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A Procedure for Evaluating Pavements With Nonuniform Paving Materials

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A procedure was developed to evaluate layered systems with nonuniform material properties. The procedure consists of defining the layered system by a physical model consisting of mass points tied together by springs and bars. The variability of the material is simulated by assigning different characteristics of the material properties to springs connecting the mass points. Assignment of values representing the material properties is done on a random basis. The random values are generated in a manner that produces a model with mean characteristics corresponding to the mean properties of the materials in the various layers of the pavement, and with a coefficient of variation compatible with the coefficient of variation of the corresponding paving material.

Results from the study show that the response of the layered system is influenced by the statistical characteristics of the materials. The statistical nature of the response is influenced by both the variability of the material and the nature of the variability. A large area with slightly less than average stiffness has a greater influence on the response of the system than a large difference in stiffness over a small area. Thus, detailed analyses are necessary to obtain a comprehensive understanding of the behavior of the system. Much work still needs to be done to obtain a complete picture of the statistical nature of pavement response. Preliminary results strongly indicate a need for the type of analysis presented in the paper as a guide for establishing realistic quality control criteria for paving materials. With results from such a procedure it is possible to establish a cost benefit from higher quality control criteria.

•PAVING MATERIALS, because of their heterogeneous nature, have natural or inherent variations in their physical properties. This natural variability is compounded by nonuniformity in the material due to construction processes and techniques. Taken together, these variations may have a profound effect on the behavior and performance of pavement systems.

Little is known about the manner and extent of influence that material variability has on the behavior of pavements. It is known that pavements do exhibit significant variations in response to loads (1, 2, 3, 4), and this variability can often be attributed to nonuniformity in the paving materials. What is not clear is what role the variability in the various layers of the pavement plays in the nonuniform behavior of the overall pavement.

To evaluate the effects of nonuniform paving materials on the behavior of pavements it is necessary to (a) know the effect of the magnitude of material variability on the behavior of the pavement system, (b) know the effects of the size of the defect on the behavior of the pavement system, and (c) know the magnitude of variability of in-place paving materials. In this paper, a procedure is developed that can deal directly with items a and b. Item c can be evaluated indirectly by correlating the observed behavior of actual pavements with the results obtained from the procedure outlined here.

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It is important to know the relationships between material variability and pavement behavior due to the increased use of statistical quality control techniques. The present statistical quality control plans are based on the present level of control being delivered. The procedure developed herein can be used to evaluate various levels of material variability so that the optimum level of control can be determined.

Knowledge of the effect of defect size and severity on pavement performance is important for aiding inspectors in applying non-statistical sampling techniques. If the material is only slightly substandard or if the quantity of defective material is small, the decision between acceptance and rejection becomes unclear. The procedure presented here can be used to determine whether the substandard or defective material is critical, i.e., whether it would cause a noticeable decrease in pavement performance. The relationship between the size of the critical defect and its deviation from the standard would define the characteristics of the critical defect.

The specific objective of this paper is to describe the procedures developed to evaluate the statistical nature of load-induced stresses, strains, and deflections in pavements having materials with variable physical characteristics. While the procedure can be applied to a three-dimensional problem, the example solutions and applications given in this paper are limited to the two-dimensional plane strain case because of limited capacity of available computer systems.

DEVELOPMENT OF THE SOLUTION PROCEDURE

The method for determining the response of the pavement system due to variable material properties is based on a mathematical discrete element model. By using this type of model to represent the pavement system, the material properties can be varied from point to point, and in this way the variability of the material properties can be incorporated into the solution. The nature of the variability of the response can be determined by randomly selecting for every point in the model values for the material properties from their respective statistical populations and repeatedly solving the layered structural system using sets of material properties selected on a random basis. A sufficient number of solutions must be developed to evaluate the statistical nature of the pavement response.

