

Viscoelastic Constitutive Equation for Sand-Asphalt Mixtures

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●ANALYSIS of stresses and displacements within a layered system subjected to a variety of surface loading is an essential step in the development of a rational method of design for flexible pavements. Such an analysis is generally achieved through the solution of partial differential equations of equilibrium, compatibility, and constitutive equations. The equations of equilibrium are obtained from the conservation of momentum, and the continuity assumption of the body leads through some geometric logic and definition of the strain tensor, to the linear strain-displacement equations. The constitutive equations, which characterize the materials, are generally assumed to be in linear form, thus avoiding the mathematical complications that may otherwise arise. The Boltzmann superposition principle is a consequence of this assumption. This principle states that the total stress (strain) at any time can be computed by simple summation of the stresses (strains) due to all of the individual strain (stress) increments that have been applied to the body in its past history. In the limit of a continuous strain (stress) history, the summation process becomes integration. This principle is often used as an alternate definition of linearity, and constitutive equations can accordingly be written as

$$\sigma_{ij} = \int_0^t E_{ijkl} d \epsilon_{kl} \quad (1)$$

or

$$\epsilon_{ij} = \int_0^t D_{ijkl} d \sigma_{kl} \quad (2)$$

where σ_{ij} and ϵ_{ij} are stress and strain tensors respectively and the coefficient functions entering the constitutive equations are called "moduli" for Eq. 1 and "compliances" for Eq. 2.

The constitutive equations used for analysis of stress and displacements in layered systems have been assumed to be that of elastic materials, for which the coefficient functions in Eqs. 1 and 2 are simply constants (1). This assumption has resulted in large discrepancies between experimentally measured stresses and displacements with those calculated theoretically. The effects of variables such as the duration of load and the rate of loading on the deflection of a pavement system cannot be accounted for by the elasticity assumptions. Similarly, the theories of elasticity cannot be used to explain the accumulation of deflection that is observed experimentally in a pavement. Such discrepancies have led many investigators to the use of constitutive equations that take into consideration the time-dependent behavior of materials (2, 3, 4).

The class of linear materials for which the coefficient functions in Eqs. 1 and 2 are time-dependent are referred to as linear viscoelastic. The additional complexity introduced into the analytical mechanics of continua by admitting time as a variable in the

constitutive equations of a material is illustrated by the fact that different mathematical formulations of the problem lead to different constitutive functions that are not related in a simple arithmetic manner (as is the case for elasticity). Because of the wide variety of materials to which the theory of linear viscoelasticity is applied, one formulation is often preferable over another from the standpoint of physical or mathematical interpretation or computational convenience. This latter point is of great importance in analysis of stresses and displacements in layered viscoelastic systems. All such analyses, regardless of loading conditions on the surface or the number of layers, make use of numerical techniques, and thus constitutive equations of the materials involved must be described mathematically in such a manner that the parameters used will appear in the final solutions. For example, if a paving material is characterized by the real and imaginary parts of its complex modulus, and these expressions are used in an analytical technique that uses the creep compliance function of the material, it would be extremely difficult from such an analysis to predict the influence of complex modulus on stress distribution or deflection of such a system. Similarly, when complicated models are used to characterize a material, determination of the influence of the numerous parameters of the model on the response of the system and the bookkeeping of these parameters in numerical solutions become very cumbersome and complicated. It is, therefore, the purpose of this paper to provide a brief review of the characterization methods of viscoelastic materials, to present the results of a study made on characterization of sand-asphalt mixture, and to show a possible technique that may be used to provide constitutive equations that render themselves easily to numerical analysis. The paper, therefore, concerns itself primarily with viscoelastic characterization of paving materials for use in stress analysis of pavement systems. It is not the intent of the paper to discuss the significance of viscoelastic properties of the materials or the use of viscoelastic concepts for design of asphaltic mixtures.

CONSTITUTIVE EQUATIONS FOR LINEAR VISCOELASTICITY

One phenomenological approach to the theory of linear viscoelasticity postulates the existence of "creep" and "relaxation" functions, defined as the observed material response for a unit applied stress or strain respectively. These functions can be determined by simple laboratory experiments; Boltzmann's superposition principle is then used to determine the strain or stress for arbitrary loading conditions.

The mathematical form of the constitutive equation employing creep functions is

$$\epsilon_{ij}(t) = \int_{-\infty}^t D_{cijkl}(t-\tau) \frac{\partial \sigma_{kl}(\tau)}{\partial \tau} d\tau \quad (3)$$

The form employing relaxation functions is

$$\sigma_{ij}(t) = \int_{-\infty}^t E_{rijkl}(t-\tau) \frac{\partial \epsilon_{kl}(\tau)}{\partial \tau} d\tau \quad (4)$$

The operational forms of the constitutive equations that are related to mechanical models can be written as

$$P(\sigma_{ij}) = Q(\epsilon_{ij}) \quad (5)$$

where

$$P = \sum_0^p p_r \frac{\partial^r}{\partial t^r} \quad (6)$$

$$Q = \sum_0^q q_r \frac{\partial^r}{\partial t^r}$$

The material characteristics are then contained in the orders of the operators, p and q , and in the values of the coefficients, p_r and q_r . Because of the large number of constants involved, such a material characterization is extremely unwieldy to use in any numerical analysis and it is considered of practical use as a conceptual step in developing a computational scheme.

Another derived constitutive relation can be obtained by letting the number of elements in the foregoing characterization tend to infinity. The result is an expression for modulus in terms of a distribution function of "retardation" times. These functions are usually expressed in logarithmic coordinates (for ease in plotting) as

$$E(t) = E_{\infty} + \int_{-\infty}^{+\infty} H(\ln \tau) e^{-t/\tau} d \ln \tau \quad (7)$$

and

$$D(t) = \frac{t}{\eta} + D_0 + \int_{-\infty}^{+\infty} L(\ln \tau) [1 - e^{-t/\tau}] d \ln \tau \quad (8)$$

where E_{∞} is the equilibrium modulus at $t = 0$, D_0 is the instantaneous compliance, and η is the equilibrium viscosity. In this manner the experimental problem has been transformed to the determination of one of the functions $H(\ln \tau)$ or $L(\ln \tau)$ rather than the coefficients of the operators in the preceding form. The distribution functions play a very useful role in computational programs, particularly in approximate methods.

