The Intervening Opportunities Model: A Theoretical Consideration

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The intervening opportunities trip distribution model, though functionally more complex than its predecessors, has been subject to very little investigation except from a utilitarian point of view. While preparing a calibration computer program, an opportunity was found for a more theoretical consideration.

It was discovered that it is sometimes mathematically impossible to derive a value for L (the calibration parameter) which will cause an interchange to assume an allowable value; that is, it is sometimes not possible to calibrate the model to a base year. The reason for this is that, for a single interzonal interchange, increasing the size of L only causes a corresponding increase in the trips generated up to a certain point (the maximum). Beyond that point, any increase in L causes a decrease in the number of trips generated. This maximum is often less than the number of trips desired in a base year.

When interchanges are grouped, the same difficulty with maxima occurs. The problem is complicated by the facts that there may be more than one maximum and that one maximum may be higher than another. It is explained how the best L-factor is chosen from such a group.

•THE intervening opportunities trip distribution model is much more complex than any of its predecessors. Possibly for this reason, it has received very little purely theoretical investigation. The authors found an opportunity for such an investigation while developing a computer program to calibrate the model. Some of the findings, together with a review of the derivation of the model, are presented here in the hope that a better understanding of travel models will result.

DERIVATION

The intervening opportunities model (1) assumes that the trip interchange between an origin and a destination zone is equal to the total trips emanating from the origin multiplied by the probability that each trip origin will find an acceptable terminal at the destination. This is expressed mathematically as follows:

$$T_{ij} = O_i P(D_j) \tag{1}$$

where

T_{ij} = the trips between origin zone i and destination zone j;

O_i = the total trip origins produced at zone i;

D_j = the total trip destinations attracted to zone j; and

$P(D_j)$ = the probability that each trip origin at i will find destination j an acceptable terminal.

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$P(D_j)$ is expressed as a function of D_j. D_j is defined as the total trip destinations attracted to zone j because the model assumes that the probability that a destination will be acceptable is determined by two zonal characteristics: the size of the destination and the order in which it is encountered as trips proceed away from the origin.

$P(D_j)$ may also be expressed as the difference between the probability that the trip origins at i will find a suitable terminal in one of the destinations, ordered by closeness to i, up to and including j, and the probability that they will find a suitable terminal in the destinations up to but excluding j; thus

$$T_{ij} = O_i \left[P(A) - P(B) \right] \tag{2}$$

where

A = the sum of all destinations for zones between, in terms of closeness, i and j and including j; and

B = the sum of all destinations for zones between i and j but excluding j.

Note that

$$A = B + D_j \tag{3}$$

It is possible to formulate the function P as follows. The probability that a trip will terminate within some volume of destination points is equal to the product of two probabilities: (a) the probability that this volume contains an acceptable destination, and (b) the probability that an acceptable destination closer to the origin of the trip has not been found. This may be expressed in differentials as follows:

$$dP = (1 - P)\, L dV \tag{4}$$

where

$$P = P(V)$$

and where

V = the volume of destination points (destination trip ends) within which the probability of a successful terminal is to be calculated; and

L = the probability density (probability per destination) of destination acceptability at the point of consideration.

Assuming L to be constant, the solution to Eq. 4 is

$$P = 1 - ke^{-LV} \tag{5}$$

where

k = the constant of integration; and

e = the constant base of the natural logarithms, 2.71828. . . .

It can be shown that k = 1 since P must be zero when V is zero. Eq. 5 thus becomes

$$\begin{cases} P(V) = 0 & V \leq 0 \\ P(V) = 1 - e^{-LV} & V > 0 \end{cases} \tag{6}$$

Eq. 6 is a cumulative probability distribution. It is nondecreasing, $P(-\infty) = 0$, and $P(\infty) = 1$. Its corresponding density function is

$$P'(V) = Le^{-LV} \tag{7}$$

The function thus derived for P(V) may be substituted into Eq. 2 letting V equal A and B; thus

$$T_{ij} = O_i \left(e^{-LB} - e^{-LA}\right) \tag{8}$$

Eq. 8 is the standard formulation of the intervening opportunities model. This formulation requires that destination zones be ordered according to their travel time from the origin zone. Thus, destinations are placed in sequence according to the contents of the skim trees associated with the origin.

