# A Theoretical Model for Determination of Expressway Usage in a Uniform Region

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•THE purpose of this paper is to present the solution to the problem of computing the theoretical "usage," in terms of mean trip density, of certain high-speed facilities that are placed in a given region containing trip origins and destinations. Certain results based on this solution are also presented. The types of facilities considered are very special, and the assumptions under which the problem is solved are quite restrictive. However, neither the specialization nor the restrictions should completely negate the applicability of the results to certain phases of the transportation planning process.

As the areas in which formal transportation studies are undertaken become larger and more complex, the limits of present planning tools become more clearly defined. These tools rest heavily upon computer simulation techniques—techniques which are unwieldy, time-consuming, and expensive at best, and which are not (at present) wholly applicable in the super-regions for which transportation planning is being attempted.

There are problem areas for which the computer is inadequate for solutions, at least at present. Among them are the basic ones of formulating alternative plans to be tested, of settling upon a desirable network geometry (to say nothing about an optimal geometry), of delineating a reasonable range of facility spacings to be more closely examined, and of solving other problems involving far too many combinations to be dealt with by methods of exhaustion. Now the machine is obviously necessary for any completely practical solution to these hypercomplex problems. The number of cases in which it is not sufficient may possibly be reduced by taking a closer look at the concepts involved in the simulation models we use. One way of doing this is to examine the consequences of the hypotheses of a model in various hypothetical control situations, with the hope that our insight into the relationships implied by the model may be increased. It is toward this gain in understanding that this paper is directed, as have been other papers in recent years (see References).

While the results of these efforts may never be used in actually locating a highway or transit line, they should give some insight into the behavior of trip distribution functions and the relationship between the various parameters inherent in trip-making patterns. It is possible that guidelines for computer model development will be found in them and even that certain broad planning decisions may be based on them.

The basic types of expressway "networks" considered here are (a) an isolated expressway with unlimited access, (b) a sheaf of parallel expressways with unlimited access, (c) an isolated pair of parallel expressways with unlimited access, and (d) an isolated expressway with limited access. The usage of these types of expressways is established rigorously under the following assumptions:

1. The given region is of constant (vehicle) trip density;
2. A constant speed is allowed on the expressway(s);
3. Unfettered movement at a constant speed is allowed throughout the remainder of the region, except that trip distance is measured as right-angle distance and all movement is at right angles;
4. A trip will follow the most economical route at all times: if costs are equal for a non-expressway route and an expressway route, then the latter will be used; if costs

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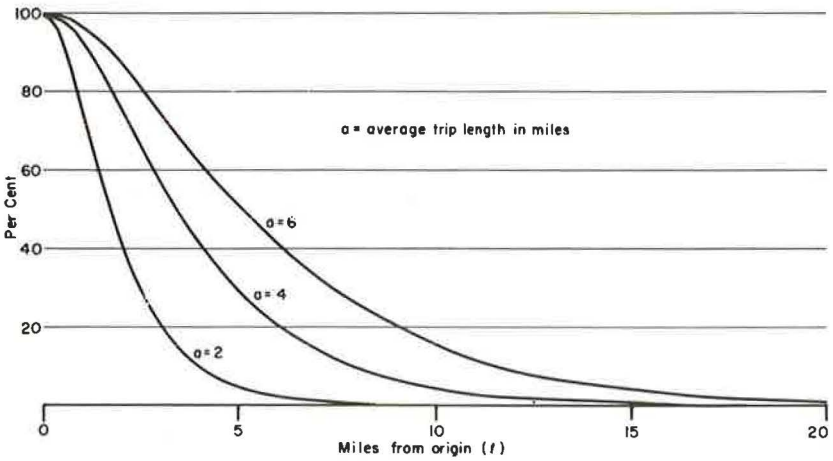


Figure 1. Percent of trips undelivered at t miles from origin.

are equal for two or more expressway routes, then the trips involved are allocated in equal proportions to the expressways concerned; and

5. Trip length is u + v, where u and v are random variables with the joint density function $k^2 \exp[-k(u+v)]$. This assumption implies that the mean trip length is 2/k and that u and v each have the (marginal) density function k exp (-kt) and mean 1/k.

Perhaps the most restrictive of these assumptions are numbers 1 and 5. The assumption of constant trip density is necessary for my peculiar attack, because it assures the requisite uniformity of volume on the various expressway networks. However, there is nothing intrinsically valuable about the method. If another can be devised which will allow alterations in assumption 1, a much more practical set of solutions will result. Of particular interest would be a solution for a region containing a finite number of "point" generators of trips.

There are a priori arguments that can be devised for the use of the trip length frequency function defined in assumption 5 (one can be found in Ref. 1). I choose the argument that this is one of the very few functions amenable to my purpose that even crudely describes available trip data. Examples of this trip length function are given in Figure 1 for various values of k.

A presentation of results and a discussion of some of the applications of these results are given in the next section. There are many implications possible which are only hinted at here (or not mentioned at all). Because a paper of this type is marginally digestible at best, it seemed discreet to hold the list of ramifications to a minimum.

## RESULTS AND APPLICATIONS

There are several parameters on which the volumes for all expressway networks considered here depend—namely, expressway speed, arterial speed, average trip length for the region, density for the region, value of time, and operating and accident costs at expressway and arterial speeds. In addition, there are special parameters on which the volumes for some of the special types of networks depend. For example, in the case of an isolated pair of expressways, the distance between them is an explicit parameter, as is the distance between the expressways in a parallel sheaf. In the case of the isolated expressway with limited access, the ramp spacing is a parameter. Each of these parameters is identified in a later section.

Tables have been generated which display the various volumes. Certain data from these tables are presented here in graphical form, accompanied by a discussion. In addition, some obvious applications of these data are discussed and illustrated.

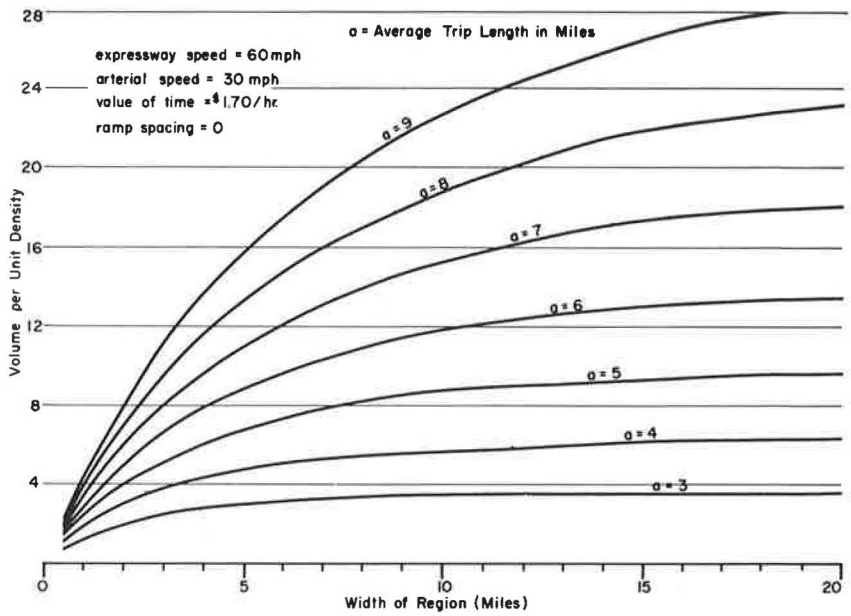


Figure 2. Isolated expressway volume vs width of region from which trips are drawn.

## Volume on Isolated Expressway vs Width of Region

Under the assumptions listed previously, the volume carried by an isolated expressway with unlimited access which draws trips from a band of trip origins extending c miles on either side of the expressway is given by

$$
\begin{aligned}
{}_cV = (D/4)a^2 \Big( & [(R + 2)/R] \\
& - \{[4R^2 - 7R + 2 + 4R(2R - 1)(R - 1)(c/a)]/[R(2R - 1)^2]\} \\
& \exp(-4Rc/a) - [4R^2/(2R - 1)^2] \exp(-2c/a) \Big)
\end{aligned}
$$

where

$2c$ = total width of region,
$D$ = trip density for the region,
$a$ = average trip length for the region,
$R = C_a/\left(C_a - C_e\right)$,
$C_a$ = trip cost/mile on arterials, and
$C_e$ = trip cost/mile on the expressway.

Another interpretation of this formula is that it gives the volumes carried by each member of a sheaf of parallel facilities spaced at 2c miles. In this situation, for each trip that originates in the c band for facility A but uses facility B, there will be a trip which originates in the c band for facility B but uses facility A. Thus, the formula gives the usage of each facility in the sheaf.

The curves in Figure 2 show the usage for both cases. We can identify the horizontal axis with the total width of the region for the first interpretation or with the expressway spacing for the second interpretation. In either case, the vertical axis represents mean trip density on the expressway(s).

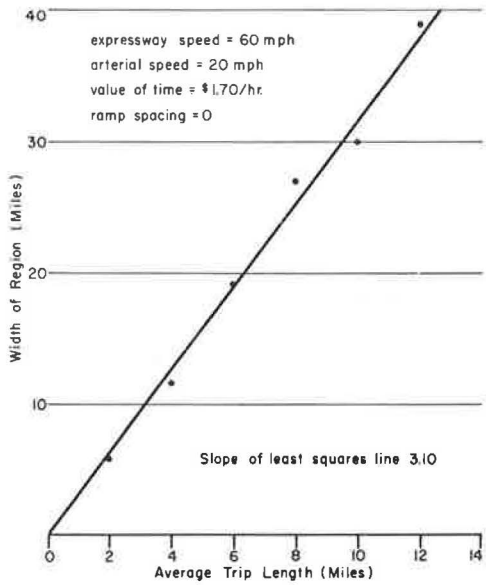


Figure 3. Width of region from which 95 percent of maximum expressway volume is drawn vs average trip length.

As c becomes infinite, $_cV$ approaches $_\infty V = (D/4)a^2 [ (R + 2)/R ]$ (which each curve in Figure 2 approaches as an asymptote for the appropriate value of a).