The operator form of the constitutive equations can be made to yield another form that is useful for analytical and experimental purposes. For sinusoidal motion the operator $\partial/\partial t$ becomes $i\omega$, where ω is the circular frequency of the motion, and Eq. 5 can be reduced to

$$\sigma_{ij} = E^*_{ijkl} \epsilon_{kl} \quad (9)$$

where the tensor E^*_{ijkl} is now complex. A complex compliance D^*_{ijkl} also results in the obvious manner. By splitting the complex modulus into real and imaginary parts (and dropping the tensorial subscripts for notational convenience),

$$E^*(\omega) = E'(\omega) + iE''(\omega) \quad (10)$$

a new variable, the "loss tangent," can be defined by

$$\tan \delta = \frac{E''}{E'} \quad (11)$$

As the designation implies, $\tan \delta$ is a measure of the specific energy loss per cycle of motion, and it is a function of frequency.

The most useful parametric characterization of viscoelastic material properties for incorporation in the "dynamic analysis" of structural systems is by the loss tangent, $\tan \delta$, and the real part of the complex modulus, E' , as functions of frequency. When the material, however, is subjected to static loading or can be assumed to be under a quasi-static loading condition, the dynamic moduli should, in general, be transformed into creep or relaxation functions for stress analysis purposes. This point raises an important question on the usefulness of complex modulus as the characterization method for asphaltic mixtures, while all the viscoelastic methods of solution of multi-layer systems have made use of creep function or operator form of constitutive equations.

Special Deformations and Experimental Methods

To apply the viscoelastic theories to a continuum with a specified geometry and loading pattern it is necessary to determine the constitutive parameters for the material experimentally. In order to attach significance to the results of the subsequent stress analysis the analyst must understand the testing process and the manner in which the test results are manipulated to obtain the constitutive parameters. It is evident that a very precise and complicated analytical program for stress determination is unwarranted unless the material properties can be specified with a precision matching that of the analysis. It is also necessary to determine experimentally the limits of linear viscoelastic behavior of the material in question in order to verify that a linear analysis is valid for the specified loading conditions. Laboratory tests are most commonly performed in a manner such that they can be characterized by one of four simple deformation patterns: "simple shear," "dilatation" (or "bulk compression"), "simple extension," or "compression."

The test most common for asphaltic mixtures is a simple compression test. In this test, it is usually assumed that the stress is uniform across the specimen. This type of test leads to changes in both volume and shape, although for most materials the change in shape is predominant. To characterize the relative importance of volume and shape change, use is made of Poisson's ratio.

For linearly elastic bodies there are simple arithmetic relationships among modulus of elasticity, E , bulk modulus, K , shear modulus, G , and Poisson's ratio, ν . For linear viscoelastic bodies this arithmetic relationship occurs in the Laplace or Fourier transform domain only. A general observation based on the relations existing among these constants is that the simple compression test theoretically contains the total characterization of the material within a single test if both E and ν are measured. From a practical standpoint, however, it is often difficult to measure ν accurately. It is a common practice in practical applications to assume either K or ν is a constant and to use measurements of E as a function of time and use the appropriate relations in Laplace or Fourier domain to find the other constant as a function of time.

A wide variety of specialized instrumentation and data reduction schemes have been employed for obtaining the various constitutive parameters of materials. Only two basic parameters are ever measured directly, however. In the time domain, constant strain or constant rate of strain tests measure directly the relaxation modulus. In the frequency domain, sinusoidal strain tests measure directly the complex modulus. (Of course, compliances could alternatively be obtained by imposing stress instead of strain conditions.)

If the distribution functions or the coefficients of an operator representation are the desired quantities, these must be derived from the measured quantities using the results of the theory. For example, the distribution function of relaxation times can be obtained from a relaxation modulus by solving the appropriate integral equation or from a complex modulus by solving one of the integral equations relating the two functions. Bland (7) outlines a technique for fitting an operator equation to measured data over a specified frequency or time range. Ferry (9) gives an exhaustive list of exact and approximate relations for interrelating other parameters. Examples of test techniques and data reduction schemes for specific situations can be found in several works (7-9 and 13-19).

It should be kept in mind when using or reviewing the results of such mathematical transformations that it is the measured data that are characteristic of the material; all derived data are only valid insofar as the theory of linear viscoelasticity is applicable. When presenting the results of experimental work on viscoelastic materials it should therefore be common practice to include the actual measured data as well as any derived data. To test the assumption that a material is linearly viscoelastic it is necessary to determine the modulus (or compliance) for a series of strains (or stresses) of different magnitudes; if the modulus exhibits a dependence on stress or strain magnitude the material is not linearly viscoelastic. The most informative manner for displaying this linearity check is to plot a family of stress-strain curves with time (or

frequency) as a parameter for selecting data from each of the tests at different magnitudes of the forcing function. The limits of linear behavior of the material will be exhibited by the region within which these parametric curves are straight lines. Such a plotting procedure will also reveal the effect of the time scale of the experiment on the region of linear behavior. Note that if a measured modulus exhibits a specific domain of linearity it cannot be assumed that any other functions or moduli derived from it will have the same domain of linearity.

Time-Temperature Superposition

Although the theory of linear viscoelasticity presented here is an isothermal theory, it is still possible within this framework to incorporate the effects of temperature. By retaining the postulate that the thermal and mechanical effects are not coupled in the constitutive equations, it is obviously possible to write the relaxation function, for example, as

$$E(t, \theta) = E_1(t) E_2(\theta) \quad (12)$$

where t is time and θ is temperature. $E_1(t)$ is the relaxation function from the isothermal linear theory already presented. It is shown that for at least one class of materials there exists a demonstrable relationship between the functions $E_1(t)$ and $E_2(\theta)$. Such materials have been termed "thermorheologically simple" and the relationship between time and temperature is contained in the "time-temperature superposition principle." Although this principle was first discovered empirically, it has since been deduced for polymers as a consequence of certain molecular state theories (9). The essential assumptions are (a) that the moduli of all molecular mechanisms are directly proportional to the absolute temperature and to the density, and (b) that all relaxation (or retardation) times (i. e., the molecular mobilities) are affected the same amount by a temperature change (13, 14). The practical consequence of these assumptions is that data obtained at different temperatures should superimpose using the appropriate shift factor a_T . Various analytical expressions have been developed, empirically and theoretically, for the shift factor a_T , but it is best determined empirically as a partial test of the validity of this approach.

The procedure for utilizing this principle for determining a modulus might be as follows. Obtain a series of relaxation moduli at different temperatures for some convenient time interval. Select one temperature as a reference temperature and display the measured moduli on a plot of $\log [E(t)(P_0 T_0 / PT)]$ vs $\log [t]$, where P and P_0 are density of material at temperatures T and T_0 . Perform a horizontal shift of all curves with $T \neq T_0$ until they superimpose to form one continuous smooth curve. The time scale of the resulting curve is now $\log [t/a_T]$. The complete presentation of the experimental data consists of the superimposed plot of the reduced variables and an additional plot of $\log a_T$ vs temperature. Ferry (9) lists as necessary (although he does not claim they are "sufficient") criteria for applicability of this method that (a) the shapes of adjacent curves must match exactly, (b) the same value of a_T must be obtained for all viscoelastic parameters, and (c) the plot of a_T vs temperature must exhibit a "reasonable form consistent with experience."