INVESTIGATION

The intervening opportunities model as derived above is unusual among trip distribution models in that it does not guarantee the utilization of 100 percent of the origins available; that is, it has been found that all origins are seldom accounted for. The reason for this is that its cumulative probability distribution as represented by Eq. 6 approaches one only as the total destinations become very large. In practice, this means that 10 or 20 percent of the origins from a zone may easily remain unaccounted for.

Another difficulty in using the model was encountered by staff members of the Traffic Research Group of Peat, Marwick, Livingston & Co.; namely, it is not always possible to calibrate the model to a base year. This was observed while implementing a contract with the U.S. Bureau of Public Roads to program the trip distribution portion of the Urban Planning System/360 Traffic Assignment Package.

It was the group's responsibility to derive and program a calibration technique as well as to program the model itself. It was decided to develop a procedure which would locate an L-factor yielding exactly the number of trips required for any combination of zonal interchanges. This approach was somewhat novel in that all previous techniques known to the authors strive to optimize some derived trip characteristic such as vehicle-miles of travel.

It was first attempted to calculate this optimum L-factor by means of several iterative approaches. These attempts were all found to be inadequate to cope with certain interchange configurations. An investigation was initiated to discover the reason for these failures. The first and second derivatives according to L were taken of Eq. 8:

$$\frac{d}{dL} T_{ij} = O_i \left(Ae^{-LA} - Be^{-LB}\right) \tag{9}$$

$$\frac{d^2}{dL^2} T_{ij} = O_i \left(B^2e^{-LB} - A^2e^{-LA}\right) \tag{10}$$

The first derivative was then set equal to zero and was solved for a value of L, L_0, which was shown to be a maximum by demonstrating that Eq. 10 is less than zero when L_0 is substituted into it.

$$L_0 = \frac{\ln (A/B)}{A - B} \tag{11}$$

A maximum for T_{ij}, T_m, may thus be calculated by substituting Eq. 11 back into Eq. 8:

$$T_m = O_i \left[\frac{B}{A}\right]^{\frac{B}{A - B}} \left[1 - \frac{B}{A}\right] \tag{12}$$

Eq. 12 reduces to an interesting and useful form when solved for the ratio T_m/O_i and when the substitutions $A = B + D_j$ and $r = B/D_j$ are made:

$$\frac{T_m}{O_i} = \frac{r^r}{(r + 1)^{r+1}} \tag{13}$$

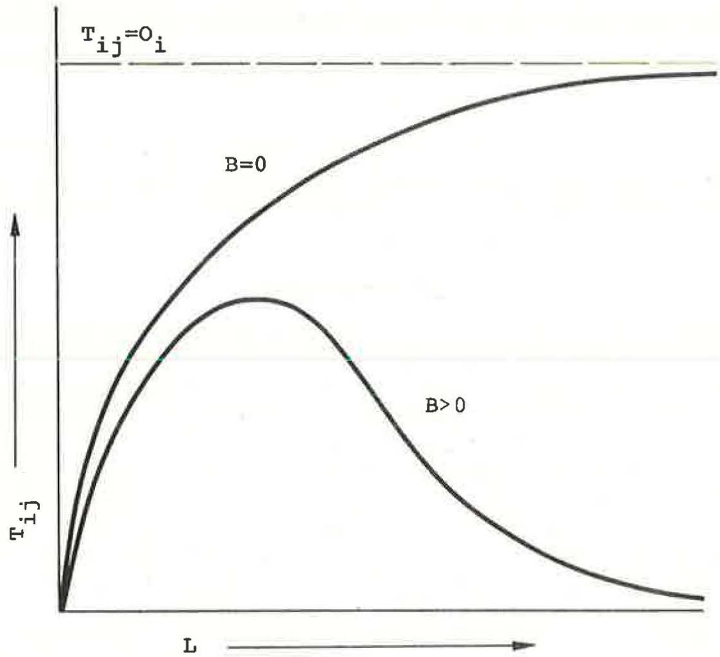

Figure 1. Graph of $T_{ij} = O_i\ (e^{-LB} - e^{-LA})$ as a function of L.