## Practical Meaning of "Isolated"

The term "isolated" has been used throughout this report to describe expressways. This term has been used in the sense that an expressway (expressway network) is isolated if no other expressway (expressway network) is near enough to compete with it for trips. In our theoretical world, this would imply that an expressway is isolated only if it is in an infinite region with no other expressways.

In order to have some practical measure of "near enough," the following calculations were made. The volume ($_\infty V$) which the expressway would carry if trips were drawn from an unbounded region was considered maximum; 95 percent of this maximum volume was calculated for each value of average trip length, after which the value of 2c (width of the region from which trips are drawn to the expressway) giving this proportion was found. (A linear interpolation was used, even though this is not strictly justifiable here. The error introduced is no more than the same type of interpolation would give in an ordinary table of logarithms.) This procedure was carried out for arterial speeds of 10, 20, 30, 40, and 50 miles per hour. Figure 3 shows the points that resulted for an arterial speed of 20 miles per hour. Because the relationship between 2c and a seemed essentially linear, a least squares line was fitted and included in Figure 3. The same was done for arterial speeds of 10, 30, 40, and 50 miles per hour. The slopes of the least square lines are 3.27, 3.05, 3.14, and 3.22 respectively, with a mean of 3.16.

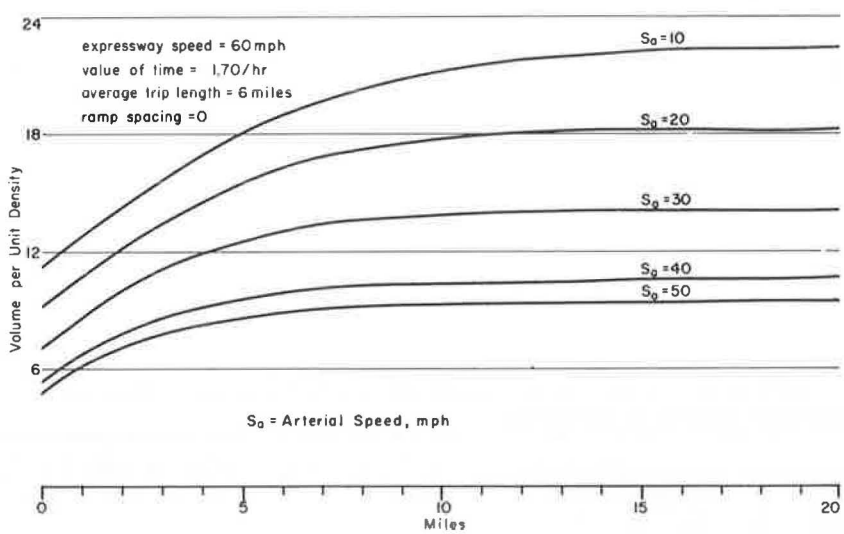


Figure 4. Volume on each of a pair of parallel expressways vs distance between expressways.

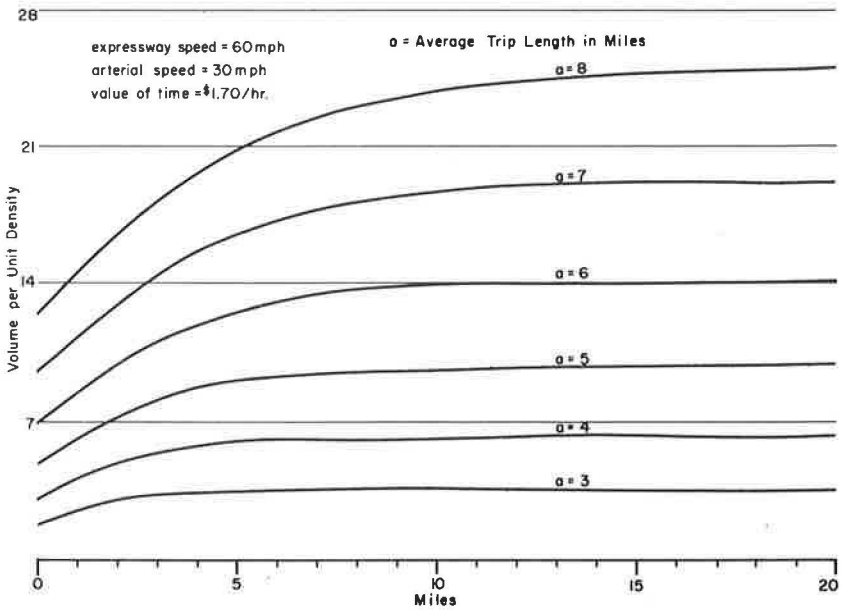


Figure 5. Volume on each of a pair of parallel expressways vs distance between expressways.

Thus, for all practical purposes an expressway could be called "isolated" in the sense needed here if there is no other expressway within approximately 3a, where a is the average trip length for the region.

## Expressway Volume vs Expressway Spacing

Figures 4 and 5 show the volume on each of an isolated pair of expressways as a function of the distance between them. In Figure 4, the average trip length has been held constant at 6 miles, while in Figure 5 the arterial speed has been fixed at 30 miles per hour. The formula in this case is

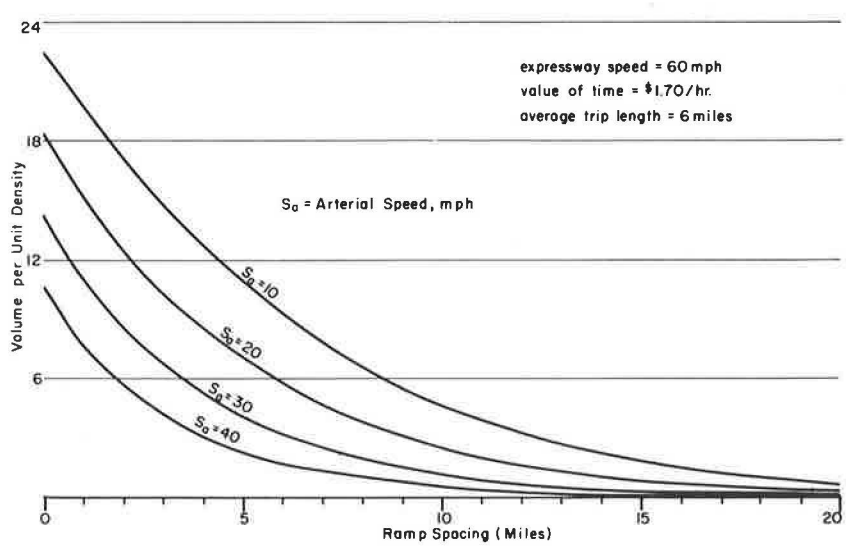


Figure 6. Expressway volume vs ramp spacing.

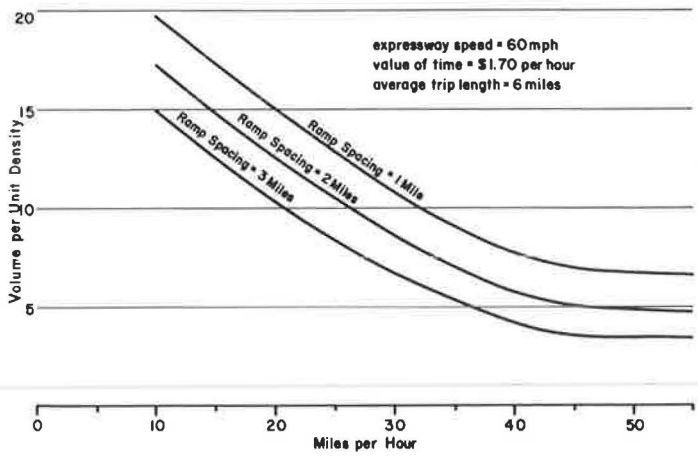


Figure 7. Expressway volume vs arterial speed.

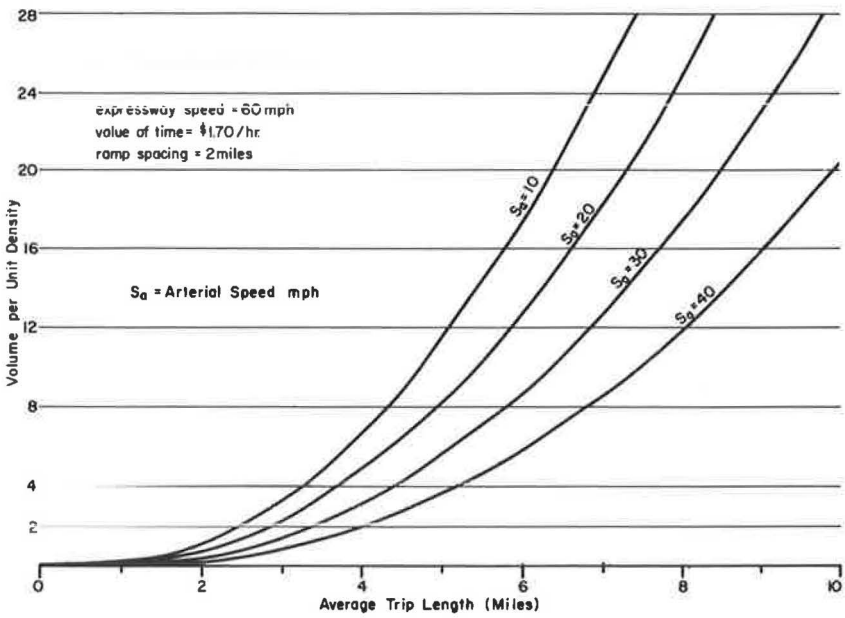


Figure 8. Expressway volume vs average trip length.