If the principle of time-temperature superposition is applicable to a specific case it permits extending to many decades of $\log (t)$ or $\log (\omega)$ data that may be gathered in a reasonable amount of laboratory time.

Application to Asphaltic Materials

Asphalts have been shown to be a linear viscoelastic liquid above the glass transition temperature T_g (21). The time-temperature superposition principle is found valid for asphalts and the WLF equation is obeyed over a wide temperature range.

Bituminous concrete mixtures have also been analyzed using rheological principles. Papazian (16) tested asphalt mixtures dynamically to arrive at general stress-strain relations of a linear viscoelastic material in the frequency domain and transformed

these relations into the time domain. Krokosky (23) found that under high stresses and strains, aggregates in bituminous mixtures introduce nonlinearity in the response because of aggregate interlock, which forces the deformation to stop in some places in the specimen. Alexander (24) performed creep, constant rate of strain, and relaxation type tests on bituminous concrete in an effort to determine the limits of linear viscoelastic behavior. He also compared the tests in both tension and compression, and as a result concluded that asphaltic concretes have different properties in tension than in compression.

The effect of temperature on the response of asphalt-aggregate composite has also been investigated. Pagen (25) determined that the asphalt mixtures used in his experiments were thermorheologically simple and that the time-temperature superposition principle could be used to obtain a master curve giving long-time viscoelastic response. Lottman (26) found that time-temperature superposition was applicable to his test results and in addition attempted to formulate road specifications from the temperature response of the bituminous mix.

MATERIALS AND TEST PROCEDURE

The sand-asphalt mixture specimens used in this study were made of Ottawa sand and asphalt cement. Ottawa sand was proportioned according to ASTM D 1663-59T as given in Table 1. The asphalt cement was an AC-20 grade asphalt coded B-3056 from the "Asphalt Institute-Bureau of Public Roads Cooperative Study of Viscosity-Graded Asphalts." The percentage of asphalt used was 9 percent by weight of aggregate, and the results of conventional tests on this asphalt are given in Table 2.

The sand and asphalt were mixed after being heated to 325 F and compacted into cylindrical specimens 1.4 in. in diameter and 3 in. high. The method of compaction was a miniature Marshall compaction apparatus, and the number of blows were such that total compaction energy delivered per unit volume of the mixture equaled that of Marshall compaction specified for medium traffic. The compacted specimens had an average void content of approximately 4 percent.

The compacted specimens were subjected to creep and relaxation tests. The variables included the test temperature, the level of stress in creep test, and the level of strain in relaxation test. The six testing temperatures varied from -5 C to 45 C. In the creep test the specimens were subjected to various levels of stress, and in relaxation tests the specimens were subjected to at least three levels of strains. The levels of stress in creep tests were chosen such that the final strains were always below 1 percent.

RESULTS AND DISCUSSION

The results presented in this section are concerned with the feasibility of deriving analytical expressions for constitutive equations of sand-asphalt mixtures, which can subsequently be used in the analysis of stresses and displacements in pavement systems. The methods of stress analysis available in the literature for multi-layer viscoelastic

TABLE 2
RESULTS OF TESTS ON ASPHALT

Test	Result
Specific gravity, 77/77 F	1.020
Penetration, 200 gm, 60 sec, 39.9 F	30
Ductility, 77 F	250 + CM
Flash point, Cleveland open cup	545 F

TABLE 1
GRADATION OF AGGREGATES

Sieve Size	Percent Passing*	
	ASTM D 1663-59T	Selected Gradation
16	85-100	100
30	70-95	75
50	45-75	45
100	20-40	26
200	9-20	15

* Percent of the total weight of material passing a given sieve size.

systems are based on the assumption that the system is in a quasi-static equilibrium state and thus the dynamic effects can be neglected. This assumption and the possibility of using correspondence principles to obtain viscoelastic solutions have led most investigators to the selection of constitutive equations that will provide the response of the material over a large time span and will also require relatively small numbers

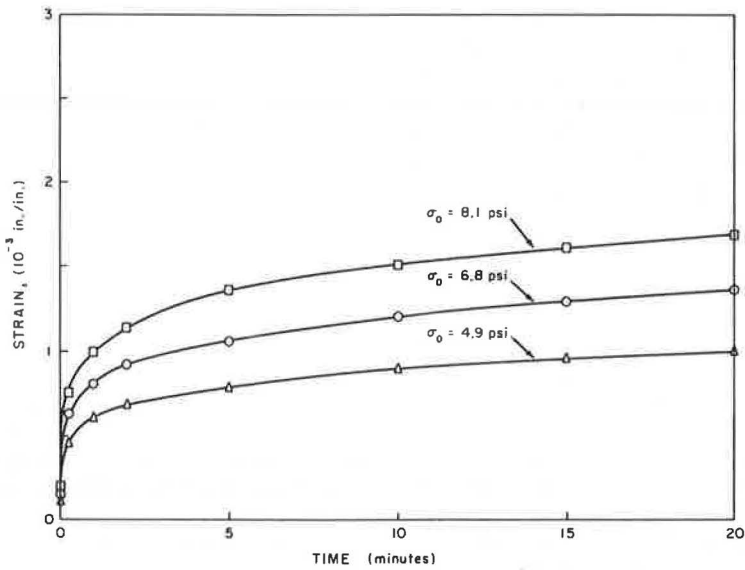


Figure 1. Typical creep curves, 35 C.

of tests to obtain the expressions for constitutive equations. The correspondence principle used by most investigators has involved the use of Laplace transformation and its consequent inversion of the desired expressions (such as stress and deflection) to the time domain. This transformation and its consequent inversion have limited the characterization of the material to that of simple models such as Kelvin solids and Burgess' model, which are greatly oversimplified and do not necessarily reflect the performance of real materials. To avoid this, and to use representative constitutive equations of

