Stochastic Models in Traffic Flow Theory: Intersection Control

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A new model of a vehicle-actuated traffic signal is introduced. This model is a natural generalization of the fixed-cycle traffic signal and can be analyzed under certain dependent input conditions. The first model applies to the intersection of 2 one-way, one-lane streets. The time-dependent and asymptotic behavior of the traffic signal and the traffic queues is determined by using results from the theory of storage. Exact distributions of green times and their moments are determined. Conditions for asymptotic stability are given, and when these conditions are satisfied the steady-state queue sizes are determined. Expected total delay per cycle is used as a criterion of optimality, and the optimal signal settings are given in certain special cases. The model is generalized to include more complicated intersections, such as the intersection of $k \geq 2$ one-way, one-lane streets, a four-way intersection with no left-turning vehicles, and a four-way intersection with a separate cycle phase for left-turning vehicles. Two types of dependent input processes are considered—Markov chains and martingales. Many of the preceding results carry over to these types of input processes.

•MUCH ATTENTION has been devoted recently to models of traffic signal behavior. Unfortunately most of these models deal with a fixed-cycle traffic signal, and relatively few apply to vehicle-actuated traffic signals. The models of vehicle-actuated traffic signals that have thus far been introduced are generally lacking in several important respects. First, the analysis usually depends heavily on the assumption of Poisson and Bernoulli input. Input of this nature may be reasonable for an isolated intersection; however, it is not reasonable in an urban setting where input tends to arrive in waves depending on the status of nearby traffic signals. Second, the models generally deal with very simple types of intersections. Ideally a model should apply to four-way intersections, perhaps with left turns allowed.

This paper introduces a new model of a vehicle-actuated traffic signal that overcomes the 2 difficulties mentioned. First, a very simple intersection will be analyzed under restrictive input conditions, then the model is later generalized to include more complicated intersections with dependent input.

THE MODEL

Consider the intersection of 2 one-way, one-lane streets controlled by a cycle that begins when the light turns green for the north flow. It remains green for at least g_1 seconds, and each car entering the intersection in the north direction extends the green time by e_1 seconds. The green time can be extended to a maximum length of g_2 seconds after which all cars subsequently crossing the intersection do not extend the green time any further. At the conclusion of the green period, the light turns red for both flows for a random length of time. This lost time simulates the yellow phase of the light and the extra start-up time of the first car in the east flow. The light then turns green for the east and stays green for at least r_1 seconds. Each car crossing the

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intersection in the east direction extends the green time by e_2 seconds to a maximum value of r_2. At the conclusion of the green period, the light turns red for both directions for a random lost time. This time simulates the yellow phase and the extra start up time of the first car in the north queue. The cycle is then complete and is repeated.

FIXED-CROSSING TIMES

Assumptions

1. g_1, g_2, r_1, and r_2 are positive integers satisfying

$$g_0 \le g_1 \le g_2 \le \infty, \quad r_0 \le r_1 \le r_2 \le \infty$$

2. e_1 and e_2 are positive integers such that e_1 divides $g_2 - g_1$ and e_2 divides $r_2 - r_1$.
3. Each car takes a fixed equal length of time to cross the intersection. This time, τ, is the same for both flows. Without loss of generality all time quantities can be measured with respect to this time unit; hence, time may be rescaled so that $\tau = 1$.
4. If the light is green for either direction, the queue for that direction is empty at time k of the n th cycle, and i cars arrive during (k, k + 1), then the queue size at time k + 1 will be i. If a car arrives during (k, k + 1), $k + 1 < g_2$, then the light will be extended to at least time k + 2.
5. $\{L_{1n}, n \ge 1\}$ and $\{L_{2n}, n \ge 1\}$ are mutually independent sequences of iid random variables. The random variables are non-negative and integer valued. L_{1n} represents the lost time to the north flow just prior to the start of the n th cycle. L_{2n} represents the lost time to the east flow during the n th cycle. $E(L_{1n}) = \ell_1$, $E(L_{2n}) = \ell_2$, $Var(L_{1n}) = \delta_1^2$, and $Var(L_{2n}) = \delta_2^2$, $\delta_1^2, \delta_2^2 < \infty$.
6. $\{Y_{in}\}$ and $\{Z_{in}\}$ are mutually independent sequences of iid random variables. Y_{in} represents the number of arrivals in the north direction during (i, i + 1) of the n th cycle, and Z_{in} represents the number of arrivals in the east direction for the same time period. The input sequences are independent of the lost times. $E(Y_{in}) = p_1$, $E(Z_{in}) = p_2$, $Var(Y_{in}) = \sigma_1^2$, and $Var(Z_{in}) = \sigma_2^2$, $\sigma_1^2, \sigma_2^2 < \infty$.
7. $g_2 = r_2 = \infty$.