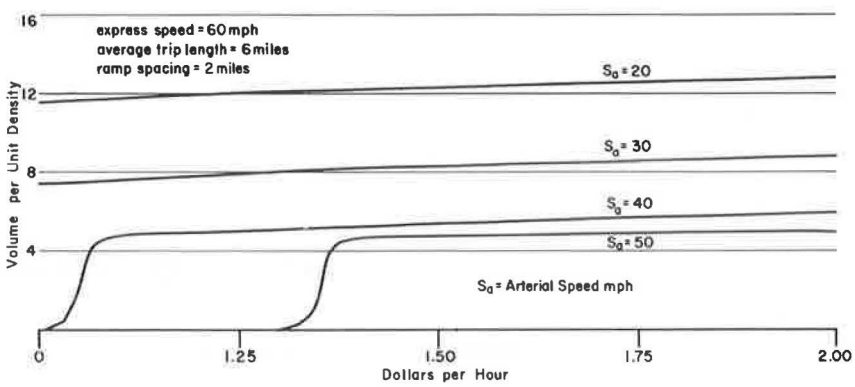


Figure 9. Expressway volume vs value of time.

$$_dV = (D/4)a^2 \Big( [(R + 2)/R] - [R^2/2(R - 1)^2] \exp(-2d/a)$$
$$+ \{ [3R - 2 + 2R(R - 1)d/a]/[2R(R - 1)^2] \} \exp(-2Rd/a) \Big)$$

As the distance between the two expressways becomes larger, the volume on each approaches as a maximum the volume ($_\infty V$) which a single isolated expressway would carry for the same choice of parameters. As d becomes smaller, the volume on each of the pair approaches as a minimum one-half of the maximum volume. Fortunately, this behavior is in agreement with intuition.

## Effect of Various Parameters on Expressway Volumes

Figures 6 through 9 show the relative dependence of expressway volumes on the various parameters involved: ramp spacing, arterial speed, average trip length, and value of time. The graphs should be self-explanatory. It is worth noting that average trip length is a particularly important parameter, but that volumes are not especially sensitive to the value of time (assuming, of course, that the value is high enough for the expressway to be used at all).

## Travel Cost Savings Due to Construction of Expressway

Figure 10 shows the savings in travel cost per trip due to the construction of an expressway in a region. The savings are graphed as a function of ramp spacing. The formula used to compute the savings S is easily derived:

$$S = C_a (Q \div D)/(wR)$$

where

$C_a$ = travel cost per mile at the given arterial speed,
$Q \div D$ = volume of trips past a point on the expressway per unit density of vehicle trips in the region,
$w$ = width of the region (region extends w/2 miles on either side of the expressway), and
$R = C_a/(C_a - C_e)$, where $C_e$ is the travel cost per mile at the given expressway speed.

Because the volumes on the expressway were obtained under the assumption that trips are drawn from an unbounded region, it was necessary to choose w large enough to guarantee that most (approximately 95 percent) of the volume comes from the bounded region under consideration. It was decided to use w = 18 miles because an average trip length of 6 miles was assumed for the region. The assumptions governing travel costs are given later.

## Expressway Construction Cost

The construction costs used here are entirely hypothetical. Under no circumstances should the cost analysis given in this report be regarded as anything but a series of examples of the use of the theory in answering certain standard questions. In order to make the examples meaningful, an attempt was made to use cost data that are at least faintly realistic, but a much more nearly precise application of the ideas illustrated here could be made in a particular economic situation.

At any rate, the expressway costs used to obtain the curves in Figure 11 are based on the following rules of thumb:

Right-of-way cost = \$(300,000 + 100D)/mile, where D = number of (vehicular) trips/mile$^2$;
Main line cost = \$1,400,000/mile;
Interchange costs = \$750,000 per ramp for D = 10,000; \$1,000,000 per ramp for D = 20,000;

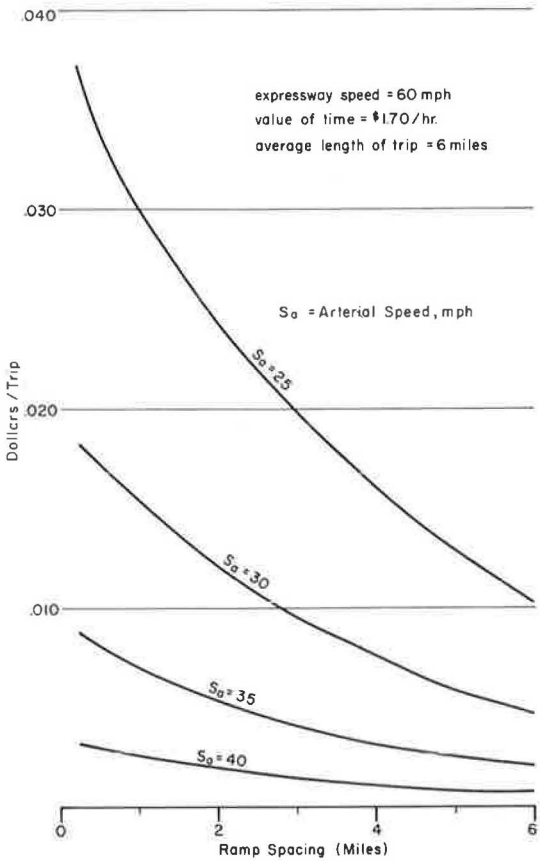


Figure 10. Savings per trip due to presence of expressway.

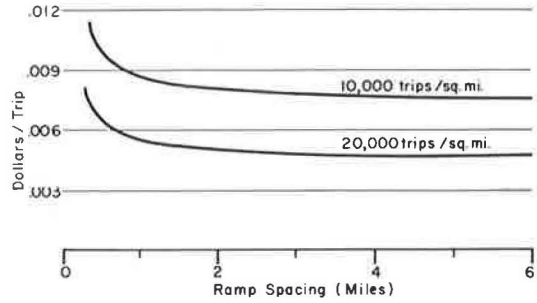


Figure 11. Expressway construction cost per trip vs ramp spacing.

Crossing cost = $1,250,000/mile (assuming 5 crossings per mile at $250,000 per crossing to maintain as closely as possible the "unfettered" movement required by the theory);

Capital recovery factor = .110127; and

Number of days per year = 340.

## Benefit/Cost vs Ramp Spacing

Figures 12 and 13 indicate that, from a benefit/cost standpoint, an optimum ramp spacing is determined that is somewhat greater than the minimum spacing dictated by flow theory considerations. Actually, the optimum ramp spacings suggested here are probably too small because no form of capacity restraint has been incorporated in the derivation of the expressway volumes. Because of the crudeness of the cost criteria, the lack of congestion considerations, and the overall restrictiveness of the hypotheses under which the expressway volumes are derived, any estimates of optimality made herein must be of the roughest sort. On the other hand, an analytical crutch is sometimes better to lean on than a less synthetic variant; at least one is propped up enough to identify and examine the assumptions involved.

Despite the foregoing disclaimer, I find it irresistible to point out that the optimum ramp spacing, whatever it may be in absolute terms, is very definite for low arterial speeds. Furthermore, this optimum seems to regress and to become less critical as arterial speeds increase.

## Benefit/Cost vs Expressway Spacing

In Figures 14, 15, 16, 17, and 18, average total trip cost was chosen as the criterion for optimum expressway spacing. Trip cost was defined as the sum of the cost per trip of expressway construction and the travel cost per trip. The comments made in reference to Figures 10, 11, 12, and 13 apply to the cost data used here.

Construction costs were defined as follows:

Right-of-way cost = $(300,000 + 100D)/mile;
Main line cost = $1,400,000/mile;
Interchange cost = $3,000,000/mile for D = 10,000,
= $4,000,000/mile for D = 20,000;
Crossing cost = $1,250,000/mile;

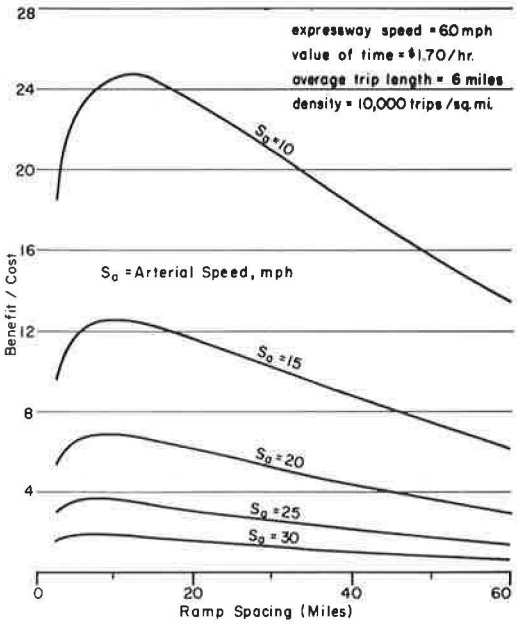


Figure 12. Benefit/cost vs ramp spacing.

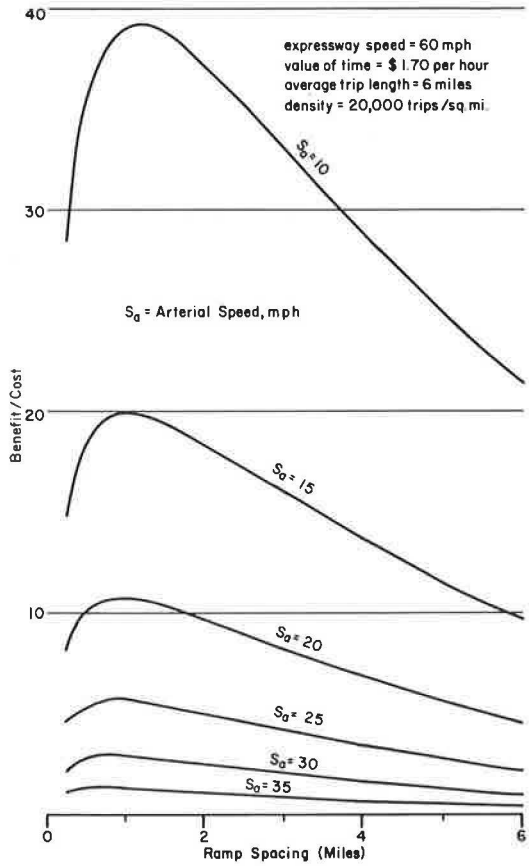


Figure 13. Benefit/cost vs ramp spacing.