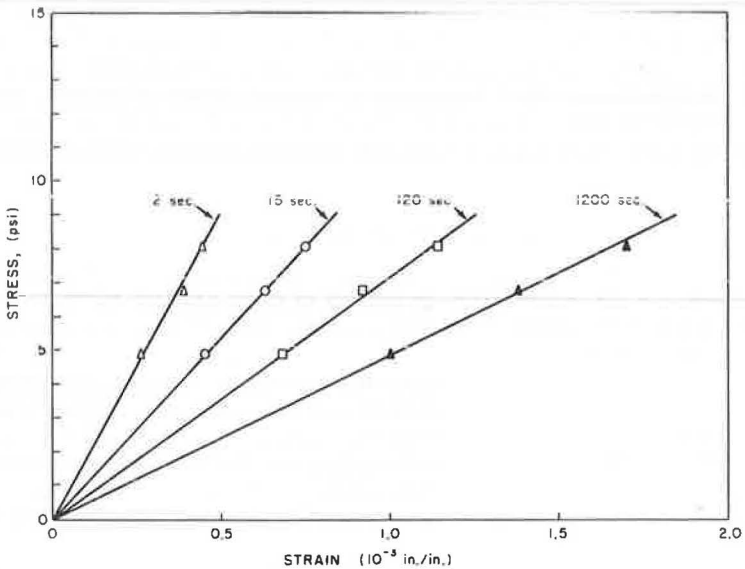


Figure 2. Isochrones of typical creep curves, 35 C.

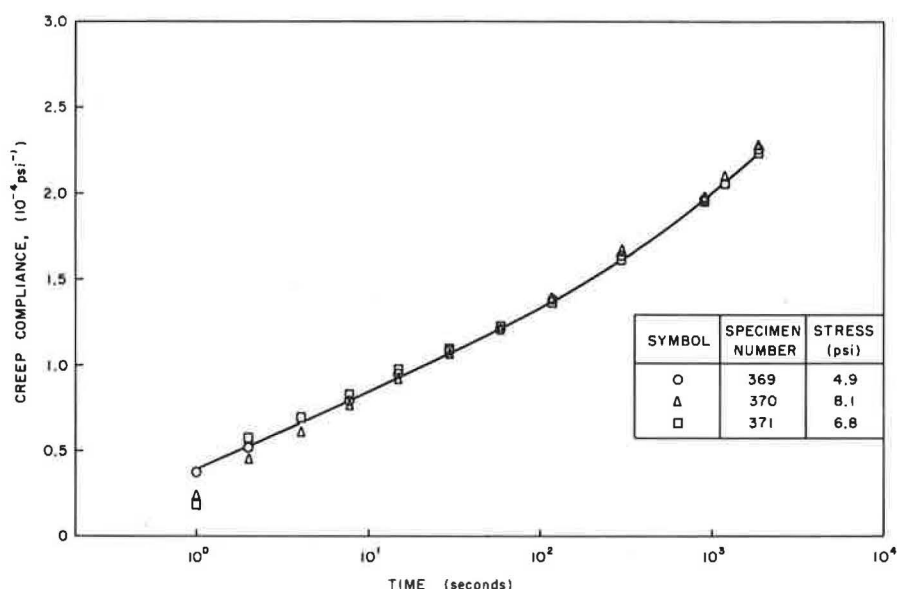


Figure 3. Typical creep compliance, 35 C.

the material in analysis of stresses and displacements, the authors have recently developed a technique based on multi-convolution integrals that does not require the use of Laplace transformations (3). This method, which has the capability of using the experimental data points, can use the experimental results in either creep or relaxation forms. The only conditions provided are that the material be linear, and the experimental data be available over a wide range of time scale. To obtain the latter point, the time-temperature superposition principle can be used.

Figure 1 shows typical results of creep tests for the sand-asphalt mixture used in this study. The testing temperature was 35 C and the results are for three stress levels. The corresponding level of stress for each test is indicated on the curve. The isochrones shown in Figure 2 were obtained by plotting the stress vs the corresponding strain at specific times of 2, 15, 120, and 1200 seconds. The linearity of these lines indicates that, within the experimental errors, the mixture used in this study behaves as a linear viscoelastic material.

In Figure 3, the creep compliance $D_C(t)$, defined as the ratio of strain $E(t)$ and stress σ_0 , is plotted vs time in a logarithmic scale. This figure shows that the creep compliances obtained at different levels of stress are superimposed and thus the material behaves as linear viscoelastic when tested in creep. The scatter observed in the data at short loading times is primarily due to the insensitivity of the measuring devices, and also due to the discontinuity that exists at zero time.

The creep compliance curves of the sand-asphalt mixture at other temperatures are shown in Figure 4. The values of compliance are corrected to a base temperature of 25 C. This correction is necessary to account for entropy changes associated with change in temperature.

In Figure 5 the time-temperature superposition principle is applied to the creep compliance data shown in Figure 4. The reference temperature was arbitrarily chosen as 25 C or 298 K for the experimental temperature range of 268 to 318 K. The effect of temperature change on the reduced compliance is a horizontal shift of the plots corresponding to a multiplication or division of the time scale by a constant factor for each temperature. Hence, by a horizontal shift on a logarithmic scale, the reduced compliance curves will superpose and form one complete curve at the reference temperature. The method of reduced variables thus extends the results over a much wider time

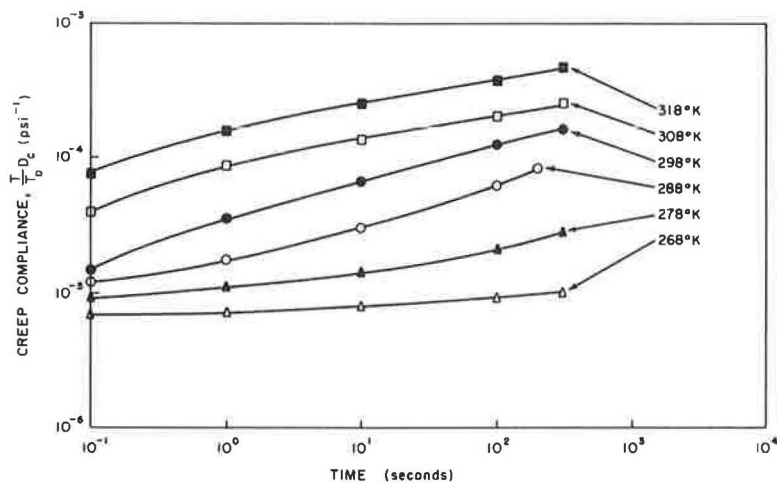


Figure 4. Typical temperature-corrected creep compliance, $T_0 = 298\text{ K}$.

scale than possible by direct testing. The master creep curve shows three different regions of viscoelastic behavior:

- 1. A glassy region where the effect of time is not very important, corresponding to low-temperature or short-time response of the material.
- 2. A transition region in which the reduced compliance changes very rapidly with time and temperature.
- 3. The onset of a third region, the rubbery region, where the time effect diminishes.

