substituted in Eq. 9c. There results

$$\left[\frac{\partial g_1(\theta)}{\partial \theta} + \int g_1(\theta)\, d\theta\right] + \left[r\frac{\partial g_2(r)}{\partial r} - g_2(r)\right] = 0 \tag{12a}$$

Inasmuch as the foregoing must be true for all r and θ,

$$\frac{\partial g_1(\theta)}{\partial \theta} + \int g_1(\theta)\, d\theta = C_1$$

$$\frac{r\partial g_2(r)}{\partial r} - g_2(r) + -C_1$$

where C_1 is the arbitrary constant of integration.

Let:

$$g_1(\theta) = H \sin\theta + L \cos\theta \tag{13a}$$

$$g_2(r) = Fr \tag{13b}$$

where F, H, L are arbitrary constant. These satisfy Eq. 12a identically for all values of r and θ.

Substituting Eqs. 13a and 13b in Eqs. 11 and 12:

$$u(r, \theta) = \frac{K_0 r}{E_1}\left[\left(1 + \frac{a^2}{r^2}\right) - \nu_1\left(1 - \frac{a^2}{r^2}\right)\right] + H \sin\theta + L \cos\theta \tag{14a}$$

$$v(r, \theta) = H \cos\theta + L \sin\theta + Fr \tag{14b}$$

To evaluate H, L, and F, the following conditions are employed:

1. By symmetry of tangential displacements $v(r, 0) = v(r, \pi) = 0$; substitution in Eq. 14b leads to F = 0.

2. On the inner boundary $v(a, \theta) = \sum_{n=1}^{\infty} (B_n \sin n\theta)$ from Eq. 3b. Comparing coefficients with Eq. 14b and setting F = 0 as a consequence of item 1 above, it follows that L = 0.

3. On the inner boundary $u(a, \theta) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta)$ from Eq. 3a. Comparing coefficients with Eq. 14a and setting L = 0 as a consequence of item 2 above, it follows that H = 0.

The following displacement field now results:

$$u(r, \theta) = \frac{K_0 r}{E_1}\left[\left(1 + \frac{a^2}{r^2}\right) - \nu_1\left(1 - \frac{a^2}{r^2}\right)\right]$$

$$v(r, \theta) = 0$$

On the inner boundary:

$$u(a, \theta) = \frac{2K_0 a}{E_1} = \frac{2a_0 ab^2}{(b^2 - a^2) E_1} \tag{15a}$$

$$v(r, \theta) = 0 \tag{15b}$$

Comparing Eqs. 3a and 3b with Eqs. 15a and 15b, respectively, it follows

$$\frac{2a_0\, ab^2}{(b^2 - a^2)\, E_1} = A_0, \quad A_n = 0, \quad B_n = 0 \text{ for } n \geq 1$$

Therefore

$$a_0 = A_0 \frac{(b^2 - a^2)}{2ab^2} E_1 \tag{16}$$

From Eqs. 4a and 4b, it now follows

$$p_0 = A_0 \frac{(b^2 - a^2)}{2ab^2} E_1 = \frac{A_0}{2a}\left(1 - \frac{a^2}{b^2}\right) E_1 \tag{17a}$$

$$q_0 = 0 \tag{17b}$$

It has now been established that for the zeroth harmonic of inner boundary displacement, the tractions of the outer boundary are described by their zeroth harmonic only. This is precisely the objective as indicated earlier. Knowing A_0, the amplitude of the zeroth harmonic of inner boundary displacement, the normal and shearing tractions of the outer boundary may be evaluated with the aid of Eqs. 17a and 17b.

Substituting Eq. 16 in Eqs. 6a, 6b, and 6c, the description of the entire state of stress throughout the culvert (due to the zeroth harmonic loading) is obtained as:

$$\tau_{rr} = \frac{A_0}{2a}\left(1 - \frac{a^2}{r^2}\right) E_1 \tag{17c}$$

$$\tau_{\theta\theta} = \frac{A_0}{2a}\left(1 + \frac{a^2}{r^2}\right) E_1 \tag{17d}$$

$$\tau_{r\theta} = 0 \tag{17e}$$

Summary—Zeroth Harmonic

The displacements to be experimentally determined are

$$u(a, \theta) = A_0$$
$$v(a, \theta) = 0$$

The corresponding outer boundary tractions are

$$p_0 = \tau_{rr}(b, \theta) = \frac{A_0}{2a}\left(1 - \frac{a^2}{b^2}\right) E_1$$

$$q_0 = \tau_{r\theta}(b, \theta) = 0$$

First Harmonic (Fig. 3)

Tractions of the first harmonic loading are described as:

$$p_1 = a_1 \cos\theta \tag{18a}$$

$$q_1 = b_1 \sin\theta \tag{18b}$$

The most general form of stress function, φ_1, satisfying the requirement that its form be that of the first harmonic was given by Michell (2, p. 116) as:

$$\varphi_1 = \overset{1}{\varphi}_1 + \overset{2}{\varphi}_1 + \overset{3}{\varphi}_1 + \overset{4}{\varphi}_1$$

where

$$\overset{1}{\varphi}_1 = \frac{C_1}{2} r\theta \sin\theta$$

$$\overset{2}{\varphi}_1 = (C_2 r^3 + C_3 r^{-1} + C_4 r \log r) \cos\theta$$

$$\overset{3}{\varphi}_1 = \frac{-C_5}{2} r\theta \cos\theta$$

$$\overset{4}{\varphi}_1 = (C_6 r^3 + C_7 r^{-1} + C_8 r \log r) \sin\theta$$

Timoshenko (2, p. 119) argues that when the resultant of all forces on each boundary is zero, then $C_1 = C_4 = C_5 = C_8 = 0$. As this is certainly the case for an empty culvert in equilibrium with the tractions applied to its outer boundary, the stress function for this case becomes:

$$\varphi_1 = \overset{2}{\varphi}_2 + \overset{4}{\varphi}_1$$

where

$$\overset{2}{\varphi}_1 = (C_2 r^3 + C_3 r^{-1}) \cos\theta$$

$$\overset{4}{\varphi}_1 = (C_6 r^3 + C_7 r^{-1}) \sin\theta$$

The stresses are given by

$$\tau_{rr} = \frac{1}{r}\frac{\partial\varphi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi_1}{\partial\theta^2} = 2(C_2 r - C_3 r^{-3}) \cos\theta + 2(C_6 r - C_7 r^{-3}) \sin\theta \quad (19a)$$

$$\tau_{\theta\theta} = \frac{\partial^2\varphi_1}{\partial r^2} = 2(3C_2 r + C_3 r^{-3}) \cos\theta + 2(3C_6 r + C_7 r^{-3}) \sin\theta \quad (19b)$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi_1}{\partial\theta}\right) = 2(C_2 r - C_3 r^{-3}) \sin\theta - 2(C_6 r - C_7 r^{-3}) \cos\theta \quad (19c)$$

The following conditions exist on the boundaries:

$$\tau_{rr}(a, \theta) = 0 \quad (20a)$$

$$\tau_{r\theta}(a, \theta) = 0 \quad (20b)$$

$$\tau_{rr}(b, \theta) = p_1 = a_1 \cos\theta \quad (20c)$$

$$\tau_{r\theta}(b, \theta) = q_1 = b_1 \sin\theta \quad (20d)$$

Matching coefficients of Eq. 19a when r = b with Eq. 20c it follows:

$$\tfrac{1}{2} a_1 = C_2 b - C_3 b^{-3} \quad (21a)$$

$$0 = C_6 b - C_7 b^{-3} \quad (21b)$$

Matching coefficients of Eq. 19c when r = b with Eq. 20d it follows:

$$\tfrac{1}{2} b_1 = C_2 b - C_3 b^{-3} \quad (21c)$$

$$0 = C_6 b - C_7 b^{-3}$$

For both Eqs. 21a and 21c to agree, it follows that

$$a_1 = b_1$$

This result is also a necessary condition for the tractions, Eqs. 18a and 18b, to be in equilibrium.