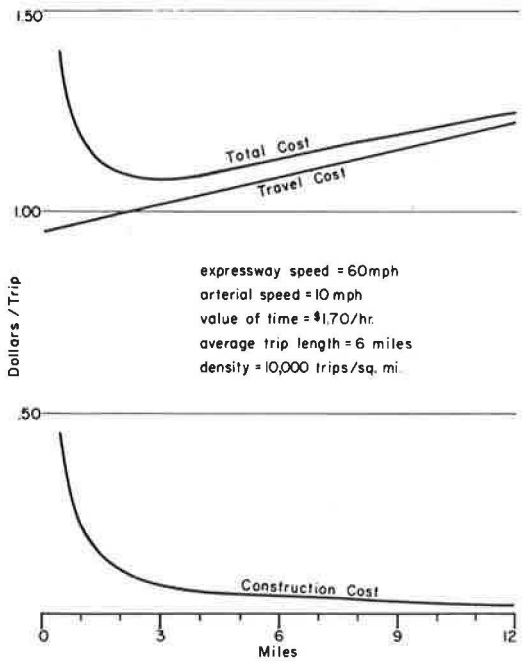


Figure 14. Trip cost vs distance between expressways.

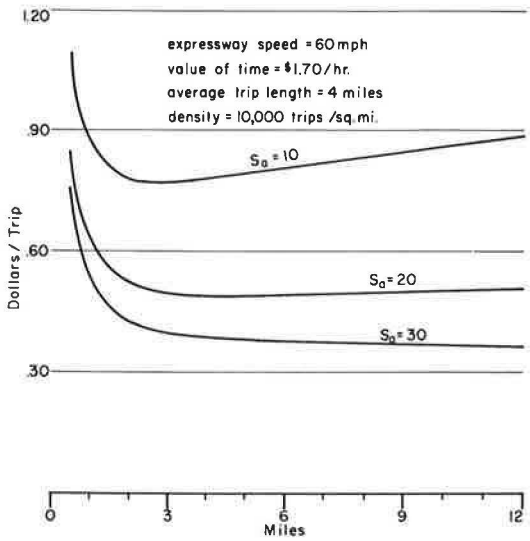


Figure 15. Trip cost vs distance between expressways.

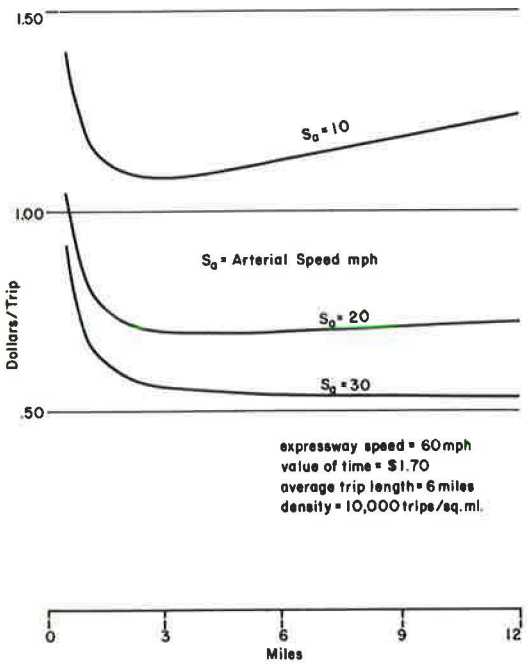


Figure 16. Trip cost vs distance between expressways.

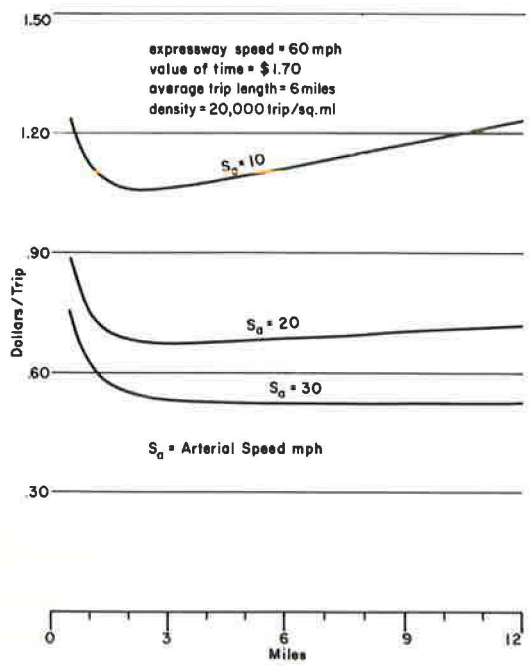


Figure 18. Trip cost vs distance between expressways.

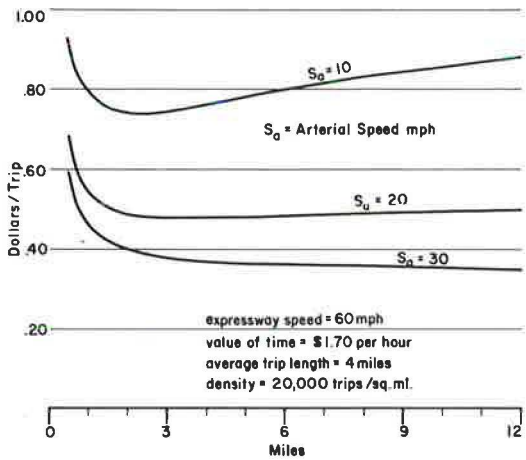


Figure 17. Trip cost vs distance between expressways.

Capital recovery factor = .110127; and
Number of days per year = 340

Travel costs were obtained from the formula

$$\text{Travel cost/trip} = C_a A - S$$

where A is the average trip length, $C_a$ is the travel cost/mile at the given arterial speed, and S is as defined in the discussion for Figure 10, except that Q has been replaced by $_cV$.

The components of the total trip cost are shown in Figure 14, but were omitted from the other figures in order to show more clearly the dependence of trip cost on the various parameters.

Again, the crudeness of the assumptions makes the drawing of conclusions about optimum spacing from the graphs a highly questionable practice. However, the hint (obscure as it may be) given in Figures 17 and 18 that optimum spacing in many cases is not at all critical, is one of several in the graphs of this report that may help to frame some sensible questions and to urge further research in the analysis of transportation systems. If this is true, then the real purpose of an investigation of this sort has been realized.

## DERIVATION OF FORMULAS

### An Isolated Expressway With Unlimited Access

It is convenient to choose coordinate axes so that the expressway coincides with

the x axis (see Fig. 19). Consider the element of area $\Delta A$ whose centroid is at (0,y), and assume that the trips originating from this element are concentrated at the centroid. Let the trip using the facility originate within c miles on either side of the facility (see Fig. 20). Each trip will be composed of a horizontal segment u and a vertical segment v, each of which can be considered a random variable of one dimension with distribution exp (-kt) and mean 1/k.

We shall classify trips according to the following scheme. Consider the world divided into two regions, 1 and 2 (see Fig. 21a). The trips that originate in region 1 can be typed as in Figure 21b. A trip in 1j will have its origin in region 1 and will be of type j. Note that trips in classes 21, 22, and 23 are mirror images, respectively, of classes 11, 12, and 13 with respect to the expressway. The volume contributed by trips in classes $2_j$ will be the same as the volume contributed by trips in classes $1_j$, so that we may concentrate on the classes $1_j$. These may be described analytically as follows:

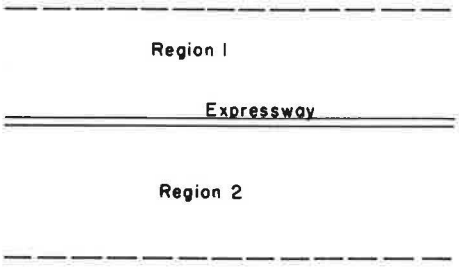


Figure 21a.

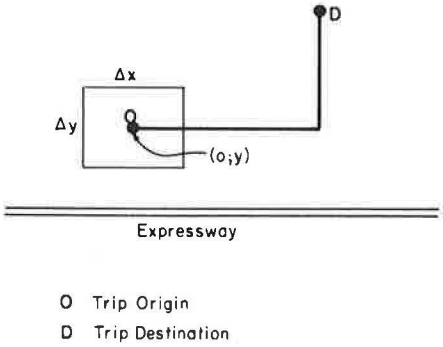


Figure 19.

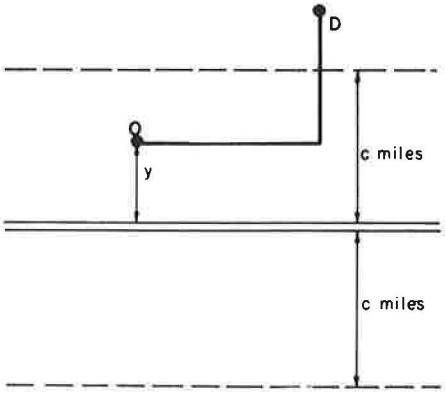


Figure 20.

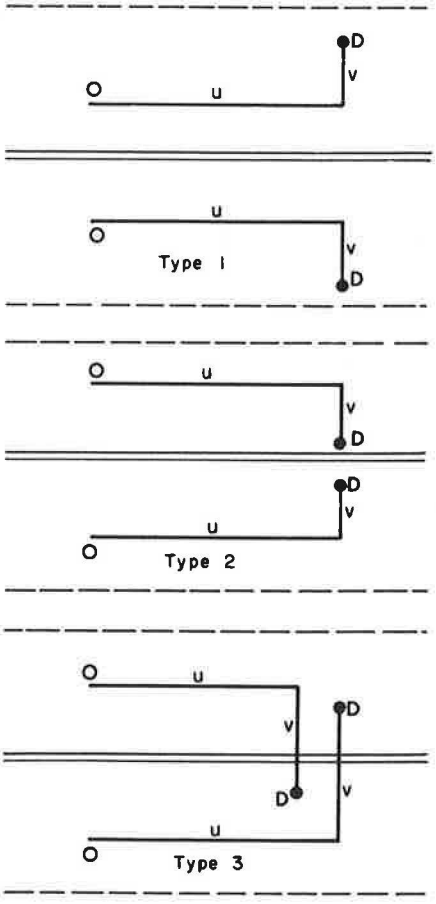


Figure 21b.