Typical results from the stress relaxation tests on the sand-asphalt mixture are shown in Figure 6. These results were obtained at -15 C for three levels of strain. To emphasize the variations of the results at short times, the time axis is plotted in logarithmic scale. The origin of time is taken at the end of the loading time. Relaxation modulus is defined as $E_r(t) = E(t)/\epsilon_0$, ϵ_0 being the strain applied at zero time as a step function. Experimentally, the application of the strain cannot be instantaneous. A high rate of loading can be used, but it may produce wave propagation and

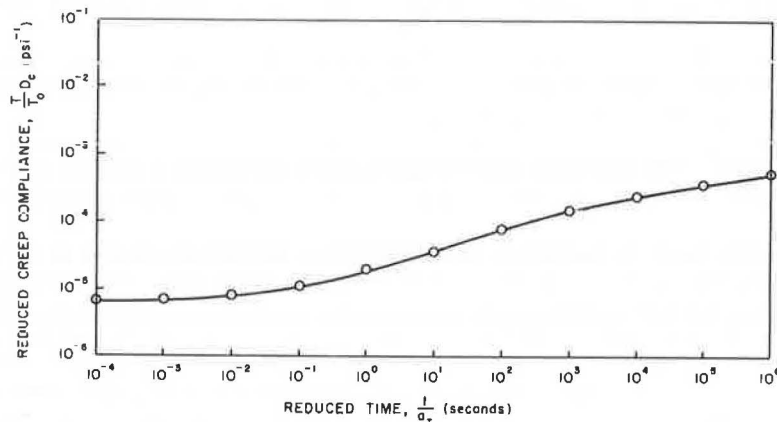
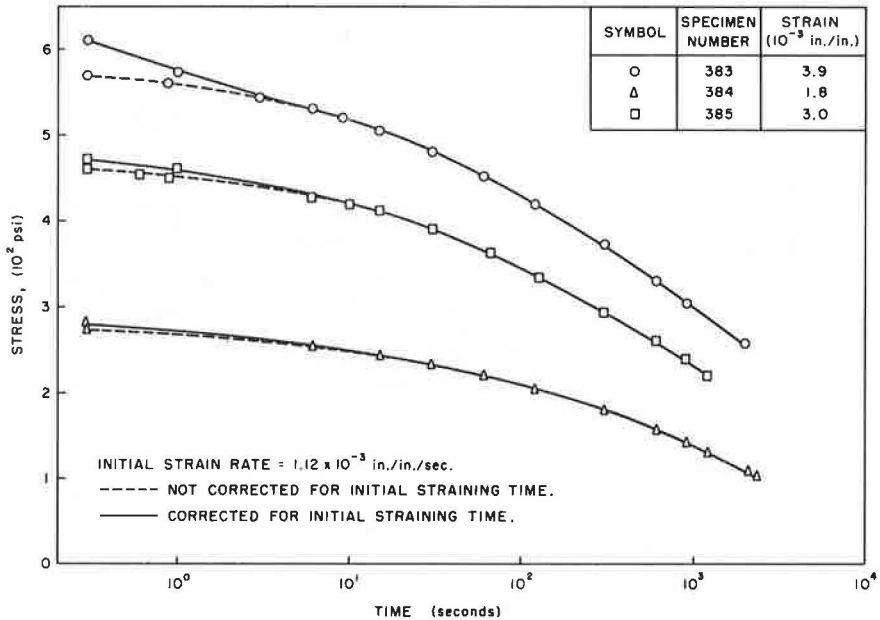
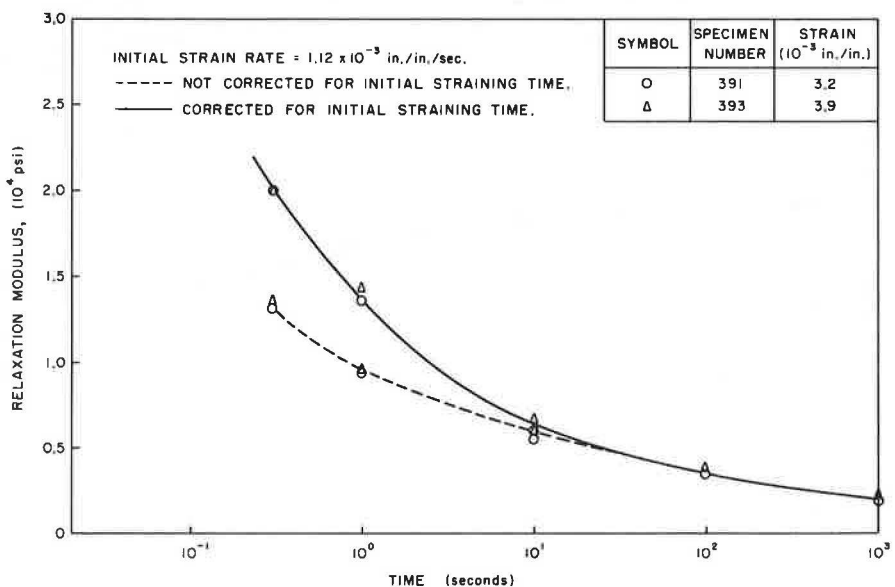


Figure 5. Typical creep compliance master curve, $T_0 = 298\text{ K}$.

Figure 6. Typical stress relaxation curves, -5°C .

the inertia forces would become significant. To avoid this difficulty the specimen was deformed at a rate of 0.2 in./min, which corresponded to a strain rate of 1.12×10^{-3} in./in./sec. A numerical method using a collocation technique was then used to correct for the error introduced by the stress relaxation occurring during the loading period.

The results of such a correction are shown in Figures 6 and 7 for two different test temperatures. Figure 6 contains the results of stress relaxation tests at -5°C . At this

Figure 7. Typical relaxation modulus, 35°C .