Time-Dependent Behavior

Let

Q_{1n} = north queue length at the start of the n th north green time;
Q_{2n} = east queue length at the start of the n th east green time;
t_n = n th north green time; and
r_n = n th east green time.

First, one can show that the cycle mechanism described earlier is equivalent to one that is mathematically easier to handle. The new mechanism assumes that cars extend the light as they join the queue, not when they cross the intersection. Letting $Y_n(j)$ be the random walk $Y_n(j) = g_1 + e_1(Q_{1n} + \sum_{i=0}^{j-1} Y_{in}) - j$, one can define a stopping time $y_n = \inf\{j \mid Y_n(j) = 0\}$. The north green time is given by $t_n = \min(y_n, g_2)$.

The stopping time y_n can be thought of as the first emptiness time of a storage process with initial height $g_1 + e_1 Q_{1n}$, release of one unit per second, and input of Y_{in} units per second. The exact distribution of y_n can be found by using a result from Lloyd's work (3); thus, by truncation

$$P(t_n = t \mid Q_{in}) = \begin{cases} \dfrac{g_1 + e_1 Q_{1n}}{i} f(i - g_1 - e_1 Q_{1n}, i \mid Q_{1n}), & i < g_2 \\ 1 - \displaystyle\sum_{j=g_1+e_1Q_{1n}}^{g_2-1} \dfrac{g_1 + e_1 Q_{1n}}{j} f(j - g_1 - e_1 Q_{1n}, j \mid Q_{1n}), & i = g_2 \end{cases} \tag{1}$$

where

$$f(i, j) = P(Y_{0n} + \ldots + Y_{j-1,n} = i).$$

The exact distribution of the north green time will have a simple or complicated form depending on the complexity of the n-fold convolution of the input distributions; however, the moments of this distribution have a particularly simple form. From an application of Wald's equation to the random walk $\{Y_n(j),\ j \geq 1\}$, one may find all moments of the stopping time y_n. Thus under assumption 7,

$$E(t_n|Q_{1n}) = \begin{cases} \dfrac{g_1 + e_1 Q_{1n}}{1 - p_1 e_1} & p_1 e_1 < 1 \\ +\infty & p_1 e_1 \geq 1 \end{cases} \tag{2}$$

$$E(t_n^2|Q_{1n}) = \begin{cases} \dfrac{(g_1 + e_1 Q_{1n})^2}{(1 - p_1 e_1)^2} + \dfrac{(g_1 + e_1 Q_{1n})\sigma_1^2 e_1^2}{(1 - p_1 e_1)^3} & p_1 e_1 < 1 \\ +\infty & p_1 e_1 \geq 1 \end{cases} \tag{3}$$

Similar results may be derived for r_n. These equations show that $p_1 e_1 < 1$ and $p_2 e_2 < 1$ are necessary conditions for t_n and r_n to have finite moments of all orders; however, stronger conditions are required for asymptotic stability when $g_2 = r_2 = \infty$.

Asymptotic Behavior

It is important to find conditions under which the queue lengths remain asymptotically stable. This problem must be broken into 3 distinct cases, each one giving different results.

$g_2 = r_2 = \infty$—By inspection it can be seen that $\{Q_{1n},\ n \geq 1\}$ and $\{Q_{2n},\ n \geq 1\}$ each form Markov chains. The transition probability matrices are very complicated with all positive entries, thus one cannot solve the steady-state equations $\pi P = \pi$ even for the simplest types of input. One can, however, compute the conditional moments and prove positive recurrence in this fashion.

$$E(Q_{1n+1}|Q_{1n}) + a + bQ_{1n} \qquad a \geq 0,\ b \geq 0 \tag{4}$$

This result is used to prove theorem 1: If $b < 1$, then $\{Q_{in},\ n \geq 1\}$ is a positive recurrent Markov chain, $i = 1, 2$. Thus there exist almost surely finite random variables Q_1 and Q_2 such that $Q_{in} \xrightarrow{W} Q_i$, and Q_i has the steady-state distribution, $i = 1, 2$.