Class 11: D above 0

$y$ in ${}_cY_{11} = [0;c]$
$v$ in ${}_cV_{11} = [0;\infty]$

Class 12: d below 0

$y$ in ${}_cY_{12} = [0;c]$
$v$ in ${}_cV_{12} = [0;y]$

Class 13: d below 0

$y$ in ${}_cY_{13} = [0;c]$
$v$ in ${}_cV_{13} = [y;\infty]$

Let ${}_cP_{ij}(y) = \Pr[\text{trip is of class } ij]$ for a given y. Then

$$
\begin{aligned}
{}_cP_{11}(y) &= \Pr[D \text{ is above } 0]\Pr[0 \le v < \infty] = 1/2 \\
{}_cP_{12}(y) &= \Pr[D \text{ is below } 0]\Pr[0 \le v < y] = (1/2)[1 - \exp(-ky)] \\
{}_cP_{13}(y) &= \Pr[D \text{ is below } 0]\Pr[0 \le v < \infty] = (1/2)\exp(-ky)
\end{aligned}
$$

Let ${}_cQ_{ij}(y,v) = \Pr[\text{trip of class } ij \text{ will use expressway}]$ for a given y,v. Let $C_a$ = trip cost per mile at arterial speed and $C_e$ = trip cost per mile at expressway speed. (See note at the end of this paper for a discussion of trip cost as used here.) Then for trips of class 11, the total cost $c_a$ for a non-expressway route is $uC_a + vC_a$ and the total cost $c_e$ for the expressway route is $2yC_a + uC_e + vC_a$, so that $c_e \le c_a$ implies $u \ge 2Ry$, where $R = C_a/(C_a - C_e)$. Thus,

$$ {}_cQ_{11}(y,v) = \Pr[u \ge 2Ry] = \exp(-2Rky) $$

Similarly,

$$
\begin{aligned}
{}_cQ_{12}(y,v) &= \Pr[u \ge 2R(y - v)] = \exp(-2Rky)\exp(2Rkv) \\
{}_cQ_{13}(y,v) &= \Pr[u \ge 0] = 1
\end{aligned}
$$

Let ${}_cm_{ij}(y,v)$ = mean distance traveled on expressway by trip of class ij, given that the expressway is used.

We recall that, for a random variable x with the exponential distribution, the statement $\Pr[x \ge a + b \text{ given } x \ge b] = \Pr[x \ge a]$ obtains. Thus,

$$
\begin{aligned}
{}_cm_{11}(y,v) &= 1/k + 2Ry \\
{}_cm_{12}(y,v) &= (1/k + 2Ry) - 2Rv \\
{}_cm_{13}(y,v) &= 1/k
\end{aligned}
$$

Let ${}_cf_{ij}(y,v) = {}_cP_{ij}(y)\,{}_cQ_{ij}(y,v)\,{}_cm_{ij}(y,v)$, so that for each y, v, ${}_cf_{ij}(y,v)$ = VMT/trip of expressway for trips of class ij. We must find the mean ${}_c\bar{f}_{ij}(y)$ of ${}_cf_{ij}(y,v)$ with respect to v for each ij. Let ${}_cF_{ij}(t)$ be the density function for v for class ij. We have

$$
\begin{aligned}
{}_cF_{11}(t) &= k\exp(-kt),\ t \text{ in } {}_cT_{11} = [0;\infty] \\
{}_cF_{12}(t) &= \{1/[2\,{}_cP_{12}(y)]\}\,k\exp(-kt),\ t \text{ in } {}_cT_{12} = [0;y] \\
{}_cF_{13}(t) &= \{1/[2\,{}_cP_{13}(y)]\}\,k\exp(-kt),\ t \text{ in } {}_cT_{13} = [y;\infty]
\end{aligned}
$$

Then

$$ {}_c\bar{f}_{ij}(y_i) = \int_{{}_cT_{ij}} {}_cf_{ij}(y,t)\,{}_cF_{ij}(t)\,dt $$

so that

$$
\begin{aligned}
{}_c\bar{f}_{11}(y) &= [1/(2k)]\exp(-2Rky) + Ry\exp(-2Rky) \\
{}_c\bar{f}_{12}(y) &= \{(4R - 1)/[2(2R - 1)^2k]\}[\exp(-ky) - \exp(-2Rky)] \\
&\quad - [R/(2R - 1)^2]\,y\exp(-2Rky)
\end{aligned}
$$

$$_c\bar{f}_{13}(y) = [1/(2k)] \exp(-ky)$$

Let

$$_cS_{ij} = \int_{_cY_{ij}} {}_c\bar{f}_{ij}(y)\,dy$$

so that $_cS_{ij}$ is the VMT/trip on the expressway for all trips of class ij entering the expressway in $\Delta x$. We have

$$_cS_{11} = 1/(2Rk^2) - [(1 + Rkc)/2Rk^2] \exp(-2Rkc)$$
$$_cS_{12} = 1/(2Rk^2) + \{[3R - 1 + R(2R - 1)kc]/[2R(2R - 1)^2k^2]\} \exp(-2Rkc)$$
$$_cS_{13} = 1/(2k^2) - [1/(2k^2)] \exp(-kc)$$

Let

$$\Delta_c V = 2k({}_cS_{11} + {}_cS_{12} + {}_cS_{13})\Delta x,$$

so that $\Delta_c V$ = VMT on expressway for trips entering expressway in $\Delta x$. Then

$$_cV = \int_0^1 d_cV = \text{mean trip density on the expressway}$$

We have

$$_cV = (D/4)a^2\Big([(R + 2)/R] - \{[4R^2 - 7R + 2 + 4R(2R - 1)(R - 1)(c/a)]/[R(2R - 1)^2]\} \exp(-4Rc/a) - [4R^2/(2R - 1)^2] \exp(-2c/a)\Big)$$

where D is the trip density for the region, a is the average trip length for the region, and $R = C_a/(C_a - C_e)$.

A Pair of Parallel Expressways (With Unlimited Access)

We are concerned here with the case of two parallel expressways that are d miles apart in an unbounded region. We find it convenient in this case to partition the world into four regions (Fig. 22). Trips originating in the various regions are then classified according to their direction, whether or not they cross an expressway. These classes are described schematically in Figure 23 and analytically in the following (notation is that used earlier):

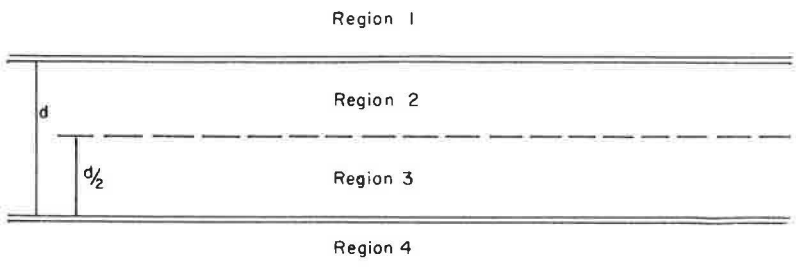


Figure 22.

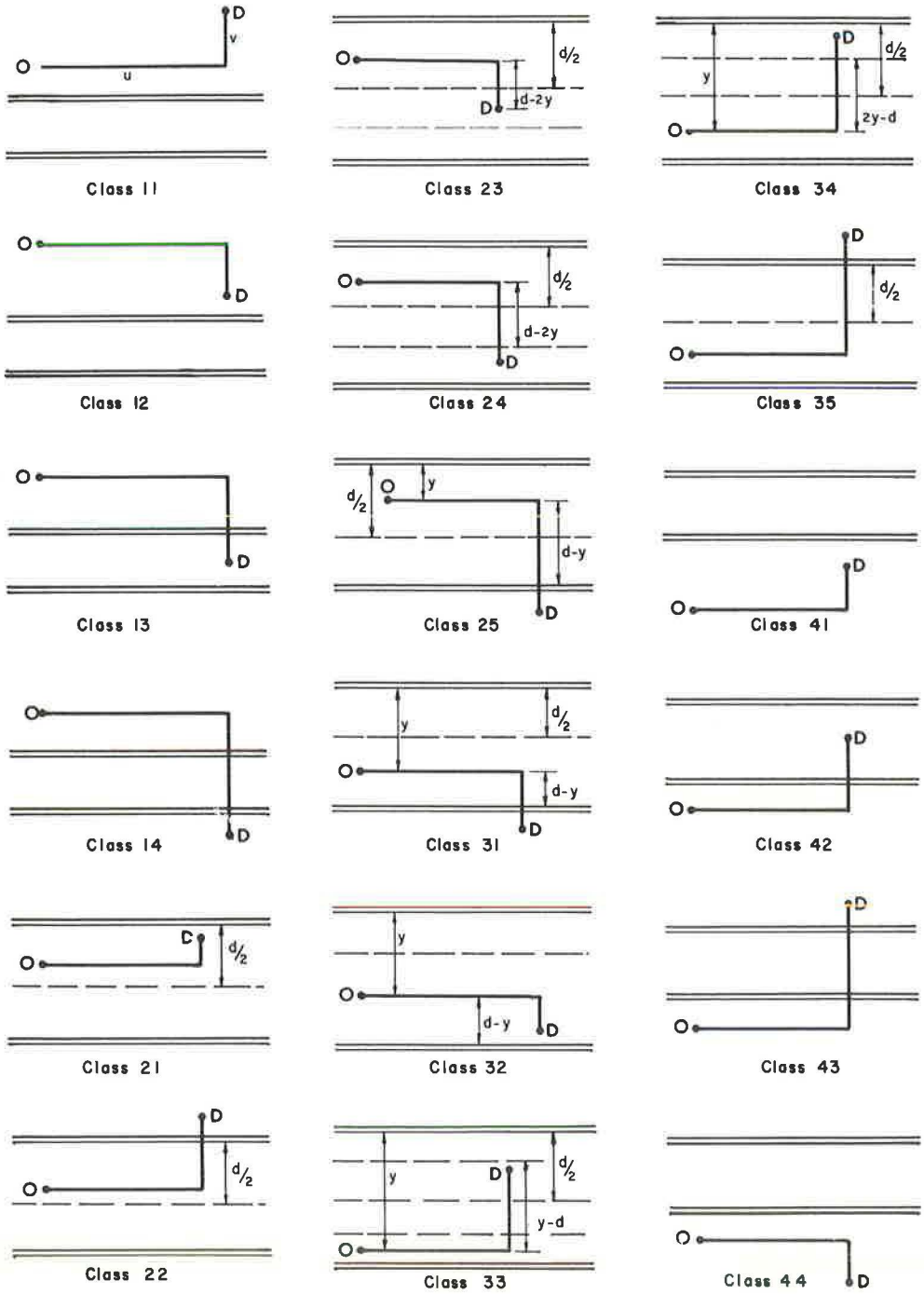
Class 11
Class 12
Class 13
Class 14
Class 21
Class 22
Class 23
Class 24
Class 25
Class 31
Class 32
Class 33
Class 34
Class 35
Class 41
Class 42
Class 43
Class 44
O
D
u
v
y
d/2
d-2y
2y-d
d-y
y-d

Figure 23.