Boundary conditions (Eq. 20a and Eq. 20b) supply the additional information necessary for the evaluation of the constants C_2, C_3, C_6, and C_7. Substituting the value $r = a$ in Eqs. 19a and 19c and matching coefficients with Eqs. 20a and 20b, it follows:

$$C_2 a - C_3 a^{-3} = 0 \tag{22a}$$

$$C_6 a - C_7 a^{-3} = 0 \tag{22b}$$

The four equations, Eqs. 21a, 21b, 22a, and 22b are solved simultaneously and yield the following values for the four constants.

$$C_2 = \frac{a_1}{2b\left[1 - \left(\frac{a}{b}\right)^4\right]}$$

$$C_3 = \frac{a_1 a^4}{2b\left[1 - \left(\frac{a}{b}\right)^4\right]} = a^4 C_2$$

$$C_6 = 0$$

$$C_7 = 0$$

These expressions for constants C_2, C_3, C_6, C_7 are now substituted into the expressions for stress Eqs. 19a, 19b and 19c. Therefore,

$$\tau_{rr}(r, \theta) = \frac{r}{b} \frac{\left(1 - \frac{a^4}{r^4}\right)}{\left(1 - \frac{a^4}{b^4}\right)} a_1 \cos\theta = k_1 r \left(1 - \frac{a^4}{r^4}\right) a_1 \cos\theta \tag{23a}$$

$$\tau_{\theta\theta}(r, \theta) = \frac{r}{b} \frac{\left(3 + \frac{a^4}{r^4}\right)}{\left(1 - \frac{a^4}{b^4}\right)} a_1 \cos\theta = k_1 r \left(3 + \frac{a^4}{r^4}\right) a_1 \cos\theta \tag{23b}$$

$$\tau_{r\theta}(r, \theta) = \frac{r}{b} \frac{\left(1 - \frac{a^4}{r^4}\right)}{\left(1 - \frac{a^4}{b^4}\right)} a_1 \sin\theta = k_1 r \left(1 - \frac{a^4}{r^4}\right) a_1 \sin\theta \tag{23c}$$

when

$$k_1 = \frac{1}{b\left(1 - \frac{a^4}{b^4}\right)}$$

These values of stress are now substituted into the stress-displacement relations, Eqs. 9a, 9b, and 9c. For Eq. 9a:

$$\frac{\partial u}{\partial r} = \frac{k_1}{E_1}\, r \left[\left(1 - \frac{a^4}{r^4}\right) - \nu_1 \left(3 + \frac{a^4}{r^4}\right)\right] a_1 \cos\theta$$

Integrating:

$$u(r, \theta) = \frac{k_1}{2E_1}\, r^2 \left[\left(1 + \frac{a^4}{r^4}\right) - \nu_1 \left(3 - \frac{a^4}{r^4}\right)\right] a_1 \cos\theta + g_3(\theta) \qquad (24)$$

Putting this result and the stresses (Eqs. 23a, 23b, and 23c) into Eq. 9b:

$$\frac{\partial v}{\partial \theta} = \frac{k_1 r^2}{2E_1} \left[\left(5 + \frac{a^4}{r^4}\right) + \nu_1 \left(1 + \frac{a^4}{r^4}\right)\right] a_1 \cos\theta - g_3(\theta)$$

Integrating:

$$v(r, \theta) = \frac{k_1 r^2}{2E_1} \left[\left(5 + \frac{a^4}{r^4}\right) + \nu_1 \left(1 + \frac{a^4}{r^4}\right)\right] a_1 \sin\theta - \int g_3(\theta)\, d\theta + g_4(r) \qquad (25)$$

Substituting Eqs. 24, 25, and 23c, in Eq. 9c:

$$\left[\frac{\partial g_3(\theta)}{\partial \theta} + \int g_3(\theta)\, d\theta\right] + \left[\frac{r \partial g_4(r)}{\partial r} - g_4(r)\right] = 0$$

Symmetry conditions are now introduced to evaluate $g_3(\theta)$ and $g_4(r)$.

<u>Symmetry Condition No. 1</u>

$v(r, 0) = 0$ for $b \geq r \geq a$ $- \int g_3(\theta)\, d\theta + g_4(r) = 0$ for $\theta = 0$ and $b \geq r \geq a$

As this must be true for all r ($b \geq r \geq a$), then $g_4(r) \equiv 0$. Consequently

$$\int g_3(\theta)\, d\theta = 0 \text{ when } \theta = 0 \qquad (26)$$

<u>Symmetry Condition No. 2</u>

$u(r, \theta) = u(r, -\theta)$

Substituting in Eq. 24:

$$g_3(\theta) = g_3(-\theta), \text{ for all } \theta \qquad (27)$$

which implies $g_3(\theta)$ is an even function. We now search for a function $g_3(\theta)$ which will satisfy Eq. 26 and Eq. 27. Let

$$G_3(\theta) = \int g_3(\theta)\, d\theta = C_1\theta \sin\theta + C_2\theta \cos\theta + C_3\theta\ C_4 + C_5 \sin\theta \cos\theta + C_6 \sin\theta \cos\theta + C_7 \sin\theta + C_8 \cos\theta \qquad (28)$$

Applying Eq. 26:

$$C_4 + C_8 = 0 \qquad (29)$$

Solving now for $g_3(\theta) = \frac{d}{d\theta} \int g_3(\theta)\, d\theta$ and applying Eqs. 27 and 29:

$C_1 = C_4 = C_6 = C_8 = 0$

Eq. 28 then becomes

$$G_3(\theta) = C_2\theta\cos\theta + C_3\theta + C_5\sin\theta\cos\theta + C_7\sin\theta = 0$$

Single-Valuedness of Displacements Condition No. 3

$$v(r,\theta) = v(r,\theta + 2\pi n) \tag{30}$$

With Eq. 30 and $g_4(r) = 0$, Eq. 25 yields

$$C_2\theta\cos\theta + C_3\theta = 0, \text{ for all } \theta$$

This is true only if $C_2 = C_3 = 0$ so that Eq. 28 becomes

$$G_3(\theta) = C_5\sin\theta\cos\theta + C_7\sin\theta$$

and thus

$$g_3(\theta) = C_5(1 - 2\sin^2\theta) + C_7\cos\theta$$

Eqs. 24 and 25 now become

$$u(r,\theta) = \frac{k_1}{2E_1} r^2 \left[\left(1 + \frac{a^4}{r^4}\right) - \nu_1\left(3 - \frac{a^4}{r^4}\right)\right] a_1\cos\theta + C_5(1 - 2\sin^2\theta) + C_7\cos\theta \tag{31}$$

$$v(r,\theta) = \frac{k_1 r^2}{2E_1}\left[\left(5 + \frac{a^4}{r^4}\right) + \nu_1\left(1 + \frac{a^4}{r^4}\right)\right] a_1\sin\theta - C_5\sin\theta\cos\theta - C_7\sin\theta$$

Symmetry Condition No. 4

$$u(a,0) = -u(a,\pi)$$

This condition will serve to keep the origin of the coordinates at the midpoint of the vertical diameter.