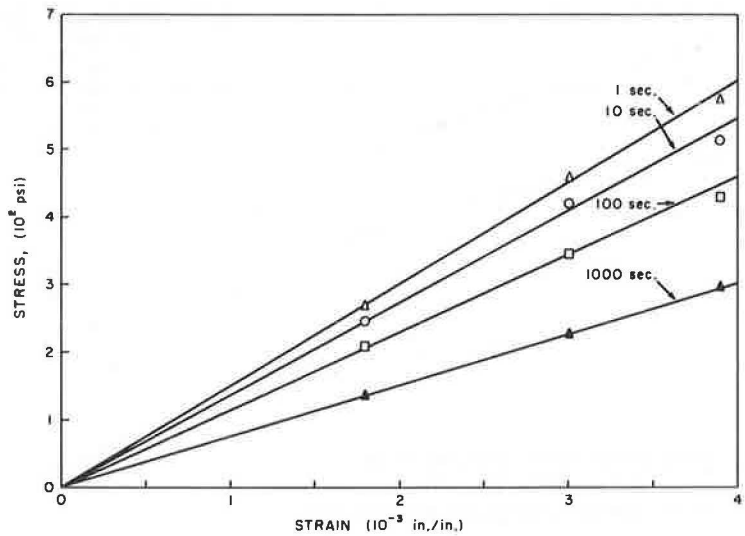


Figure 8. Isochrones of typical stress relaxation curves, -5 C.

temperature, response of material does not vary significantly with time and it is referred to as the glassy region, and hence the correction for ramp loading is not very significant. Figure 7, however, represents the relaxation modulus at 35 C, the transition region. The correction is significant and thus should be accounted for. The accuracy of the correction method was tested on a three-element mechanical model, and the plot of the master curve, using the time-temperature superposition principles, showed that the relaxation curves, when corrected for the ramp loading effect, superpose better than the uncorrected results. Thus all the relaxation results in this study are corrected for ramp loading, and are then used for subsequent analysis. Figure 8

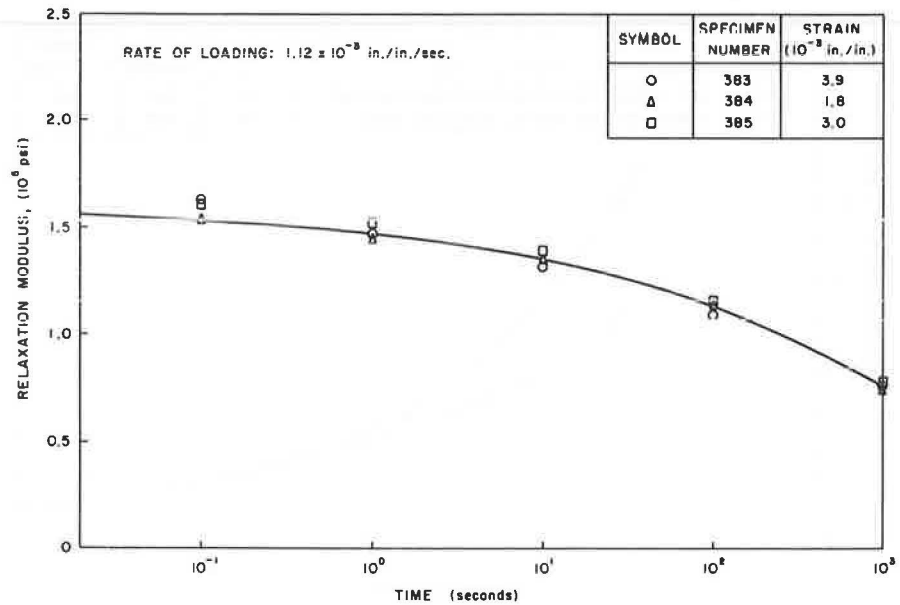


Figure 9. Typical relaxation modulus, -5 C.

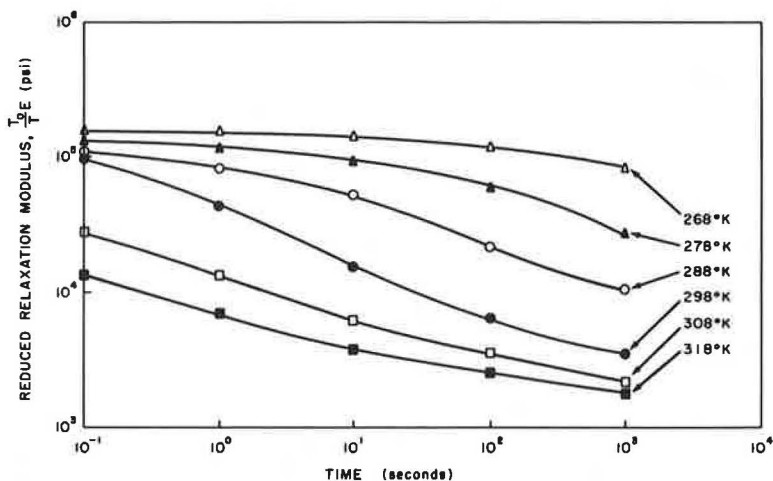


Figure 10. Typical temperature-controlled relaxation modulus, $T_0 = 298$ K.

shows the linearity of the relaxation results. This figure is similar to Figure 2, which shows the linearity examination of creep data. The relaxation results used in Figure 8 were obtained at -5 C. The relaxation modulus at -5 C is plotted in Figure 9, which also shows the linearity of the response under the strain levels applied.

In Figure 10, all the reduced relaxation moduli are plotted vs time to an arbitrary base temperature of 298 K. By appropriate horizontal shifts of these reduced curves, the relaxation master curve at the reference temperature of 25 C is constructed in Figure 11. The procedure used here is similar to that used for the creep compliance master curve. The master curve of relaxation modulus shows also three different regions: glassy region, transition region, and rubbery region. The values of the shift factors obtained from both the creep and the relaxation results are compared in Figure 12. This figure shows that the variation of the shift factor with temperature is a smooth curve. This is a necessary condition for the application of the time-temperature superposition principle. Moreover, within experimental errors, the shift factors obtained from the relaxation and the creep tests are the same. This indicates that the sand-asphalt mixture used is a thermorheologically simple material. The shift factor

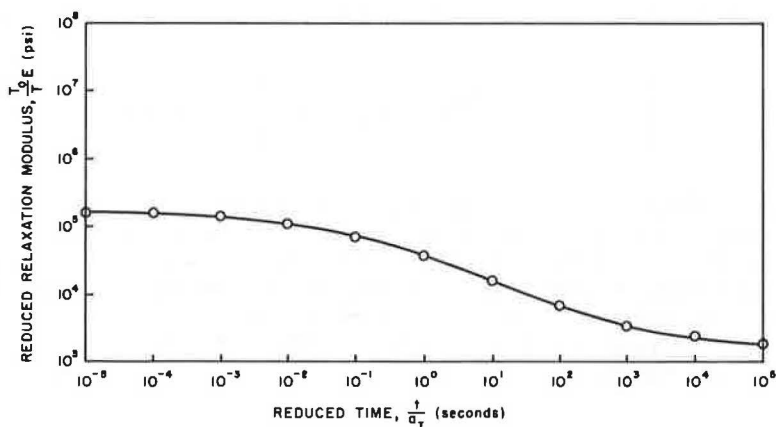


Figure 11. Typical relaxation modulus master curve, $T_0 = 298$ K.