Class 11: D above 0
y in ${}_dY_{11}$ = [0;∞]
v in ${}_dV_{11}$ = [0;∞]

Class 12: D below 0
y in ${}_dY_{12}$ = [0;∞]
v in ${}_dV_{12}$ = [0;y]

Class 13: D below 0
y in ${}_dY_{13}$ = [0;∞]
v in ${}_dV_{13}$ = [y;y + d]

Class 14: D below 0
y in ${}_dY_{14}$ = [0;∞]
v in ${}_dV_{14}$ = [y + d;∞]

Class 21: D above 0
y in ${}_dY_{21}$ = [0;d/2]
v in ${}_dV_{21}$ = [0;y]

Class 22: D above 0
y in ${}_dY_{22}$ = [0;d/2]
v in ${}_dV_{22}$ = [y;∞]

Class 23: D below 0
y in ${}_dY_{23}$ = [0;d/2]
v in ${}_dV_{23}$ = [0;d - 2y]

Class 24: D below 0
y in ${}_dY_{24}$ = [0;d/2]
v in ${}_dV_{24}$ = [d - 2y; d - y]

Class 25: D below 0
y in ${}_dY_{25}$ = [0;d/2]
v in ${}_dV_{25}$ = [d - y;∞]

Class 31: D below 0
y in ${}_dY_{31}$ = [d/2;d]
v in ${}_dV_{31}$ = [d - y;∞]

Class 32: D below 0
y in ${}_dY_{32}$ = [d/2;d]
v in ${}_dV_{32}$ = [0;d - y]

Class 33: D above 0
y in ${}_dY_{33}$ = [d/2;d]
v in ${}_dV_{33}$ = [0;2y - d]

Class 34: D above 0
y in ${}_dY_{34}$ = [d/2;d]
v in ${}_dV_{34}$ = [2y - d;y]

Class 35: D above 0
y in ${}_dY_{35}$ = [d/2;d]
v in ${}_dV_{35}$ = [y;∞]

Class 41: D above 0
y in ${}_dY_{41}$ = [d;∞]
v in ${}_dV_{41}$ = [0;y - d]

Class 42: D above 0
y in ${}_dY_{42}$ = [d;∞]
v in ${}_dV_{42}$ = [y - d;y]

Class 43: D above 0
y in ${}_dY_{43} = [d;\infty]$
v in ${}_dV_{43} = [y;\infty]$

Class 44: D below 0
y in ${}_dY_{44} = [d;\infty]$
v in ${}_dV_{44} = [0;\infty]$

Now the volume will certainly be the same on each of the expressways, so that we may concentrate on one of them, say the "upper" one. For this reason, we shall omit classes 24, 25, 31, 32, 33, 41, 42, and 44 from most of our listings below, since these are classes of trips that will never use the "upper" expressway.

Let ${}_dP_{ij}(y)$ = Pr [a trip is in class ij] for a given y. We have

$${}_dP_{11}(y) = \Pr[D \text{ is above } 0]\Pr[v \text{ is in } {}_dV_{11}] = 1/2$$

In general ${}_dP_{ij}(y) = (1/2)\Pr[v \text{ is in } {}_dV_{ij}]$, so that

$${}_dP_{12}(y) = (1/2)[1 - \exp(-ky)]$$
$${}_dP_{13}(y) = (1/2)[1 - \exp(-kd)]\exp(-ky)$$
$${}_dP_{14}(y) = (1/2)\exp(-kd)\exp(-ky)$$
$${}_dP_{21}(y) = (1/2)[1 - \exp(-ky)]$$
$${}_dP_{22}(y) = (1/2)\exp(-ky)$$
$${}_dP_{23}(y) = (1/2)[1 - \exp(-kd)\exp(-ky)]$$
$${}_dP_{34}(y) = (1/2)[\exp(kd)\exp(-2Ky) - \exp(-ky)]$$
$${}_dP_{35}(y) = (1/2)\exp(-ky)$$
$${}_dP_{43}(y) = (1/2)\exp(-ky)$$

Given that a trip is in class ij, let ${}_dQ_{ij}(y, v)$ = Pr [the trip will use an expressway route] for given y, v. Now for trips in class 11, the total cost $c_a$ for a non-expressway route is $uC_a + vC_a$ and the total cost $c_e$ for an expressway route is $2yC_a + uC_e + vC_a$, so that $c_e \le c_a$ implies that $u \ge 2Ry$, where $R = C_a/(C_a - C_e)$, as before. (In this case it was clear that the "lower" expressway did not compete for the trip in the sense that $c_e$ for the "upper" was obviously smaller than $c_a$ for the "lower" facility. For some other classes this may not be so obvious, in fact the "lower" facility may be as likely a candidate for the trip as the "upper." Consider, for example, class 14. In these cases the trips were split 1:1 between the two expressways. In all other cases, the reader will observe that the classes themselves have been defined to guarantee that the use of the "upper" expressway will give a smaller $c_e$ than the use of the "lower.")

Thus,

$${}_dQ_{11}(y, v) = \exp(-2Rky)$$

Similarly,

$${}_dQ_{12}(y, v) = \exp(-2Rky)\exp(2Rkyv)$$
$${}_dQ_{13}(y, v) = 1$$
$${}_dQ_{14}(y, v) = 1$$
$${}_dQ_{21}(y, v) = \exp(-2Rky)\exp(2Rkv)$$
$${}_dQ_{22}(y, v) = 1$$
$${}_dQ_{23}(y, v) = \exp(-2Rky)$$
$${}_dQ_{34}(y, v) = \exp(-2Rky)\exp(2Rkv)$$
$${}_dQ_{35}(y, v) = 1$$
$${}_dQ_{43}(y, v) = 1$$

Let $P'_{ij}$ = Pr [trip in class ij which uses an expressway route will use the "upper" expressway]. We have $P'_{11} = P'_{12} = P'_{13} = P'_{21} = P'_{23} = P'_{34} = P'_{35} = 1$, while $P'_{14} = P'_{43} = 1/2$. (Of course, $P'_{31} = P'_{32} = P'_{33} = P'_{41} = P'_{42} = P'_{44} = 0$.)

Let ${}_dm_{ij}(y, v)$ = mean distance traveled on expressway by trip of class ij, given that the expressway is used. We have

$$
\begin{aligned}
{}_dm_{11}(y, v) &= 1/k + 2Ry \\
{}_dm_{12}(y, v) &= (1/k + 2Ry) - 2Rv \\
{}_dm_{13}(y, v) &= 1/k \\
{}_dm_{14}(y, v) &= 1/k \\
{}_dm_{21}(y, v) &= (1/k + 2Ry) - 2Rv \\
{}_dm_{22}(y, v) &= 1/k \\
{}_dm_{23}(y, v) &= 1/k + 2Ry \\
{}_dm_{34}(y, v) &= (1/k + 2Ry) - 2Rv \\
{}_dm_{35}(y, v) &= 1/k \\
{}_dm_{43}(y, v) &= 1/k
\end{aligned}
$$

Let ${}_df_{ij}(y, v) = (P'_{ij})\, {}_dP_{ij}(y, v)\, {}_dQ_{ij}(y, v)\, {}_dm_{ij}(y, v)$. We need the mean ${}_d\bar{f}_{ij}(y)$ of ${}_df_{ij}(y, v)$ with respect to v for each ij. Let ${}_dF_{ij}(t)$ be the density function for v on ${}_dV_{ij}$, so that ${}_dF_{ij}(t) = [1/2\ {}_dP_{ij}(y, t)]\, k \exp(-kt)$, t in ${}_dV_{ij}$, then

$$
{}_d\bar{f}_{ij}(y) = \int_{{}_dV_{ij}} {}_df_{ij}(y, t)\, {}_dF_{ij}(t)\, dt
$$

We have

$$
\begin{aligned}
{}_d\bar{f}_{11}(y) &= [1/(2k)] \exp(-2Rky) + Ry \exp(-2Rky) \\
{}_d\bar{f}_{12}(y) &= \{(4R - 1)/[2(2R - 1)^2 k]\} [\exp(-ky) - \exp(-2Rky] \\
&\quad - [R/(2R - 1)]\, y \exp(-2Rky) \\
{}_d\bar{f}_{13}(y) &= [k/(2k)][1 - \exp(-kd) \exp(-ky)] \\
{}_d\bar{f}_{14}(y) &= [1/(4k)] \exp(-kd) \exp(-ky) \\
{}_d\bar{f}_{21}(y) &= {}_df_{12}(y) \\
{}_d\bar{f}_{22}(y) &= [1/(2k)] \exp(-ky) \\
{}_d\bar{f}_{23}(y) &= [1/(2k)] \exp(-2Rky) + Ry \exp(-2Rky) \\
&\quad - [1/(2k)] \exp(-kd) \exp[-2(R - 1)ky] \\
&\quad - R \exp(-kd)\, y \exp[-2(R - 1)ky] \\
{}_d\bar{f}_{34}(y) &= \{(4R - 1)/[2(2R - 1)^2 k]\} \exp(-ky) \\
&\quad - \{[4R - 1 + 2R(2R - 1)kd]/[2(2R - 1)^2 k]\} \exp[-(2R - 1) \\
&\quad\ kd] \exp[2(R - 1)ky] \\
&\quad + [R/(2R - 1)] \exp[-(2R - 1)kd]\, y \exp[2(R - 1)ky] \\
{}_d\bar{f}_{35}(y) &= [1/(2k) \exp(-ky)] \\
{}_d\bar{f}_{43}(y) &= [1/(4k)] \exp(-ky)]
\end{aligned}
$$