Substituting in Eq. 31:

$$C_5 + C_7 = -(C_5 - C_7)$$

This implies $C_5 = 0$. The following displacement field now results:

$$u(r,\theta) = \frac{k_1 r^2}{2E_1}\left[\left(1 + \frac{a^4}{r^4}\right) - \nu_1\left(3 - \frac{a^4}{r^4}\right)\right] a_1\cos\theta$$

$$v(r,\theta) = \frac{k_1 r^2}{2E_1}\left[\left(5 + \frac{a^4}{r^4}\right) + \nu_1\left(1 + \frac{a^4}{r^4}\right)\right] a_1\sin\theta$$

On the inner boundary: (recalling $a_1 = b_1$)

$$u(a,\theta) = \frac{a^2}{b}\cdot\frac{b^4}{(b^4 - a^4)}\left(\frac{1 - \nu_1}{E_1}\right) a_1\cos\theta \tag{32a}$$

$$v(a,\theta) = \frac{a^2}{b}\cdot\frac{b^4}{(b^4 - a^4)}\left(\frac{3 + 2\nu_1}{E_1}\right) b_1\sin\theta \tag{32b}$$

Comparing Eqs. 3a and 3b with Eqs. 32a and 32b, respectively, it follows

$$A_0 = 0$$

$$A_1 = \frac{a^2}{b} \cdot \frac{b^4}{(b^4 - a^4)} \left(\frac{1 - \nu_1}{E_1}\right) a_1$$

$$B_1 = \frac{a^2}{b} \cdot \frac{b^4}{(b^4 - a^4)} \left(\frac{3 + 2\nu_1}{E_1}\right) b_1$$

$$A_n = B_n = 0 \text{ for } n \geq 2$$

Therefore,

$$a_1 = A_1 \frac{b}{a^2} \cdot \frac{(b^4 - a^4)}{b^4} \left(\frac{E_1}{1 - \nu_1}\right) \tag{33a}$$

$$b_1 = B_1 \frac{b}{a^2} \cdot \frac{(b^4 - a^4)}{b^4} \left(\frac{E_1}{3 + 2\nu_1}\right) \tag{33b}$$

From Eqs. 18a and 18b it now follows

$$p_1 = A_1 \frac{b}{a^2} \left(1 - \frac{a^4}{b^4}\right) \left(\frac{E_1}{1 - \nu_1}\right) \cos\theta \tag{34a}$$

$$q_1 = B_1 \frac{b}{a^2} \left(1 - \frac{a^4}{b^4}\right) \left(\frac{E_1}{3 + 2\nu_1}\right) \sin\theta \tag{34b}$$

It has now been established that for the first harmonics of inner boundary displacements, the tractions of the outer boundary are described by the first harmonics only. Knowing A_1 and B_1, the amplitudes of the first harmonics of the inner boundary displacements, the normal and shearing tractions on the boundary may be evaluated with the aid of Eqs. 34a and 34b.

Substituting Eqs. 33a and 33b in Eqs. 23a, 23b, and 23c; we have

$$\tau_{rr}(r, \theta) = A_1 \frac{r}{a^2} \left(1 - \frac{a^4}{r^4}\right) \left(\frac{E_1}{1 - \nu_1}\right) \cos\theta \tag{34c}$$

$$\tau_{\theta\theta}(r, \theta) = A_1 \frac{r}{a^2} \left(3 + \frac{a^4}{r^4}\right) \left(\frac{E_1}{1 - \nu_1}\right) \cos\theta \tag{34d}$$

$$\tau_{r\theta}(r, \theta) = B_1 \frac{r}{a^2} \left(1 - \frac{a^4}{r^4}\right) \left(\frac{E_1}{3 + 2\nu_1}\right) \sin\theta \tag{34e}$$

Summary—First Harmonic

The displacements to be determined experimentally are

$$u(a, \theta) = A_1 \cos\theta$$

$$v(a, \theta) = B_1 \sin\theta$$

The corresponding outer boundary tractions are

$$p_1 = \tau_{rr}(b, \theta) = \frac{b}{a^2} \left(1 - \frac{a^4}{b^4}\right) \left(\frac{E_1}{1 - \nu_1}\right) A_1 \cos\theta$$

$$q_1 = \tau_{r\theta}(b, \theta) = \frac{b}{a^2} \left(1 - \frac{a^4}{b^4}\right) \left(\frac{E_1}{3 + 2\nu_1}\right) B_1 \sin\theta$$

<u>n th Harmonic, $n \geq 2$</u>

The tractions of the n th harmonic of loading are:

$$p_n = a_n \cos n\theta$$
$$q_n = b_n \sin n\theta$$

The most general form of stress function, φ_n, satisfying the requirement that its form be that of the n th harmonic was given by Michell (2, p. 116) as:

$$\varphi_n = \overset{1}{\phi}_n + \overset{2}{\phi}_n$$

where

$$\overset{1}{\phi}_n = (A_n r^n + B_n r^{n+2} + A'_n r^{-n} + B'_n r^{-n+2}) \cos n\theta \qquad (35a)$$

$$\overset{2}{\phi}_n = (C_n r^n + D_n r^{n+2} + C'_n r^{-n} + D'_n r^{-n+2}) \sin n\theta \qquad (35b)$$

The stresses are given by:

$$\tau_{rr} = \frac{1}{r}\frac{\partial \varphi_n}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi_n}{\partial \theta^2} \qquad (36a)$$

$$\tau_{\theta\theta} = \frac{\partial^2 \varphi_n}{\partial r^2} \qquad (36b)$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi_n}{\partial \theta}\right) \qquad (36c)$$

Substituting Eqs. 35a and 35b in 36a, we obtain:

$$\tau_{rr} = h_1(r) \cos n\theta + h_2(r) \sin n\theta \qquad (37a)$$

where

$$h_1(r) = -n(n-1) A_n r^{n-2} - (n-2)(n+1) B_n r^n - n(n+1) A'_n r^{-n-2} - (n+2)(n-1) B'_n r^{-n}$$

$$h_2(r) = -n(n-1) C_n r^{n-2} - (n-2)(n+1) D_n r^n - n(n+1) C'_n r^{-n-2} - (n+2)(n-1) D'_n r^{-n}$$