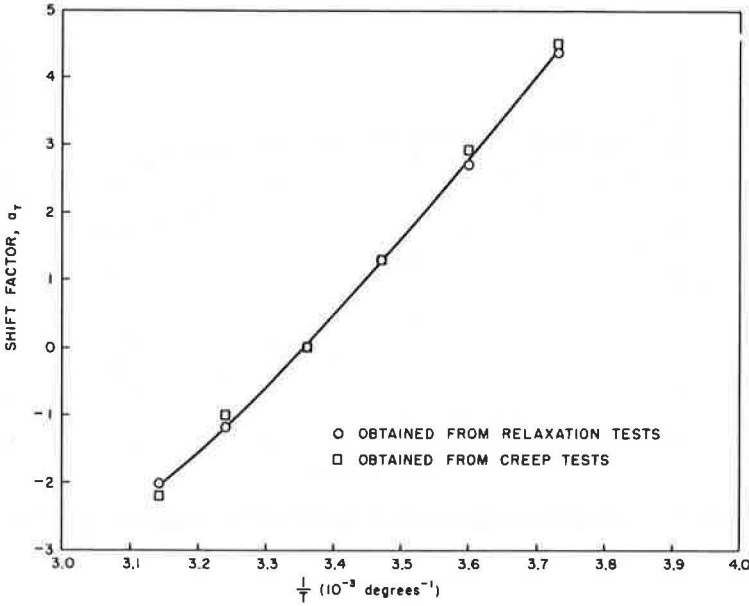


Figure 12. Shift factor as a function of inverse temperature.

represents the function of temperature at a given time, while the master curve represents the function of time at a given temperature.

To express the relaxation and the creep characteristic functions of a material analytically, the experimental data can be handled using mathematical expressions that would fit closely the obtained data in the experimental range and that would permit a reasonable approximation of the functions for any extrapolation. Schapery (28) has suggested the use of Dirichlet series for this purpose. The technique involves selection of points (X_i, t_i) one decade apart in the transition region, and expressing the data as

$$X(t) = X_{\infty} + \sum_{i=1}^n X_i e^{-t/2t_i}$$

Assuming that n points of the curves are given, a linear system of n equations will result:

$$\left[\sum_{i=1}^n X_i e^{-t_j/2t_i} = X(t_j) - X_{\infty} \right] \text{ for } 1 \leq j \leq n$$

This system can then be solved easily for the coefficients.

This technique not only allows one to express the viscoelastic characteristic functions of a material in an analytical form, but it also provides an excellent means of establishing the relations that exist among the various viscoelastic functions of a material. The numerical values obtained for coefficients E_i and D_i of relaxation modulus and creep compliance, shown in master curve forms in Figures 5 and 11, are given in Table 3. The expressions for the relaxation modulus $E_r(t)$ and the creep compliance $D_c(t)$ are

$$E_r(t) = E_{\infty} + \sum_{i=1}^{10} E_i e^{-\alpha_i t}$$

TABLE 3
NUMERICAL VALUES OF COEFFICIENTS OF RELAXATION MODULUS AND CREEP COMPLIANCE

(a) for relaxation $E_r(t) = E_\infty + \sum_{i=1}^{10} E_i e^{-\alpha_i t}$	(b) for creep $D_c(t) = D_\infty + \sum_{i=1}^{11} D_i e^{-\alpha_i t}$
where: $E_1 = 6.65 \times 10^3$ $\alpha_1 = 5 \times 10^3$ $E_2 = 2.32 \times 10^4$ $\alpha_2 = 5 \times 10^2$ $E_3 = 4.19 \times 10^4$ $\alpha_3 = 5 \times 10^1$ $E_4 = 3.30 \times 10^4$ $\alpha_4 = 5 \times 10^0$ $E_5 = 3.01 \times 10^4$ $\alpha_5 = 5 \times 10^{-1}$ $E_6 = 1.26 \times 10^4$ $\alpha_6 = 5 \times 10^{-2}$ $E_7 = 4.30 \times 10^3$ $\alpha_7 = 5 \times 10^{-3}$ $E_8 = 1.44 \times 10^3$ $\alpha_8 = 5 \times 10^{-4}$ $E_9 = 8.36 \times 10^2$ $\alpha_9 = 5 \times 10^{-5}$ $E_{10} = 4.03 \times 10^2$ $\alpha_{10} = 5 \times 10^{-6}$ $E_\infty = 1.60 \times 10^3$	where: $D_1 = 8.87 \times 10^{-6}$ $\alpha_1 = 5 \times 10^3$ $D_2 = -1.25 \times 10^{-6}$ $\alpha_2 = 5 \times 10^2$ $D_3 = 4.42 \times 10^{-7}$ $\alpha_3 = 5 \times 10^1$ $D_4 = -8.09 \times 10^{-8}$ $\alpha_4 = 5 \times 10^0$ $D_5 = -7.42 \times 10^{-8}$ $\alpha_5 = 5 \times 10^{-1}$ $D_6 = -2.52 \times 10^{-8}$ $\alpha_6 = 5 \times 10^{-2}$ $D_7 = -5.43 \times 10^{-9}$ $\alpha_7 = 5 \times 10^{-3}$ $D_8 = -8.21 \times 10^{-9}$ $\alpha_8 = 5 \times 10^{-4}$ $D_9 = -9.95 \times 10^{-9}$ $\alpha_9 = 5 \times 10^{-5}$ $D_{10} = -1.19 \times 10^{-4}$ $\alpha_{10} = 5 \times 10^{-6}$ $D_{11} = -1.97 \times 10^{-4}$ $\alpha_{11} = 5 \times 10^{-7}$ $D_\infty = 6.0 \times 10^{-4}$

and

$$D_c(t) = D_\infty + \sum_{i=1}^{11} D_i e^{-\alpha_i t}$$

where E_∞ and D_∞ are the values of the relaxation modulus and the creep compliance at infinite time.

The number of terms was chosen approximately equal to the number of decades of time where the curves have a significant slope. The use of this technique in certain cases may result in some coefficients of different signs, which means that too many terms were used in the corresponding region, or that the curve is not smooth in that region. This point may not be important in the fitting procedure but it results in a considerable deviation once the data are used to extrapolate the results. The different terms of the series cannot be considered as having a real physical meaning as in the case of a mechanical model. In the series representation the number of elements is chosen arbitrarily, and the representation should be considered as a purely mathematical approximation.