Let

$$
{}_dS_{ij} = \int_{{}_dY_{ij}} {}_d\bar{f}_{ij}(y)\, dy
$$

so that

$$
\begin{aligned}
{}_dS_{11} &= 1/(2Rk^2) \\
{}_dS_{12} &= 1/(2Rk^2) \\
{}_dS_{13} &= [1/(2k^2)]\,[1 - \exp(-kd)] \\
{}_dS_{14} &= [1/(4k^2)] \exp(-kd) \\
{}_dS_{21} &= 1/(2Rk^2) - \{(4R - 1)/[2(2R - 1)^2k^2\} \exp(-kd/2) \\
&\quad + \{[2(3R - 1) + R(2R - 1)kd]/[4R(2R - 1)^2k^2]\} \exp(-Rkd) \\
{}_dS_{22} &= [1/(2k^2)]\,[1 - \exp(-kd/2)] \\
{}_dS_{23} &= 1/(2Rk^2) - \{(2R - 1)/[4(R - 1)^2k^2]\} \exp(-kd) \\
&\quad + \{[3R - 2 + R(R - 1)kd]/[4R(R - 1)^2k^2]\} \exp(-Rkd) \\
{}_dS_{34} &= \{(4R - 1)/[2(2R - 1)^2k^2]\} \exp(-kd/2) \\
&\quad - \{(2R - 1)/[4(R - 1)^2k^2]\} \exp(-kd) \\
&\quad + \{[6R^2 - 6R + 1 + R(R - 1)(2R - 1)kd]/[4(R - 1)^2k^2]\} \exp(-Rkd) \\
{}_dS_{35} &= [1/(2k^2)]\,[\exp(-kd/2) - \exp(-kd)] \\
{}_dS_{43} &= [1/(4k^2)] \exp(-kd)
\end{aligned}
$$

Let

$$\Delta\,{}_dV = \left(\sum_{ij} {}_dS_{ij}\right) \Delta x$$

and let

$${}_dV = D \int_0^1 d\,{}_dV = \text{mean trip density on the expressway}$$

We have

$$
\begin{aligned}
{}_dV = (D/4)a^2 \Big( &[R + 2)/R] - [R^2/2(R - 1)^2] \exp(-2d/a) \\
&+ \{[3R - 2 + 2R(R - 1)d/a]/[2R(R - 1)^2]\} \exp(-2Rd/a) \Big)
\end{aligned}
$$

where D is the trip density for the region, $a = 2/k$ is the average trip length for the region, d is the distance between the expressways, and $R = C_a/(C_a - C_e)$.

## Isolated Expressway With Limited Access

We are concerned here with the case of an isolated expressway which is placed in an unbounded region and to which there is limited access. The major assumptions are the same as for the preceding situations. Because the techniques for this case differ somewhat from those used for unlimited access expressways, a more nearly complete discussion will be given here than in previous sections.

We assume that access to the expressway is allowed at equal intervals of d miles. One of these access points is selected at random and labeled "exit 0." The origin of a rectangular coordinate system is identified with this point, with the x-axis coincident with the expressway. For ease of discussion, the positive directions of the x-axis and y-axis will be called "east" and "north," respectively (Fig. 24).

The access points are number 0, 1, 2,..., q,..., looking east. For many of our arguments, it will be convenient to group these exits in pairs, the first pair consisting of exits 0 and 1; the second, 2 and 3, etc. The (k + 1) pair will have its western and eastern exits labeled $W_k$ and $E_k$, respectively.

The general typing of trips is done as for an isolated expressway with unlimited access, and the same symmetry considerations as in that case allow us here to concentrate

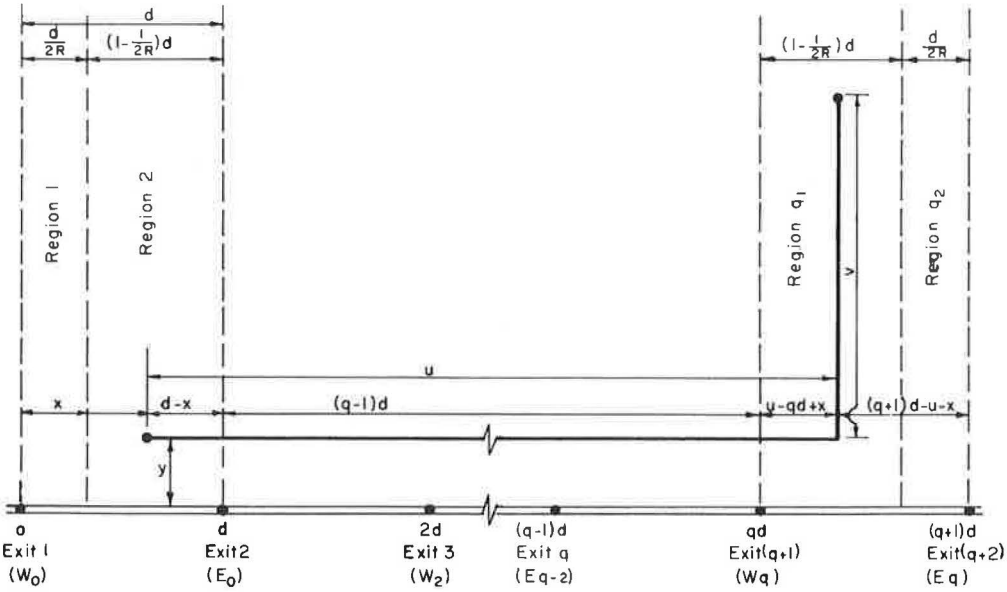


Figure 24.

on trips whose origin is north of the expressway. However, we must stratify the world more finely in this case with regard to the lateral position of trip origins and destinations.

Consider a trip whose origin has coordinates (x, y) with $0 \le x \le d$ and whose destination has coordinates $(x', y')$ with $qd \le x' \le (q+1)d$. Now, if the trip uses the expressway, it may enter at $W_0$ or $E_0$ and leave at $W_q$ or $E_q$. At any rate, the trip (assuming that the expressway is used) will pass the points $E_0$ and $W_q$. If, for the leg of the trip from the origin $E_0$, costs are computed, first directly to $E_0$ from the origin and second to $E_0$ from the origin via $W_0$, it will be observed that trips for which the x-coordinate of the origin lies between 0 and $d/2R$ $[R = C_a/(C_a - C_e)]$ will use $W_0$ while those for which $[1 - 1/(2R)]d \le x \le d$ will use $E_0$. A similar computation for the leg of the trip from $W_k$ to the destination $(x', y')$ will show that $kd \le x' \le kd + [1 - 1/(2R)]d$ implies that $W_k$ will be used, while $kd + [1 - 1/(2R)]d \le x' \le (k+1)d$ implies that $E_k$ will be used. Finally, these considerations lead us to a classification of trips from region to region, the regions depicted in Figure 24 and defined analytically by

region 1: $0 \le x \le d/(2R)$

region 2: $d/(2R) \le x \le [1 - 1/(2R)]d$

region $k_1$: $kd \le x' \le kd + [1 - 1/(2R)]d$

region $k_2$: $kd + [1 - 1/(2R)]d \le x' \le (k+1)d$

The classification itself proceeds as follows: Consider those trips whose origin lies within the vertical strip with base from $W_0$ to $E_0$ and whose destination lies within the vertical strip with base from $W_q$ to $E_q$. These trips are classified according to the scheme:

Class 11: D above 0
0 in region 2
D in region $q_1$
v in $[0;\infty]$

Class 12: D above 0
0 in region 1
D in region $q_1$
v in $[0;\infty]$

Class 13: D above 0
0 in region 2
D in region $q_2$
v in $[0;\infty]$

Class 14: D above 0
0 in region 1
d in region $q_2$
v in $[0;\infty]$

Class 21: D below 0
0 in region 2
d in region $q_1$
v in $[0;y]$

Class 22: D below 0
0 in region 1
d in region $q_1$
v in $[0;y]$

Class 23: D below 0
0 in region 2
D in region $q_2$
v in $[0;y]$

Class 24: D below 0
0 in region 1
D in region $q_2$
v in $[0;y]$

Class 31: D below 0
0 in region 2
D in region $q_1$
v in $[y;\infty]$

Class 32: D below 0
0 in region 1
D in region $q_1$
v in $[y;\infty]$

Class 33: D below 0
0 in region 2
D in region $q_2$
v in $[y;\infty]$

Class 34: D below 0
0 in region 1
D in region $q_2$
v in $[y;\infty]$

We will calculate the VMT on the expressway contributed by each of these classes, add these over $q = 1, 2, \ldots$, multiply by 4 to obtain the total VMT for westbound and eastbound trips as well as those which originate south of the facility, and then divided by d to obtain the mean trip density on the facility.

A detailed discussion of the computation of the VMT on the expressway contributed by class 13 should indicate the general method. Let $P_{ij}(x, y, q) = \Pr[\text{trip is in class ij for given x, y, given that 0 is in the appropriate region for class ij}]$. Let $a = \{q + [(2R - 1)/(2R)]\}d - x$ and $c = (q + 1)d - x$. We have

$$P_{13}(x, y, q) = \Pr[D \text{ is in region } q_2 \text{ given that } 0 \text{ is in region } 2]\ \Pr[0 \le v < \infty]$$
$$= \Pr[D \text{ is above and to the east of } 0]\ \Pr[a \le u \le c]$$
$$= (1/4)\ \Pr[a \le u \le c]$$

Now for trips of class 13, the trip cost $c_a$ for a non-expressway route is $uC_a + vC_a$, while the trip cost $c_e$ for the expressway route is

$$(d - x)C_a + yC_a + qdC_e + yC_a + [(q + 1)d - u - x]\,C_a + vC_a$$

(See Fig. 24.) Thus $c_e \le c_a$ if, and only if, $u \ge b$ where

$$b = y - x + \{1 + [2R - 1)/(2R)]q\}d,\ R = C_a/(C_a - C_e)$$

Hence, if we let $Q_{ij}(x, y, q) = \Pr[\text{trip in class ij for given x, y will use the expressway}]$, then $Q_{13}(x, y, q) = \Pr[b \le u \le c \text{ given that } a \le u \le c]$. Now for $a \le b \le c$, we have $\Pr[b \le u \le c \text{ given that } a \le u \le c] = \Pr[b \le u \le c]/\Pr[a \le u \le c]$. Thus

$$Q_{13}(x, y, q) = \begin{cases} \Pr[b \le u \le c]/\Pr[a \le u \le c] \text{ for } a \le b \le c \\ 1 \text{ for } b < a \end{cases}$$

So far, we have that

$$P_{13}(x, y, q)\,Q_{13}(x, y, q) = \begin{cases} (1/4)\Pr[b \le u \le c] \text{ for } a \le b \le c \\ (1/4)\Pr[a \le u \le c] \text{ for } b < a \end{cases}$$

But $b \le c$ implies that $y \le qd/(2R)$, while $a \le b$ implies $y \ge (q - 1)d/(2R)$. Thus, $a \le b \le c$ implies $(q - 1)d/(2R) \le y \le qd/(2R)$. Of course $b < a$ implies $y < (q - 1)d/(2R)$.