Substituting Eqs. 35a and 35b into Eq. 36c, we obtain:

$$\tau_{r\theta} = h_3(r) \cos n\theta + h_4(r) \sin n\theta \qquad (37b)$$

where

$$h_3(r) = -n(n-1) C_n r^{n-2} - n(n+1) D_n r^n + n(n+1) C'_n r^{-n-2} + n(n-1) D'_n r^{-n}$$

$$h_4(r) = n(n-1) A_n r^{n-2} + n(n+1) B_n r^n - n(n+1) A'_n r^{-n-2} - n(n-1) B'_n r^{-n}$$

$$\tau_{\theta\theta} = h_5(r)\cos n\theta + h_6(r)\sin n\theta \tag{37c}$$

where

$$h_5(r) = n(n-1)A_n r^{n-2} + (n+2)(n+1)B_n r^n + n(n+1)A_n' r^{-n-2} + (n-2)(n-1)B_n' r^{-n}$$

$$h_6(r) = n(n-1)C_n r^{n-2} + (n+2)(n+1)D_n r^n + n(n+1)C_n' r^{-n-2} + (n-2)(n-1)D_n' r^{-n}$$

Study the boundary conditions:

$$\begin{aligned}
&\text{(i)} & \tau_{rr}(a,\theta) &= h_1(a)\cos n\theta + h_2(a)\sin n\theta = 0\\
&\text{(ii)} & \tau_{r\theta}(a,\theta) &= h_3(a)\cos n\theta + h_4(a)\sin n\theta = 0\\
&\text{(iii)} & \tau_{rr}(b,\theta) &= h_1(b)\cos n\theta + h_2(b)\sin n\theta = a_n\cos n\theta\\
&\text{(iv)} & \tau_{r\theta}(b,\theta) &= h_3(b)\cos n\theta + h_4(b)\sin n\theta = b_n\sin n\theta
\end{aligned}$$

The following relationships must be satisfied for these conditions to hold true for all θ.

$$\begin{aligned}
h_1(a) &= 0\\
h_4(a) &= 0\\
h_1(b) &= a_n\\
h_4(b) &= b_n\\
h_2(a) &= 0\\
h_3(a) &= 0\\
h_2(b) &= 0\\
h_3(b) &= 0
\end{aligned}$$

The foregoing system of eight equations and eight unknowns yields the following results.

$$C_n = D_n = C_n' = D_n' = 0 \tag{38}$$

$$A_n = \frac{-a_n}{2n(n-1)b^{n-2}}\left\{\frac{(n-1)\left[n-(n+2)\beta_n\right] + \left[n+(n-2)\beta_n\right]\alpha^{-2n} - n^2(1-\beta_n)\alpha^2}{\alpha^{-2n}+\alpha^{2n}-n^2(\alpha^{-2}+\alpha^2)+2(n^2-1)}\right\} \tag{39a}$$

$$B_n = \frac{-a_n}{2(n+1)b^n}\left\{\frac{\alpha^{-2}\left[(n+2)\beta_n - n\right] - \alpha^{-2n}(1+\beta_n) + (1-\beta_n)(n+1)}{\alpha^{-2n}+\alpha^{2n}-n^2(\alpha^{-2}+\alpha^2)+2(n^2-1)}\right\} \tag{39b}$$

$$A_n' = \frac{a_n b^{n+2}}{2n(n+1)}\left\{\frac{\alpha^{2n}\left[n-(n+2)\beta_n\right] - n^2\left[1-\alpha^2(1+\beta_n)\right] - n - (n+1)(n-2)\beta_n}{\alpha^{-2n}+\alpha^{2n}-n^2(\alpha^{-2}+\alpha^2)+2(n^2-1)}\right\} \tag{39c}$$

$$B_n' = \frac{a_n b^n}{2(n-1)}\left\{\frac{\alpha^{-2}\left[n+(n-2)\beta_n+\alpha^{2n+2}(-1+\beta_n)\right] - (n-1)(n+\beta_n)}{\alpha^{-2n}+\alpha^{2n}-n^2(\alpha^{-2}+\alpha^2)+2(n^2-1)}\right\} \tag{39d}$$

where

$$\alpha \equiv \text{ratio of pipe diameters} = \frac{a}{b} \tag{40a}$$

and

$$\beta_n \equiv \text{ratio of tangential to normal amplitudes of the nth harmonic} = \frac{b_n}{a_n} \tag{40b}$$

Returning to Eqs. 37a, 37b, and 37c and introducing Eqs. 38, and 39a through 39d, we obtain:

$$\tau_{rr} = \frac{a_n}{2D}\left[\left(\frac{r}{b}\right)^{n-2} N_1 + \left(\frac{r}{b}\right)^{n} (n-2)\, N_2 - \left(\frac{r}{b}\right)^{-n-2} N_3 - \left(\frac{r}{b}\right)^{-n} (n+2)\, N_4\right] \cos n\theta \tag{41a}$$

$$\tau_{\theta\theta} = -\frac{a_n}{2D}\left[\left(\frac{r}{b}\right)^{n-2} N_1 + \left(\frac{r}{b}\right)^{n} (n+2)\, N_2 - \left(\frac{r}{b}\right)^{-n-2} N_3 - \left(\frac{r}{b}\right)^{-n} (n-2)\, N_4\right] \cos n\theta \tag{41b}$$

$$\tau_{r\theta} = -\frac{a_n}{2D}\left[\left(\frac{r}{b}\right)^{n-2} N_1 + \left(\frac{r}{b}\right)^{n} (n)\, N_2 + \left(\frac{r}{b}\right)^{-n-2} N_3 + \left(\frac{r}{b}\right)^{-n} (n)\, N_4\right] \sin n\theta \tag{41c}$$

where

$$D = \alpha^{-2n} + \alpha^{2n} - n^2 (\alpha^{-2} + \alpha^2) + 2 (n^2 - 1) \tag{41d}$$

$$N_1 = (n-1)\left[n - (n+2)\,\beta_n\right] + \left[n + (n-2)\,\beta_n\right] \alpha^{-2n} - n^2 (1 - \beta_n)\, \alpha^2 \tag{41e}$$

$$N_2 = \alpha^{-2}\left[(n+2)\,\beta_n - n\right] - \alpha^{-2n} (1 + \beta_n) + (1 - \beta_n)\, (n+1) \tag{41f}$$

$$N_3 = \alpha^{2n}\left[n - (n+2)\,\beta_n\right] - n^2\left[1 - \alpha^2 (1 + \beta_n)\right] - n - (n+1)(n-2)\,\beta_n \tag{41g}$$

$$N_4 = \alpha^{-2}\left[n + (n-2)\,\beta_n + \alpha^{2n+2} (-1 + \beta_n)\right] - (n-1)(1 + \beta_n) \tag{41h}$$

and where α and β_n are as defined in Eqs. 40a and 40b.