The condition $b < 1$ is equivalent to $p_1 e_1 + p_2 e_2 < 1$, thus $p_1 e_1 + p_2 e_2 < 1$ is a necessary and sufficient condition for asymptotic stability of the intersection. All moments of the steady-state queue length may be found even though the exact distribution cannot be. The expected steady-state cycle length is $\dfrac{g_1 + r_1 + \ell_1 + \ell_2}{1 - p_1 e_1 - p_2 e_2}$ for $p_1 e_1 + p_2 e_2 < 1$.

$g_2 = \infty$, $r_2 < \infty$ or $g_2 < \infty$, $r_2 = \infty$—When $g_2 = \infty$ and $r_2 < \infty$, the north queue will remain stable if $p_1 e_1 < 1$, because the east green time is at most r_2 units long, and this time can be regarded as an extra lost time that does not affect asymptotic stability.

Again by the consideration of conditional moments one may prove theorem 2: If $r_2 > \dfrac{p_2(g_1 + \ell_1 + \ell_2)}{1 - p_1 e_1 - p_2}$ and $p_1 e_1 + p_2 < 1$, then the north and east queue lengths are asymptotically stable.

The case $g_2 < \infty$ and $r_2 = \infty$ can be treated in an identical manner. The conditions for asymptotic stability are $g_2 > \dfrac{p_1(r_1 + \ell_1 + \ell_2)}{1 - p_1 - p_2 e_2}$ and $p_1 + p_2 e_2 < 1$.

$g_2 < \infty$, $r_2 < \infty$—This case required the definition of a bivariate Markov chain. By consideration of conditional moments, one may determine conditions for positive recurrence.

Theorem 3 is as follows: If $p_1 < \dfrac{g_2}{g_2 + r_2 + \ell_1 + \ell_2}$ and $p_2 < \dfrac{r_2}{g_2 + r_2 + \ell_1 + \ell_2}$, then the north and east queue lengths are asymptotically stable.

Comparison of Theorems—It is of interest to compare the conditions for asymptotic stability given in these 3 theorems.

Theorem 1: $g_2 = \infty$, $r_2 = \infty$

$$p_1 e_1 + p_2 e_2 < 1$$

Theorem 2: $g_2 = \infty$, $r_2 < \infty$

$$p_1 e_1 + p_2 < 1, \quad r_2 > \frac{p_2(\ell_1 + \ell_2 + g_1)}{1 - p_1 e_1 - p_2}$$

or $g_2 < \infty$, $r_2 = \infty$

$$p_1 + p_2 e_2 < 1, \quad g_2 > \frac{p_1(\ell_1 + \ell_2 + r_1)}{1 - p_1 - p_2 e_2}$$

Theorem 3: $g_2 < \infty$, $r_2 < \infty$

$$p_1 + p_2 < 1, \quad p_1 < \frac{g_2}{g_2 + r_2 + \ell_1 + \ell_2}, \quad p_2 < \frac{r_2}{g_2 + r_2 + \ell_1 + \ell_2}$$

In the first case, e_1 and e_2 behave like scale parameters; however, when either g_2 or $r_2 < \infty$, then e_1 or e_2 respectively has no effect on the asymptotic stability of the intersection. This illustrates that the special case $g_2 = r_2 = \infty$ cannot be treated as a limiting case of $g_2 < \infty$ and $r_2 < \infty$.

Total Delay per Cycle

The criterion of expected total delay per cycle is used as a criterion of optimality. This criterion has the drawback of ignoring the effects of the lost times; however, most reasonable criteria will be related to the total delay per cycle.