Finally, then,

$$P_{13}(x, y, q)Q_{13}(x, y, q) = \begin{cases} (1/4)\exp(-kd)\{\exp[kd/(2R)] - 1\}\exp(-qkd) \\ \quad \exp(kx), \text{ for y in } [0;(q - 1)d/(2R)) \\ (1/4)\exp(-kd)\exp(kx)\{\exp[-qkd\,(2R-1)/ \\ \quad (2R)]\exp(-ky) - \exp(-qkd)\}, \\ \quad \text{for y in } [(q - 1)d/(2R);\ qd/(2R)] \\ 0, \text{ for y in } [qd/(2R);\infty] \end{cases}$$

Let $m_{ij}(q)$ = distance traveled on facility by trip in class ij, given that the facility is used. We have $m_{13}(q) = qd$. For each q, let $V_{ij}(q)$ = VMT on facility per trip for trips in class ij, so that

$$V_{ij}(q) = \int_Y \int_X P_{ij}(x, y, q)\,Q_{ij}(x, y, q)\,m_{ij}(q)\,dxdy$$

We have

$$V_{13}(q) = \int_0^{(q-1)d/(2R)} \int_{d/(2R)}^{d} P_{13}(x, y, q)\,Q_{13}(x, y, q)\,m_{13}(q)\,dxdy$$
$$+ \int_{(q-1)d/(2R)}^{qd/(2R)} \int_{d/(2R)}^{d} P_{13}(x, y, q)\,Q_{13}(x, y, q)\,m_{13}(q)\,dxdy$$
$$= [d^2/(8Rk)]\{\exp[kd/(2R)] - 1\}(1 - \exp\{[-(2R - 1)/(2R)]kd\})$$
$$q^2\exp(-qkd)$$

$$- [d^2/(8Rk)] \exp [kd/(2R)] (1 - \exp \{[ - (2R - 1)/(2R)] kd\}) q \exp (-qkd)$$
$$+ [d/(4k^2)] [\exp [kd/(2R)] - 1](1 - \exp \{[ - (2R - 1)/(2R)] kd\}) q \exp (-qkd)$$

Let $S_{ij}$ = total VMT on the facility per trip for trips in class ij for all q, so that

$$S_{ij} = \sum_{q=1}^{\infty} V_{ij}(q)$$

In particular,

$$S_{13} = [d^2/(8Rk)] \{\exp [kd/(2R)] - 1\} \left(1 - \exp \{[ - (2R - 1)/(2R)] kd\}\right) \exp (-kd) [1 + \exp (-kd)]/[1 - \exp (-kd)]^3$$
$$+ [d/(4K^2)] \{\exp [kd/(2R)] - 1\} \left(1 - \exp \{[ - (2R - 1)/(2R)] kd\}\right) [\exp (-kd)]/[1 - \exp (-kd)]^2$$
$$- [d^2/(8Rk)] \left(1 - \exp \{[ - (2R - 1)/(2R)] kd\}\right) \exp [kd/(2R)] [\exp (-kd)]/[1 - \exp (-kd)]^2$$

Before listing for the remaining classes the values of the various quantities defined above, we note that symmetry gives $S_{i2} = S_{i3}$, i = 1, 2, 3, and state that it can be shown that $S_{1j} = S_{2j}$, j = 1, 2, 3, 4. With these omissions, then, we have

$$P_{11}(x, y, q) Q_{11}(x, y, q) = \begin{cases} (1/4) \left(1 - \exp \{[ - (2R - 1)/(2R)] kd\}\right) \exp (kx) \exp (-qkd), \text{ for } y \text{ in } [0; (q - 1)d/(2R)] \\ 0, \text{ for } y \text{ in } [(q - 1)d/(2R); \infty] \end{cases}$$

$$P_{14}(x, y, q) Q_{14}(x, y, q) = \begin{cases} (1/4) \exp (-kd) \{\exp [kd/(2R)] - 1\} \exp (kx) \exp (-qkd), \text{ for } y \text{ in } [0; qd/(2R) - x] \\ (1/4) \exp (-kd) [\exp [kd/(2R)] \exp \{[ - (2R - 1)/(2R)] qkd\} \exp (-ky) - \exp (-qkd) \exp (kx)], \text{ for } y \text{ in } [qd/(2R) - x; (q + 1)d/(2R) - x] \\ 0, \text{ for } y \text{ in } [(q + 1)d/(2R) - x; \infty] \end{cases}$$

$$P_{31}(x, y, q) Q_{31}(x, y, q) = (1/4)(1 - \exp \{[ - (2R - 1)/(2R)] kd\}) \exp (kx) \exp (-ky) \exp (-qkd), \text{ for } y \text{ in } [0; \infty]$$

$$P_{32}(x, y, q) Q_{32}(x, y, q) = P_{31}(x, y, q) Q_{31}(x, y, q)$$

$$P_{34}(x, y, q)\, Q_{34}(x, y, q) = (1/4) \exp(-kd) \{ \exp[kd/(2R)] - 1 \} \exp(kx) \exp(-ky) \exp(-qkd), \text{ for } y \text{ in } [0;\infty]$$

$$m_{11} = m_{31} = (q - 1)d$$

$$m_{32} = qd$$

$$m_{14} = m_{34} = (q + 1)d$$

$$V_{11}(q) = [d^2/(8Rk)] (1 - \exp\{[-(2R - 1)/(2R)] kd^2\})^2 (q - 1)^2 \exp[-(q - 1)kd]$$

$$\begin{aligned} V_{14}(q) = {} & [d^2/(8Rk)] \{ \exp[kd/(2R)] - 1)^2 (q + 1) \exp[-(q + 1)kd] \\ & - [d^2/(8Rk)] [\exp(kd/R) - 1] (q + 1) \exp[-(q + 1)kd] \\ & + [d/(2k^2)] \{ \exp[kd/(2R)] - 1 \}^2 (q + 1) \exp[-(q + 1)kd] \end{aligned}$$

$$V_{31}(q) = [d/(4k^2)] (1 - \exp\{[-(2R - 1)/(2R)] kd\}) \{ \exp(kd) - \exp[kd/(2R)] \} (q - 1) \exp(-qkd)$$

$$V_{32}(q) = [d/(4k^2)] (1 - \exp\{[-(2R - 1)/(2R)] kd\}) \{ \exp[kd/(2R)] - 1 \} q \exp(-qkd)$$

$$V_{34}(q) = [d/(4k^2)] \exp(-kd) \{ \exp[kd/(2R)] - 1 \}^2 (q + 1) \exp(-qkd)$$

We shall omit a listing of $S_{ij}$. They are easily gotten, as in the example for class 13, by summing the $V_{ij}(q)$ over q.

Finally, we sum the $S_{ij}$ over all i and j, multiply the result by 4 and divide by d (the reasons for doing this were explained earlier) to obtain the mean trip density Q on the facility. We find that

$$\begin{aligned} Q = {} & [Dda/(8R)] (1 - \exp\{[-(2R - 1)/R] d/a\})^2 \exp(-2d/a) [1 + \exp(-2d/a)]/[1 - \exp(-2D/a)]^3 \\ & + [Dda/(4R)] [\exp(d/Ra) - 1] (1 - \exp\{[-(2R - 1)/R] d/a\}) \exp(-2d/a) [1 + \exp(-2d/a)]/[1 - \exp(-2d/a)]^3 \\ & + (3Da^2/8) \{ \exp[d/(Ra)] - 1 \} (1 - \exp\{[-(2R - 1)/R] d/a\}) \exp(-2d/a)/[1 - \exp(-2d/a)]^2 \\ & + [Dda/(4R)] \{ \exp[d/(Ra)] - 1 \}^2 \exp(-4d/a)/[1 - \exp(-2d/a)]^3 \\ & + (5Da^2/16) \{ \exp[d/(Ra)] - 1 \}^2 \exp(-4d/a) [2 - \exp(-2d/a)]/[1 - \exp(-2d/a)]^2 \\ & + (Da^2/16) (1 - \exp\{[-(2R - 1)/R] d/a\})^2 \exp(-2d/a)/[1 - \exp(-2d/a)]^2 \\ & - [Dda/(4R)] (1 - \exp\{[-(2R - 1)/R] d/a\}) \exp[d/(Ra)] \exp(-2d/a)/[1 - \exp(-2d/a)]^2 \\ & - [Dda/(8R)] \{ \exp(2d/(Ra)] - 1 \} \exp(-4d/a) [2 - \exp(-2d/a)]/[1 - \exp(-2d/a)]^2 \end{aligned}$$

where

D = trip density for the region,
a = average trip length for the region,

$R = C_a/(C_a - C_e)$, and
$d$ = distance between exits on the facility (ramp spacing).

Note on trip cost: Let

$0(S)$ = operating cost per mile in dollars for a given speed S in miles per hour,
$A(S)$ = accident cost per mile in dollars for a given speed S in miles per hour,
$V$ = value of time in dollars per hour, and
$C(S)$ = total cost per mile in dollars for a trip segment at speed S in miles per hour.

We have

$$C(S) = 0(S) + A(S) + V/S$$

For the purpose of preparing tables, $0(S)$ and $A(S)$ were taken from an earlier paper to which minor adjustments were made (11); V was varied from \$1.00 to \$2.00 step \$.10.

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