Substituting Eqs. 41a through 41h in the stress displacement relations (Eqs. 9a, 9b and 9c) and integrating, we obtain the following:

$$u(r,\theta) - \frac{r}{2DE_1}\left\{(1+\nu_1)\left(\frac{r}{b}\right)^{n-2} \frac{N_1}{(n-1)} + \left[(n-2) + \nu_1 (n+2)\right] \left(\frac{r}{b}\right)^{n} \frac{N_1}{(n+1)} + (1+\nu_1)\left(\frac{r}{b}\right)^{-n-2} \frac{N_3}{(n+1)} + \left[(n+2) + \nu_1 (n-2)\right] \left(\frac{r}{b}\right)^{-n} \frac{N_4}{(n-1)}\right\} a_n \cos n\theta + h_7(\theta) \tag{42}$$

$$v(r,\theta) = \frac{r}{2DE_1\,\beta_n}\left\{-(1+\nu_1)\left(\frac{r}{b}\right)^{n-2}\frac{N_1}{(n-1)} - \left[n\,(1+\nu_1)+4\right]\left(\frac{r}{b}\right)^{n}\frac{N_2}{(n+1)} + (1+\nu_1)\left(\frac{r}{b}\right)^{-n-2}\frac{N_3}{(n+1)} + \left[n\,(1+\nu_1)-4\right]\left(\frac{r}{b}\right)^{-n}\frac{N_1}{(n-1)}\right\} b_n \sin n\theta - \int h_7(\theta)\,d\theta + h_8(r) \quad (43)$$

and

$$\left[\frac{\partial h_7(\theta)}{\partial\theta} + \int h_7(\theta)\,d\theta\right] + \left[r\,\frac{\partial h_8(r)}{\partial r} - h_8(r)\right] = 0$$

and where D, N_1, N_2, N_3, and N_4 are described by Eqs. 41a through 41h.

As in the case when n = 1, introduction of the following conditions leads to the conclusion that

$$h_7(\theta) = h_8(r) \equiv 0 \quad (44)$$

Symmetry condition No. 1: $v(r,0) = 0$ for $b \geq r \geq a$
Symmetry condition No. 2: $u(r,\theta) = u(r,-\theta)$
Single-valuedness of displacements condition No. 3: $v(r,\theta) = v(r,\theta + 2\pi n)$
Symmetry condition No. 4: $u(a,0) = -u(a,\pi)$

Substituting Eq. 44 into Eq. 42 and comparing with Eqs. 3a and 3b, it follows that:

$$A_0 = A_1 = B_1 = 0$$

$$A_n = \frac{a}{2DE_1}\,\overset{u}{\gamma}_n\,a_n$$

$$B_n = \frac{a}{2DE_1\,\beta_n}\,\overset{v}{\gamma}_n\,b_n \qquad \text{for } n \geq 2.$$

Therefore

$$a_n\,\overset{u}{\gamma}_n = \frac{2DE_1}{a}\,A_n \quad (45a)$$

$$\frac{b_n\,\overset{v}{\gamma}_n}{\beta_n} = \frac{2DE_1}{a}\,B_n \quad (45b)$$

where

$$\frac{\overset{u}{\gamma}_n}{4} = \left[\frac{n}{(n+1)}\alpha^{n-2} + \frac{n}{(n-1)}\alpha^{-n-2} - \frac{n}{(n-1)}\alpha^{n} - \frac{n}{(n+1)}\alpha^{-n}\right] + \left[-\frac{(n+2)}{(n+1)}\alpha^{n-2} + \frac{(n-2)}{(n-1)}\alpha^{-n-2} + \frac{n}{(n-1)}\alpha^{n} - \frac{n}{(n+1)}\alpha^{-n}\right]\beta_n$$

$$\frac{\overset{v}{\gamma}_n}{4} = \left[\frac{n}{(n+1)}\alpha^{n-2} - \frac{n}{(n-1)}\alpha^{-n-2} - \frac{(n-2)}{(n-1)}\alpha^{n} + \frac{(n+2)}{(n+1)}\alpha^{-n}\right] + \left[-\frac{(n+2)}{(n+1)}\alpha^{n-2} - \frac{(n-2)}{(n-1)}\alpha^{-n-2} + \frac{(n-2)}{(n-1)}\alpha^{n} + \frac{(n+2)}{(n+1)}\alpha^{-n}\right]\beta_n$$

It should be noted that $\overset{u}{\gamma}_n$, $\overset{v}{\gamma}_n$, and β_n are functions of a_n and b_n. Simultaneous solution of Eqs. 45a and 45b yield the values for a_n and b_n indicated in the summary that follows. Refer to Appendix B for the substantiating calculations.

Summary—n th Harmonic, $n \geq 2$

Given the experimental displacements

$$u(a, \theta) = A_n \cos n\theta$$
$$v(a, \theta) = B_n \sin n\theta$$

$$a_n = \frac{E}{2a\,(1 - \nu^2)} \left[\frac{(C_{1n})^3 A_n + (C_{2n})^3 B_n}{(C_{3n})^3 A_n + (C_{4n})^3 B_n}\right] A_n$$

$$b_n = \frac{E}{2a\,(1 - \nu^2)} \left[\frac{(C_{5n})^3 A_n + (C_{6n})^3 B_n}{(C_{7n})^3 A_n + (C_{8n})^3 B_n}\right] B_n$$

These values are then substituted in the equations below

$$p_n = a_n \cos n\theta$$
$$q_n = b_n \sin n\theta$$

For definition and evaluation of the coefficients $C_{1n}, \ldots, C_{8n}$ see Appendix B.

Reference

2. Timoshenko and Goodier. Theory of Elasticity. McGraw-Hill, 2nd Ed., 1951.

Appendix B

COMPUTER CALCULATIONS OF COEFFICIENTS $C_{1n}, C_{2n}, \ldots, C_{8n}$

Recall from Appendix A

$$a_n = \frac{2DE_1}{a\,\overset{u}{\gamma}_n} A_n \tag{46a}$$

$$b_n = \frac{2DE_1}{a\,\overset{v}{\gamma}_n} \beta_n\, \beta_n \tag{46b}$$

where

$$E_1 = \frac{E}{(1 - \nu^2)}$$

$$D = \alpha^{-2n} + \alpha^{2n} - n^2\,(\alpha^{-2} + \alpha^2) + 2\,(n^2 - 1)$$

$$\beta_n = \frac{M_1 A_n + M_2 B_n}{M_3 A_n + M_4 B_n}$$

$$\overset{u}{\gamma}_n = M_5 + M_6 \beta_n$$

$$\overset{v}{\gamma}_n = M_7 + M_8 \beta_n$$

and where

$$M_1 = \Big[-n(n-1)\alpha^{n-2} + n(n+1)\alpha^{-n-2} + (n+1)(n-2)\alpha^n - (n-1)(n+2)\alpha^{-n}\Big]$$

$$M_2 = \Big[n(n-1)\alpha^{n-2} + n(n+1)\alpha^{-n-2} - n(n+1)\alpha^n - n(n-1)\alpha^{-n}\Big]$$

$$M_3 = \Big[-(n-1)(n+2)\alpha^{n-2} - (n-2)(n+1)\alpha^{-n-2} + (n+1)(n-2)\alpha^n + (n-1)(n+2)\alpha^{-n}\Big]$$

$$M_4 = \Big[(n-1)(n+2)\alpha^{n-2} - (n-2)(n+1)\alpha^{-n-2} - n(n+1)\alpha^n + n(n-1)\alpha^{-n}\Big]$$

$$M_5 = \left[\frac{n}{(n+1)}\alpha^{n-2} + \frac{n}{(n-1)}\alpha^{-n-2} - \frac{n}{(n-1)}\alpha^n - \frac{n}{(n+1)}\alpha^{-n}\right]$$

$$M_6 = \left[-\frac{(n+2)}{(n+1)}\alpha^{n-2} + \frac{(n-2)}{(n-1)}\alpha^{-n-2} + \frac{n}{(n-1)}\alpha^n - \frac{n}{(n+1)}\alpha^{-n}\right]$$

$$M_7 = \left[\frac{n}{(n+1)}\alpha^{n-2} - \frac{n}{(n-1)}\alpha^{-n-2} - \frac{(n-2)}{(n-1)}\alpha^n + \frac{(n+2)}{(n+1)}\alpha^{-n}\right]$$

$$M_8 = \left[-\frac{(n+2)}{(n+1)}\alpha^{n-2} - \frac{(n-2)}{(n-1)}\alpha^{-n-2} + \frac{(n-2)}{(n-1)}\alpha^n + \frac{(n+2)}{(n+1)}\alpha^{-n}\right]$$