Explicit formulas for the mean value and variance of the total delay per cycle can be found, and from these it is easily shown that setting $g_1 = g_0$, $r_1 = r_0$, and $e_1 = e_2 = 1$ when $g_2 = \infty$ and $r_2 = \infty$ simultaneously minimizes expected total delay to the north and east flows. When $p_1 e_1 + p_2 e_2$ is nearly 1, then the expected total delay behaves like $\dfrac{1}{(1 - p_1 e_1 - p_2 e_2)^2}$; however, the variance behaves like $\dfrac{1}{(1 - p_1 e_1 - p_2 e_2)^3}$. This shows that, for values of $p_1 e_1 + p_2 e_2$ near saturation, the mean delay is a misleading criterion because of the large variance; however, setting $g_1 = g_0$, $r_1 = r_0$, and $e_1 = e_2 = 1$ minimizes the variance when $g_2 = r_2 = \infty$.

RANDOM-CROSSING TIMES

The following model of vehicle actuation is assumed. A cycle begins when the light turns green for the north flow. The light will stay green until the north queue empties at which time it changes to red. The light is red for both flows for a random lost time and then changes to green for the east. The light stays green until the east queue first empties, at which time it turns red for both directions. After a random lost time the cycle is completed.

Assumptions

Assumptions 1, 2, 4, and 7 are removed. Assumption 3 is modified as follows: $\{C_{in}, i \geq 1, n \geq 1\}$ and $\{D_{in}, i \geq 1, n \geq 1\}$ are mutually independent sequences of

iid positive integer valued random variables. $C_{in}(D_{in})$ represents the crossing time of the i th northbound (eastbound) vehicle during the n th cycle. Let $P(C_{in} \geq 1) = P(D_{in} \geq 1) = 1$; $E(C_{in}) = \mu_1$; $E(D_{in}) = \mu_2$; $Var(C_{in}) = \sigma_1^2$; and $Var(D_{in}) = \sigma_2^2$.

Time-Dependent Behavior

Let

t_n = north green time during n th cycle;
Q_{1n} = north queue length at the start of the n th cycle; and
N_n = number of northbound cars crossing during the n th cycle.

By considering t_n to be the first emptiness time of a dam and by using a result from Lloyd's work (3), it can be shown that

$$P(t_n = t,\ N_n = k|Q_{1n}) = \frac{Q_{1n}}{k} f(k - Q_{1n},\ t|Q_{1n})h(t,\ k),\quad t \geq k \geq Q_{1n} \tag{5}$$

where

$f(i,\ j) = P(Y_{0n} + \ldots + Y_{j-1n} = i)$, and
$h(i,\ j) = P(C_{1n} + \ldots + C_{jn} = i)$.

The moments of t_n and N_n given Q_{1n} can be determined by using Wald's equation applied to bivariate martingales.

$$E(N_n|Q_{1n}) = \begin{cases} \dfrac{Q_{1n}}{1 - p_1\mu_1} & p_1\mu_1 < 1 \\ +\infty & p_1\mu_1 \geq 1 \end{cases} \tag{6}$$

$$E(t_n|Q_{1n}) = \begin{cases} \dfrac{Q_{1n}\mu_1}{1 - p_1\mu_1} & p_1\mu_1 < 1 \\ +\infty & p_1\mu_1 \geq 1 \end{cases} \tag{7}$$

Higher moments of t_n and N_n given Q_{1n} may be calculated in this same manner. The results are surprisingly simple independent of how complex the exact distributions may be.

Asymptotic Behavior

The asymptotic behavior of the queues can be treated in a manner identical to that used earlier. $\{Q_{1n},\ n \geq 1\}$ and $\{Q_{2n},\ n \geq 1\}$ are Markov chains, and $E(Q_{1n+1}|Q_{1n}) = c + dQ_{1n}$ with $c \geq 0$, $d \geq 0$.

This form for the conditional moments is important in the proof of theorem 4: If $d < 1$, then $\{Q_{1n},\ n \geq 1\}$ and $\{Q_{2n},\ n \geq 1\}$ are positive recurrent Markov chains. Hence, there exist almost surely finite random variables Q_1 and Q_2 such that $Q_{in} \xrightarrow{W} Q_i$, and Q_i has the steady-state distribution, $i = 1,\ 2$.

The condition $d < 1$ is equivalent to $p_1\mu_1 + p_2\mu_2 < 1$. When this condition holds, the steady-state moments of the queue length can be calculated exactly.