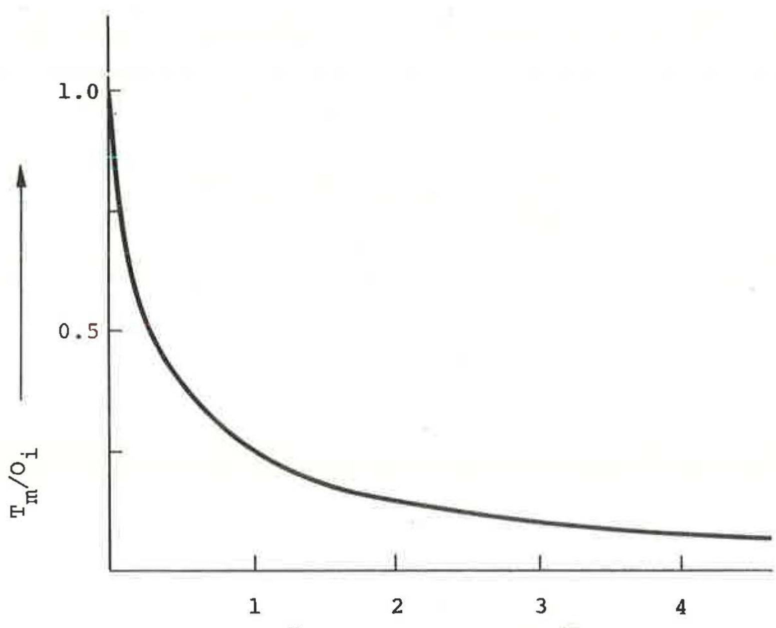

Figure 2. Graph of $\frac{T_m}{O_i} = \frac{r^r}{(r+1)^{r+1}}$.

For any interzonal interchange, so long as O_i, B and A are held constant, an increase in the value of L only causes a corresponding increase in the number of trips generated by the model up to the point L_0. Beyond that point, any increase in L causes instead a decrease in the number of trips generated. This is shown graphically in Figure 1. Note that intrazonal interchanges (B = 0) are a special case where an increase in L always causes an increase in T_{ij}. This is part of the reason that the model has been thought to overestimate intrazonal trips.

It is thus evident that there are allowable values of T_{ij} which the intervening opportunities model cannot produce. The seriousness of this limitation may be judged by studying Figure 2, a graph of Eq. 13. Where the ratio B/D_j is greater than 1, no more than 0.25 of the origins may be included in any interchange. Where the ratio is greater than 2, no more than 0.15 of the origins may be included; greater than 3, no more than 0.11; greater than 4, no more than 0.08, etc. When the ratio becomes quite large, as is usual, the maximum number of trips which the model will produce for an interchange becomes very small. As an example, when the ratio is 1000, the model will include no more than 0.0004 of the origins in any interchange.

The calibration technique actually adopted for inclusion in the System/360 Trip Distribution Package takes the limitations of the model into account. For any group of interchanges, G, the so-called best L-factor is derived. The best L-factor is defined as the first which yields the desired number of trips for interchange group G. If there is none which yields the desired number, then the L-factor yielding the most trips for group G is supplied.

It is important to understand that, although the function for one interchange may have only one maximum, the function for interchange group G may have as many maxima as there are interchanges in G. Thus, there may be a multitude of possible L-factors. The program chooses only the first in each case.

CONCLUSION

It is evident that the intervening opportunities model is mathematically limited in two ways in spite of the fact that it is a genuine probability model. In the first place, it is seldom able to account for 100 percent of the origins at any zone. In the second, it is so structured that there is a maximum number of trips which may be generated for any given interzonal interchange with an accompanying tendency to overestimate intrazonal interchanges. Thus the model is more difficult both to calibrate and to apply than has generally been found to be the case with other distribution techniques.

REFERENCE

1. Schneider, Morton. Appendix to Panel Discussion on Inter-Area Travel Formulas. HRB Bull. 253, p. 136-138, 1960.

Discussion

EARL R. RUITER, Massachusetts Institute of Technology—This discussion is written for two purposes: (a) to comment on the problem of calibrating the intervening opportunities model in the light of the limitations of the model discovered by the authors; and (b) to develop, using probability theory, an intervening opportunities model that does guarantee utilization of 100 percent of the origins available.