Eqs. 46a and 46b now become

$$a_n = \frac{E}{2a(1-\nu^2)}\left[\frac{(C_{1n})^3 A_n + (C_{2n})^3 B_n}{(C_{3n})^3 A_n + (C_{4n})^3 B_n}\right] A_n \tag{47a}$$

$$b_n = \frac{E}{2a(1-\nu^2)}\left[\frac{(C_{5n})^3 A_n + (C_{6n})^3 B_n}{(C_{7n})^3 A_n + (C_{8n})^3 B_n}\right] B_n \tag{47b}$$

where the $C_{1n}, \ldots, C_{8n}$ are as given below and are displayed in the charts which follow.

$$C_{1n} = (DM_3)^{1/3} \tag{48a}$$

$$C_{2n} = (DM_4)^{1/3} \tag{48b}$$

$$C_{3n} = (M_3 M_5 + M_1 M_6)^{1/3} \tag{48c}$$

$$C_{4n} = (M_4 M_5 + M_2 M_6)^{1/3} \tag{48d}$$

$$C_{5n} = (DM_1)^{1/3} \tag{48e}$$

$$C_{6n} = (DM_4)^{1/3} \tag{48f}$$

$$C_{7n} = (M_1 M_8 + M_3 M_8)^{1/3} \tag{48g}$$

$$C_{8n} = (M_2 M_8 + M_4 M_7)^{1/3} \tag{48h}$$

```
C           CALCULATION OF CONSTANTS  C1 THRU C8


      PUNCH 100
  100 FORMAT(1X,1HN,1X,5HALPHA,5X,2HC1,6X,2HC2,6X,2HC3,6X,2HC4,6X,2HC5
     16X,2HC6,6X,2HC7,6X,2HC8)
      DIMENSION A(20)
      G=1./3.
      READ 1, (A(I), I=1,20)
    1 FORMAT (20F4.2)
      DO 2 N=2,9
      DO 2 I=1,20
      X=N
      V=X-2.
      W=X-1.
      Y=X+1.
      Z=X+2.
      DON=A(I)**(-2.*X)+A(I)**(2.*X)-(X**2.)*((A(I)**(-2.))+(A(I)**2.))+
     12.*((X**2.)-1.)
      B=A(I)**V
      C=A(I)**(-Z)
      D=A(I)**X
      E=A(I)**(-X)
      XN1N=-X*W*B    +X*Y*C    +Y*V*D    -W*Z*E
      XN2N=X*W*B    +X*Y*C    -X*Y*D    -X*W*E
      XN3N=-W*Z*B    -V*Y*C    +Y*V*D    +W*Z*E
      XN4N=W*Z*B    -V*Y*C    -X*Y*D    +X*W*E
      XN5N = ((X*B)/Y)+((X*C)/W)-((X*D)/W)-((X*E)/Y)
      XN6N = -((Z*B)/Y)+((V*C)/W)+((X*D)/W)-((X*E)/Y)
      XN7N = ((X*B)/Y)-((X*C)/W)-((V*D)/W)+((Z*E)/Y)
      XN8N = -((Z*B)/Y)-((V*C)/W)+((V*D)/W)+((Z*E)/Y)
      CAU=DON*XN3N
      IF(CAU)11,12,13
11    CAU=CAU*(-1.)
      C1=CAU**G
      C1=C1*(-1.)
      GO TO 4
12    C1=0
      GO TO 4
13    C1=CAU**G
4     CBU=DON*XN4N
      IF(CBU)15,16,17
15    CBU=CBU*(-1.)
      C2=CBU**G
      C2=C2*(-1.)
      GO TO 5
16    C2=0
      GO TO 5
17    C2=CBU**G
5     CAL=XN5N*XN3N+XN6N*XN1N
      IF(CAL)18,19,20
18    CAL=CAL*(-1.)
      C3=CAL**G
      C3=C3*(-1.)
```

```
      GO TO 6
19    C3=0
      GO TO 6
20    C3=CAL**G
6     CBL =(XN5N*XN4N+XN6N*XN2N)
      IF(CBL)21,22,23
21    CBL=CBL*(-1.)
      C4=CBL**G
      C4=C4*(-1.)
      GO TO 7
22    C4=0
      GO TO 7
23    C4=CBL**G
    7 CA2U = DON * XN1N
      IF(CA2U)24,25,26
24    CA2U=CA2U*(-1.)
      C5=CA2U**G
      C5=C5*(-1.)
      GO TO 8
25    C5=0
      GO TO 8
26    C5=CA2U**G
    8 CB2U = DON * XN2N
      IF(CB2U) 27, 28, 29
   27 CB2U = CB2U * (-1.)
      C6 = CB2U ** G
      C6 = C6 * (-1.)
      GO TO 9
   28 C6 = 0
      GO TO 9
   29 C6 = CB2U ** G
    9 CA2L=XN1N*XN8N+XN3N*XN7N
      IF (CA2L) 30, 31, 32
   30 CA2L = CA2L * (-1.)
      C7 = CA2L ** G
      C7 = C7 * (-1.)
      GO TO 10
   31 C7 = 0
      GO TO 10
   32 C7 = CA2L ** G
   10 CB2L = XN2N * XN8N + XN4N* XN7N
      IF (CB2L) 33, 34, 35
   33 CB2L=CB2L*(-1.)
      C8 = CB2L ** G
      C8 = C8 * (-1.)
      GO TO 2
   34 C8 = 0
      GO TO 2
   35 C8 = CB2L ** G
    3 FORMAT (I2, F6.4, 8F8.3)
    2 PUNCH 3,N,A(I),C1,C2,C3,C4,C5,C6,C7,C8
      CALL EXIT
      END
+.75+.76+.77+.78+.79+.80+.81+.82+.83+.84+.85+.86+.87+.88+.89+
 .90+.91+.92+.93+.'
```