Total Delay per Cycle

The total expected delay per cycle for both the north and the east flows can be determined exactly by techniques that are a natural extension of those used for the fixed-crossing time. When $p_1\mu_1 + p_2\mu_2$ is nearly equal to 1, the expected total delays for both the north and east behave like $\dfrac{1}{(1 - p_1\mu_1 - p_2\mu_2)^2}$; however, the variance behaves like

$\frac{1}{(1 - p_1\mu_1 - p_2\mu_2)^3}$. Thus for values of $p_1\mu_1 + p_2\mu_2$ near saturation, the mean delay becomes a misleading criterion of optimality because of the large variance.

COMPLICATED INTERSECTIONS

Many of the results stated carry over to more complicated types of intersections.

Intersection of k One-Way, One-Lane Streets

The fixed-crossing time model given before extends easily to this situation. The signal mechanism works in the same manner except there are now 3 sets of k parameters: $\{g_1, \ldots, g_k\}$ the minimum green times, $\{r_1, \ldots, r_k\}$ the maximum green times, and $\{e_1, \ldots, e_k\}$ the extension times. One can easily determine the time-dependent behavior of the queues by using methods identical to those used earlier. The asymptotic behavior is also easily handled.

Theorem 5—If $\sum_{i=1}^{k} p_i e_i < 1$ and $r_1 = r_2, \ldots, r_k = \infty$, then the intersection is asymptotically stable.

Suppose that m of the r_i's are finite and the rest are infinite, $m \geq 0$. The conditions for asymptotic stability are somewhat more complex. Let us relabel the parameters so that $r_1 = \ldots = r_{k-m} = \infty$.

Theorem 6—If

$$\sum_{i=1}^{k-m} p_i e_i + \sum_{i=k-m+1}^{k} p_i < 1$$

and

$$r_j > \frac{p_j \left(\sum_{i=k-m+1}^{k} r_i + \sum_{i=1}^{k} \ell_i + \sum_{i=1}^{k-m} g_i \right)}{1 - \sum_{i=1}^{k-m} p_i e_i - \sum_{i=k-m+1}^{k} p_i}, \quad j = k - m + 1, \ldots, k$$

then the intersection is asymptotically stable.

When the conditions for asymptotic stability are satisfied, the moments of the steady-state queue lengths can be calculated.

This model can be changed to apply to random-crossing times also. The exact distribution of green times and cars crossing the intersection can be determined. The asymptotic behavior is treated the same way as for $k = 2$.

Theorem 7—If $\sum_{i=1}^{k} p_i\mu_i < 1$, then the intersection is asymptotically stable. When $\sum_{i=1}^{k} p_i\mu_i < 1$, then the moments of the steady queue lengths can be derived explicitly.

Four-Way Intersection With No Left-Turning Cars

An intersection of this type cannot be controlled in a reasonable way so that all 4 flows exercise control over the traffic signal. Instead it is assumed that only 2 of the flows (either east or west and either north or south) control the cycle lengths. One must choose the flow that controls the light in such a way that the queues remain asymptotically stable. In the fixed-crossing time, the choice is made so that the flow with the highest average input controls the light. In the random-crossing time, the

choice is made so that the flow with the highest value of $p\mu$ (the average input times the average crossing time) controls. Thus the traffic signal will behave as if there were just 2 flows, and the intersection will remain stable if the controlling flows are chosen in the manner described before.

Four-Way Intersections With Left-Turning Cars

This situation may be treated in a way analogous to that in the preceding paragraph. The traffic signal now has 4 phases instead of the usual 2: north-south, north-south left turn, east-west, and east-west left turn. From each of these 4 pairs, one is chosen to control the light by the procedure just described. The analysis described for an intersection of k one-way, one-lane streets can now be applied to prove asymptotic stability and to find the steady-state queue length moments.

Dependent Input

The green times for the fixed-crossing time were found to be equivalent to the first emptiness time of a dam. This problem has been solved under more general conditions including Markov chains (1, 2) and martingales. In addition, the total delay per cycle can be determined when the input is in the form of a Markov chain. Thus many of the results for iid input will be carried over for some dependent types of input.

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