Since the results of creep and relaxation tests are not directly comparable in the time domain, the associated values of the relaxation modulus and the creep compliance may be compared in the Laplace domain. This can be achieved using the Laplace transform of the foregoing expressions for $E_r(t)$ and $D_c(t)$, that is,

$$\bar{E}(p) = p \bar{E}_r(p) = p \int_0^\infty E_r(t) e^{-pt} dt$$

$$\bar{E}(p) = p \left[\frac{E_\infty}{p} + \sum_{i=1}^{10} \frac{E_i}{p + \alpha_i} \right]$$

and

$$\bar{D}(p) = p \bar{D}_c(p) = p \int_0^\infty D_c(t) e^{-pt} dt$$

$$\bar{D}(p) = p \left[\frac{D_\infty}{p} + \sum_{i=1}^{11} \frac{D_i}{p + \alpha_i} \right]$$

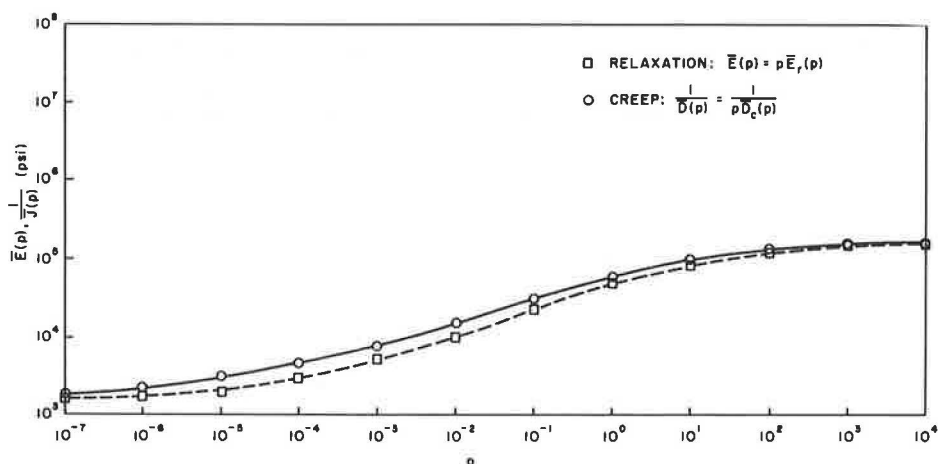


Figure 13. Laplace transform of creep and relaxation master curves.

In Laplace domain, these quantities are simply the inverse of each other, such that $\bar{E}(p) = [\bar{D}(p)]^{-1}$ or $p\bar{E}(p) = [p\bar{D}_c(p)]^{-1}$. The values of $\bar{E}(p)$ and $[\bar{D}(p)]^{-1}$ are plotted vs p in Figure 13. This figure shows that a good correspondence exists between the results of the creep and relaxation in all regions except in certain transition parts, where the two values deviate from each other. This deviation may be due to experimental error at one of the intermediate test temperatures used in this study. Substituting $i\omega$, where ω is the frequency of a dynamic testing, for p on the abscissa of Figure 13, this figure can then be considered as a plot of complex modulus vs frequency. This indicates that large values of p correspond to high frequencies or short times (for constant temperature) or for low temperatures (for constant time), and low values of p represent the behavior at low frequencies and long loading time (for constant temperature) or at high temperature (for constant time).

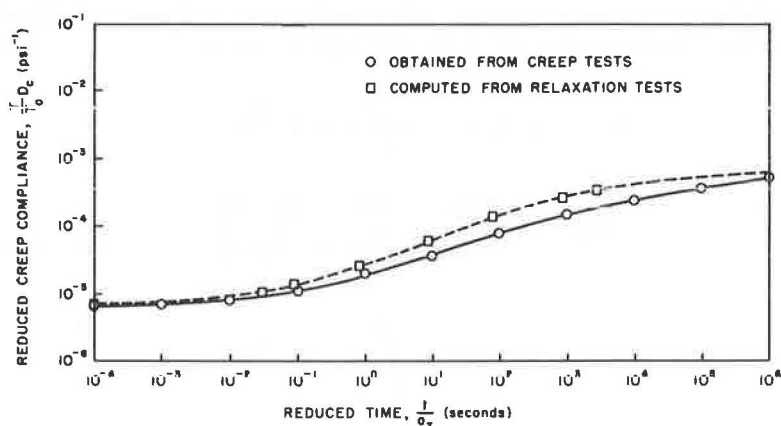


Figure 14. Creep compliance, $T_0 = 298$ K.

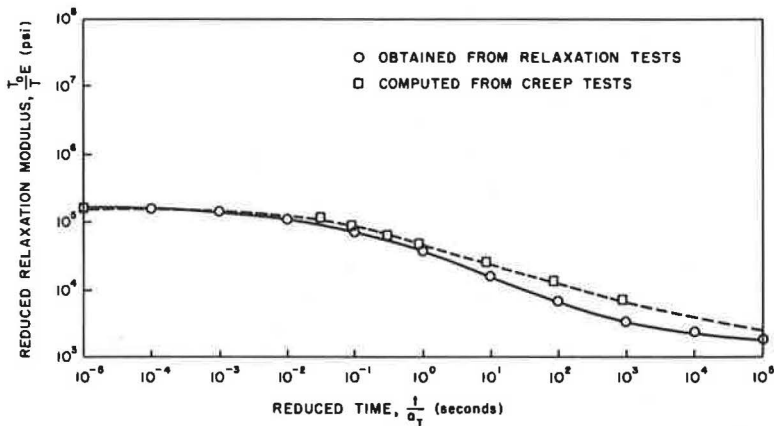


Figure 15. Relaxation modulus, $T_0 = 298$ K.

Figure 14 shows how a direct comparison in the time domain can be obtained using numerical integration of Volterra equations:

$$\int_0^{\infty} E_r(t - \tau) D_c(\tau) d\tau = t$$

or

$$\int_0^{\infty} D_c(t - \tau) E_r(\tau) d\tau = t$$

Figure 15 shows a plot of the relaxation modulus as observed and as computed from the creep compliance curve. Comparisons of creep and relaxation results in Figures 13, 14, and 15 in Laplace and time domains show the same trend: a good comparison in the glassy and rubbery regions, and some deviations occurring in the transition region.

CONCLUSIONS

1. The response of a sand-asphalt mixture to creep and relaxation modes of testing is similar to that of a linear viscoelastic material, providing the levels of imposed stresses and/or strains are small.
2. Time-temperature superposition is applicable to the response of such material in creep or relaxation tests, and the necessary shift factors in these two modes of testing are almost identical.
3. Exponential series representation can be very useful and provide easy transformations of the different viscoelastic functions. They allow a good representation of the experimental data over a very wide time or frequency range, and the mathematical transformation using direct formula or numerical methods can be used to obtain the necessary functional transformations to a high degree of accuracy.

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