PROGRAM OUTPUT

N	ALPHA	C1	C2	C3	C4	C5	C6	C7	C8
2	.7500	.711	.785	.937	.010	1.044	1.175	.010	.937
2	.7600	.653	.725	.871	.010	.952	1.076	-.006	.871
2	.7700	.598	.668	.807	.012	.866	.982	-.008	.807
2	.7800	.546	.613	.747	.010	.785	.894	.004	.747
2	.7900	.497	.561	.690	.010	.710	.812	-.005	.690
2	.8000	.451	.512	.635	0.000	.640	.735	.007	.635
2	.8100	.408	.465	.582	-.006	.574	.662	.004	.582
2	.8200	.367	.421	.533	0.000	.513	.594	0.000	.533
2	.8300	.329	.379	.485	0.000	.457	.531	0.000	.485
2	.8400	.293	.339	.440	.008	.404	.472	-.007	.440
2	.8500	.259	.302	.397	.008	.355	.417	-.006	.397
2	.8600	.227	.267	.356	.007	.310	.365	-.006	.356
2	.8700	.198	.233	.318	0.000	.268	.318	0.000	.318
2	.8800	.171	.202	.281	.008	.230	.274	-.007	.281
2	.8900	.146	.174	.246	.004	.195	.233	-.006	.246
2	.9000	.123	.147	.214	.007	.163	.196	.002	.214
2	.9100	.101	.122	.183	.007	.134	.162	-.005	.183
2	.9200	.082	.099	.154	.005	.108	.131	.002	.154
2	.9300	.065	.079	.127	.005	.085	.103	0.000	.127
2	.9400	.049	.060	.102	.002	.064	.079	.002	.102
3	.7500	.912	-.524	1.734	-.016	2.603	2.972	-.014	1.734
3	.7600	.887	.513	1.609	-.012	2.356	2.707	-.008	1.609
3	.7700	.850	.660	1.490	.008	2.127	2.459	-.014	1.490
3	.7800	.805	.708	1.377	.015	1.916	2.229	-.014	1.377
3	.7900	.756	.716	1.269	.012	1.720	2.015	-.011	1.269
3	.8000	.704	.703	1.167	.012	1.539	1.816	-.010	1.167
3	.8100	.650	.676	1.069	-.007	1.372	1.631	-.007	1.069
3	.8200	.596	.640	.977	-.007	1.218	1.458	-.008	.977
3	.8300	.543	.599	.889	.007	1.076	1.299	-.006	.889
3	.8400	.490	.554	.806	0.000	.945	1.150	-.009	.806
3	.8500	.439	.507	.726	.006	.825	1.013	-.007	.726
3	.8600	.390	.459	.651	.005	.716	.886	-.005	.651
3	.8700	.343	.411	.580	.005	.615	.769	-.010	.580
3	.8800	.299	.364	.513	.010	.524	.661	-.008	.513
3	.8900	.257	.318	.449	0.000	.441	.562	-.005	.449
3	.9000	.217	.273	.389	-.008	.367	.471	-.005	.389
3	.9100	.181	.231	.333	-.007	.300	.389	.004	.333
3	.9200	.147	.191	.280	-.006	.240	.315	-.005	.280
3	.9300	.117	.153	.231	.005	.187	.248	-.007	.231
3	.9400	.089	.119	.185	-.005	.141	.189	-.005	.185

N	ALPHA	C1	C2	C3	C4	C5	C6	C7	C8
4	.7500	-2.548	-3.369	2.659	-.025	5.158	5.828	.026	2.659
4	.7600	-2.208	-2.972	2.461	.018	4.634	5.272	.012	2.461
4	.7700	-1.899	-2.610	2.274	.014	4.155	4.761	-.012	2.274
4	.7800	-1.619	-2.278	2.097	.015	3.716	4.290	-.008	2.097
4	.7900	-1.363	-1.976	1.929	.015	3.313	3.856	.012	1.929
4	.8000	-1.130	-1.699	1.770	-.010	2.945	3.456	-.008	1.770
4	.8100	-.914	-1.445	1.620	-.014	2.608	3.088	-.004	1.620
4	.8200	-.711	-1.213	1.477	-.010	2.300	2.748	-.008	1.477
4	.8300	-.507	-.999	1.343	-.008	2.019	2.436	-.011	1.343
4	.8400	-.220	-.800	1.215	-.014	1.762	2.149	-.010	1.215
4	.8500	.351	-.610	1.094	-.009	1.529	1.885	-.013	1.094
4	.8600	.392	-.414	.980	.009	1.317	1.642	.007	.980
4	.8700	.387	.185	.872	-.011	1.125	1.420	-.007	.872
4	.8800	.363	.334	.770	0.000	.952	1.217	-.010	.770
4	.8900	.328	.351	.674	-.010	.797	1.032	.008	.674
4	.9000	.289	.335	.584	-.010	.657	.863	.006	.584
4	.9100	.248	.303	.499	.007	.533	.711	.007	.499
4	.9200	.206	.263	.420	-.010	.424	.574	.004	.420
4	.9300	.167	.220	.346	-.011	.328	.452	-.005	.346
4	.9400	.129	.176	.277	-.008	.245	.344	-.002	.277
5	.7500	-5.628	-6.946	3.731	0.000	9.084	10.118	.027	3.731
5	.7600	-4.920	-6.144	3.443	.021	8.102	9.084	.023	3.443
5	.7700	-4.285	-5.418	3.172	-.027	7.211	8.142	-.018	3.172
5	.7800	-3.716	-4.761	2.917	.021	6.404	7.285	-.030	2.917
5	.7900	-3.206	-4.167	2.677	.021	5.671	6.504	.029	2.677
5	.8000	-2.749	-3.630	2.451	-.022	5.007	5.792	.020	2.451
5	.8100	-2.342	-3.145	2.238	.012	4.406	5.143	-.015	2.238
5	.8200	-1.978	-2.707	2.037	.010	3.861	4.552	.016	2.037
5	.8300	-1.655	-2.312	1.848	.023	3.368	4.012	-.015	1.848
5	.8400	-1.367	-1.957	1.669	-.018	2.922	3.521	.013	1.669
5	.8500	-1.112	-1.638	1.501	.020	2.520	3.073	0.000	1.501
5	.8600	-.886	-1.352	1.343	.014	2.158	2.666	-.009	1.343
5	.8700	-.685	-1.096	1.193	-.008	1.833	2.296	-.011	1.193
5	.8800	-.504	-.868	1.053	.010	1.542	1.960	-.010	1.053
5	.8900	-.333	-.663	.920	.007	1.282	1.655	-.009	.920
5	.9000	.054	-.478	.796	.005	1.051	1.381	-.009	.796
5	.9100	.222	-.297	.680	.010	.848	1.134	-.009	.680
5	.9200	.222	.162	.572	.010	.669	.913	.003	.572
5	.9300	.195	.222	.471	-.010	.515	.717	-.005	.471
5	.9400	.160	.205	.377	.006	.382	.544	.005	.377