Calibration Aspects

The authors' empirical discovery and theoretical validation of the fact that it is not possible to duplicate any arbitrary group of trip interchanges under all conditions will be useful to future users of the opportunity model. The authors pose a real calibration difficulty, given their calibration objective, and prescribe a method of overcoming this difficulty; however, the authors' calibration objective can be improved upon. This can be done in such a fashion that the model limitation no longer poses calibration problems. If the fact that there will be errors in calculated trip interchanges is recognized, then the objective of the calibration procedure can be to minimize these errors. Thus, rather than attempting to match exactly a limited number of groups of interchanges, the error in the prediction of all interchanges can be minimized. This can be done using the concepts underlying multiple regression, by finding the L-value which will minimize the sum of the squares of the deviations between observed interchanges and calculated interchanges.

In equation form, if T_{ij}^o and T_{ij}^c are, respectively, an observed and a calculated interchange, then the error, D_{ij}, is, using the notation developed by the authors:

$$D_{ij} = T_{ij}^o - T_{ij}^c = T_{ij}^o - O_i \left(e^{-LB_j} - e^{-LA_j}\right)$$

And the sum of the squares of all errors, S, is:

$$S = \sum_i \sum_j D^2{}_{ij} = \sum_i \sum_j \left[T_{ij}^o - O_i \left(e^{-LB_j} - e^{-LA_j}\right)\right]^2$$

S is minimized when its derivative, with respect to L, is zero:

$$\frac{dS}{dL} = \sum_i \sum_j \left\{ 2\left[T_{ij}^o - O_i\left(e^{-LB_j} - e^{-LA_j}\right)\right] \cdot \left[O_i \left(B_j\, e^{-LB_j} - A_j\, e^{-LA_j}\right)\right] \right\} = 0$$

The solution of this equation for L may be a formidable task, but an iterative technique not unlike that which the authors have implemented could be used. The advantages of this calibration procedure are that all relevant data can be used and that a measure of the total error introduced by specifying an L-value is minimized.

Revised Model Derivation

An intervening opportunities model which will guarantee utilization of 100 percent of the trip origins available can be derived using the mathematical principles of random variables and probability functions. The mathematical developments in this section are based on the probability principles as presented by Wadsworth and Bryan (2).

As the authors point out, the cumulative probability distribution which underlies the intervening opportunities model (the authors' Eq. 6) is defined for all positive values of V, from zero to infinity. When the maximum value of V is V_n, Eq. 6 states that the probability of an origin finding an acceptable destination is:

$$1 - e^{-LV_n}$$

which approaches 1 as V_n approaches infinity. When it is known that all trip origins do end before V_n destinations are considered, trips should be a function of $P(V/V_n)$, the probability that a trip will end before reaching V destinations, given that it ends before reaching V_n destinations. $P(V/V_n)$ can be developed as follows:

$$P(V/V_n) = \frac{P(V, V_n)}{P(V_n)}$$

where

$P(V, V_n)$ = the probability that a trip will end before reaching V destinations and that it will end before reaching V_n destinations;

$P(V, V_n)$ = $P(V)$ because all V destinations are included in V_n;

$P(V_n)$ = the probability that a trip will end before reaching V_n destinations.

Using $P(V) = 1 - e^{-LV}$

$$P(V/V_n) = \frac{1 - e^{-LV}}{1 - e^{-LV_n}} \qquad (0 < V < V_n)$$

This function will distribute all trips, as can be shown by setting $V = V_n$ and observing that $P(V_n/V_n) = 1$.

The erroneous use of P(V) rather than $P(V/V_n)$ in operational opportunity model programs can be corrected. For example, the Chicago Area Transportation Study has developed a corrected version, which they refer to as their "forced interchange" opportunity model (3).

In summary, the authors' discovery of limitations of the opportunity model is important, but is not an invalidation of the model. Valid calibration procedures can be devised which recognize the limitations of the model. Also, mathematically valid revisions of the model can be implemented to guarantee the utilization of 100 percent of the trip origins available at each zone.

References

2. Wadsworth and Bryan. Introduction to Probability and Random Variables. McGraw-Hill, 1960
3. Walker, S. A., III. Recent Developments in the Simulation of Transit Travel in the Chicago Area. CATS Research News, March-April 1968.