N	ALPHA	C1	C2	C3	C4	C5	C6	C7	C8
6	.7500	-10.279	-12.249	4.978	.043	14.890	16.373	.039	4.978
6	.7600	-8.959	-10.786	4.577	.041	13.174	14.574	-.027	4.577
6	.7700	-7.785	-9.475	4.202	-.039	11.636	12.956	0.000	4.202
6	.7800	-6.741	-8.301	3.852	-.031	10.257	11.501	.031	3.852
6	.7900	-5.815	-7.249	3.524	.027	9.019	10.190	.021	3.524
6	.8000	-4.992	-6.307	3.217	.034	7.908	9.009	.029	3.217
6	.8100	-4.264	-5.464	2.930	-.027	6.912	7.944	-.021	2.930
6	.8200	-3.619	-4.709	2.660	.021	6.018	6.984	.026	2.660
6	.8300	-3.050	-4.034	2.407	-.012	5.216	6.117	-.024	2.407
6	.8400	-2.548	-3.431	2.170	.027	4.499	5.336	.018	2.170
6	.8500	-2.108	-2.895	1.948	.019	3.857	4.631	.021	1.948
6	.8600	-1.723	-2.418	1.739	.020	3.284	3.996	-.012	1.739
6	.8700	-1.387	-1.995	1.543	0.000	2.773	3.424	-.013	1.543
6	.8800	-1.096	-1.622	1.359	.012	2.319	2.910	-.013	1.359
6	.8900	-.845	-1.293	1.187	.010	1.918	2.447	.012	1.187
6	.9000	-.629	-1.007	1.025	.020	1.564	2.034	.007	1.025
6	.9100	-.444	-.757	.875	.013	1.254	1.664	-.011	.875
6	.9200	-.282	-.541	.735	.010	.984	1.336	-.010	.735
6	.9300	-.096	-.353	.604	.013	.752	1.046	.007	.604
6	.9400	.149	-.168	.484	.011	.555	.792	-.008	.484
7	.7500	-17.193	-19.975	6.434	.079	23.277	25.328	.027	6.434
7	.7600	-14.894	-17.467	5.890	.046	20.418	22.337	.021	5.890
7	.7700	-12.871	-15.246	5.385	-.046	17.885	19.680	0.000	5.385
7	.7800	-11.091	-13.278	4.917	.050	15.640	17.319	-.021	4.917
7	.7900	-9.526	-11.535	4.483	.041	13.647	15.218	0.000	4.483
7	.8000	-8.149	-9.989	4.079	0.000	11.878	13.347	.031	4.079
7	.8100	-6.939	-8.619	3.702	0.000	10.308	11.679	.021	3.702
7	.8200	-5.877	-7.404	3.352	.034	8.914	10.192	.026	3.352
7	.8300	-4.947	-6.327	3.025	-.021	7.677	8.866	.024	3.025
7	.8400	-4.134	-5.375	2.720	0.000	6.579	7.682	-.022	2.720
7	.8500	-3.425	-4.532	2.435	.028	5.606	6.626	-.018	2.435
7	.8600	-2.808	-3.789	2.170	.020	4.745	5.683	-.022	2.170
7	.8700	-2.275	-3.135	1.921	.028	3.985	4.843	-.012	1.921
7	.8800	-1.815	-2.562	1.689	-.015	3.314	4.093	.020	1.689
7	.8900	-1.422	-2.061	1.473	.024	2.726	3.427	.011	1.473
7	.9000	-1.088	-1.626	1.271	.022	2.211	2.835	.013	1.271
7	.9100	-.807	-1.251	1.083	.017	1.763	2.310	-.016	1.083
7	.9200	-.574	-.930	.908	-.008	1.376	1.848	-.012	.908
7	.9300	-.381	-.659	.746	-.008	1.046	1.442	-.010	.746
7	.9400	-.220	-.433	.597	-.014	.767	1.090	-.012	.597

N	ALPHA	C1	C2	C3	C4	C5	C6	C7	C8
8	.7500	-27.268	-31.061	8.145	-.084	35.219	38.000	.088	8.145
8	.7600	-23.443	-26.933	7.418	-.084	30.606	33.181	-.058	7.418
8	.7700	-20.116	-23.323	6.751	.073	26.570	28.956	0.000	6.751
8	.7800	-17.221	-20.163	6.137	0.000	23.034	25.247	.069	6.137
8	.7900	-14.700	-17.395	5.571	0.000	19.934	21.987	0.000	5.571
8	.8000	-12.506	-14.968	5.049	.031	17.214	19.119	.039	5.049
8	.8100	-10.596	-12.840	4.566	-.027	14.826	16.593	.027	4.566
8	.8200	-8.935	-10.972	4.120	0.000	12.729	14.366	-.027	4.120
8	.8300	-7.492	-9.334	3.706	-.027	10.886	12.402	.027	3.706
8	.8400	-6.241	-7.898	3.323	-.021	9.268	10.669	.029	3.323
8	.8500	-5.158	-6.639	2.967	-.027	7.848	9.139	-.018	2.967
8	.8600	-4.224	-5.537	2.637	-.027	6.603	7.788	.021	2.637
8	.8700	-3.420	-4.575	2.330	-.023	5.512	6.595	-.015	2.330
8	.8800	-2.733	-3.738	2.044	.015	4.560	5.543	.017	2.044
8	.8900	-2.148	-3.011	1.779	.010	3.729	4.615	.014	1.779
8	.9000	-1.654	-2.383	1.533	.021	3.009	3.799	.019	1.533
8	.9100	-1.242	-1.846	1.304	-.023	2.387	3.082	-.011	1.304
8	.9200	-.900	-1.388	1.092	.019	1.854	2.455	.016	1.092
8	.9300	-.623	-1.004	.897	.013	1.402	1.909	.013	.897
8	.9400	-.401	-.687	.717	.010	1.022	1.438	-.003	.717
9	.7500	-41.737	-46.801	10.167	-.100	52.060	55.792	-.058	10.167
9	.7600	-35.574	-40.199	9.208	-.079	44.791	48.208	-.092	9.208
9	.7700	-30.277	-34.501	8.334	.084	38.511	41.645	.046	8.334
9	.7800	-25.720	-29.575	7.538	-.066	33.080	35.956	.046	7.538
9	.7900	-21.797	-25.311	6.810	.109	28.376	31.019	.046	6.810
9	.8000	-18.419	-21.617	6.144	-.066	24.297	26.728	.043	6.145
9	.8100	-15.509	-18.413	5.534	0.000	20.758	22.994	-.031	5.534
9	.8200	-13.004	-15.634	4.974	-.021	17.684	19.741	.039	4.974
9	.8300	-10.848	-13.221	4.459	-.034	15.014	16.905	.027	4.459
9	.8400	-8.994	-11.126	3.984	.021	12.693	14.431	0.000	3.984
9	.8500	-7.404	-9.309	3.547	0.000	10.677	12.270	.024	3.547
9	.8600	-6.042	-7.733	3.143	.031	8.926	10.383	.021	3.143
9	.8700	-4.880	-6.368	2.770	.031	7.407	8.735	-.017	2.770
9	.8800	-3.892	-5.189	2.424	-.022	6.092	7.295	.017	2.424
9	.8900	-3.058	-4.174	2.105	-.012	4.955	6.039	-.012	2.105
9	.9000	-2.357	-3.304	1.810	-.025	3.977	4.944	-.009	1.810
9	.9100	-1.774	-2.562	1.538	-.025	3.139	3.991	-.013	1.538
9	.9200	-1.295	-1.934	1.286	-.010	2.426	3.165	-.004	1.286
9	.9300	-.907	-1.410	1.054	.010	1.825	2.452	.011	1.054
9	.9400	-.600	-.979	.842	-.021	1.324	1.840	.007	.842