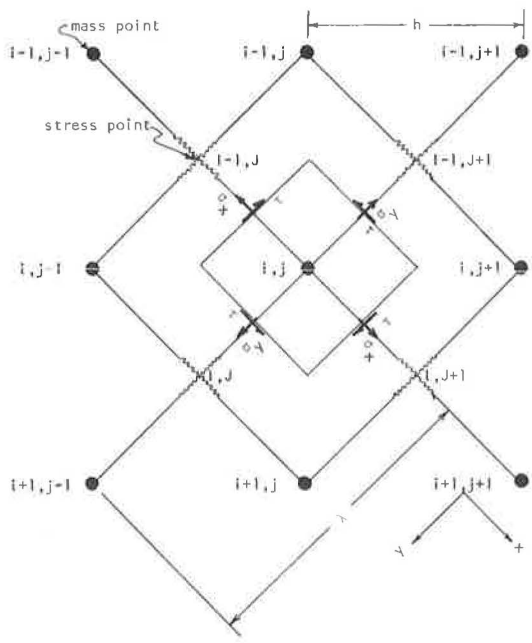


Figure 1. A typical section of the interior of the discrete element model.

Discrete Element Model

The discrete element mathematical model used in this work was developed by Ang and Harper (5, 6). The physical analog of the mathematical model consists of a two-dimensional rectangular grid of mass points connected on the diagonals by stress-strain elements. The mass points form the basis for deflection analysis while stresses, strains, and their relationships are defined at the intersections of the stress-strain elements, i.e., at the stress points. A typical section of the physical model is shown in Figure 1.

In mathematical terms, the model used is the central finite difference approximation of the basic differential equations from the theory of elasticity assuming plane strain. From the theory of elasticity the differential equations for relating deflection and strain are

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \text{and } \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

where x and y are the two coordinate axes (Fig. 1), u and v are the deformations in the x and y directions respectively, E_x and E_y are the axial strains in the x and y directions respectively, and Y_{xy} is the shear strain (7). The corresponding central finite difference equations relating strain and deflection at stress point I, J (Fig. 1) are

$$\epsilon_{x_{I,J}} = \frac{u_{i+1,j} - u_{i,j-1}}{\lambda}$$

$$\epsilon_{Y_{I,J}} = \frac{v_{i+1,j+1} - v_{i,j}}{\lambda}$$

and

$$\gamma_{xy_{I,J}} = \gamma_{I,J} = \frac{u_{i+1,j-1} - u_{i,j}}{\lambda} + \frac{v_{i+1,j} - v_{i,j-1}}{\lambda}$$

where λ is the diagonal spacing of the mass points.

Again from elastic theory the equations relating stress and strain are

$$\sigma_x = C\epsilon_x + B\epsilon_y$$

$$\sigma_y = C\epsilon_y + B\epsilon_x$$

and

$$\tau_{xy} = G\gamma_{xy}$$

where

$$C = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}$$

$$B = \frac{E\mu}{(1+\mu)(1-2\mu)}$$

$$G = \frac{E}{2(1+\mu)}$$

σ_x and σ_y are the normal stresses in the x and y directions respectively, τ is the shear stress, E is Young's modulus of elasticity, and μ is Poisson's ratio (7). Since no differentials are involved in these equations, the central finite difference forms for these equations are identical with those for the elastic theory.

This problem is being solved in terms of deflections, and therefore the only other equations needed are the equilibrium equations. Assuming that body forces and accelerations are zero (4), the differential equation for equilibrium in the x direction is (7)

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0$$

The corresponding central finite difference approximation for this equation at mass point i, j (Fig. 1) is

$$\left(\sigma_{x_{I,J+1}}\right)\frac{\lambda}{2} + (\tau_{I,J})\frac{\lambda}{2} - \left(\sigma_{x_{I-1,J}}\right) - (\tau_{I-1,J+1})\frac{\lambda}{2} = 0$$

Dividing by the volume represented by each mass point $(\lambda^2/2)$ and rearranging gives

$$\frac{\sigma_{x_{I,J+1}} - \sigma_{x_{I-1,J}}}{\lambda} + \frac{\tau_{I,J} - \tau_{I-1,J+1}}{\lambda} = 0$$

The corresponding differential and finite difference equations for equilibrium in the y direction are

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

and

$$\frac{\tau_{I,J+1} - \tau_{I-1,J}}{\lambda} + \frac{\sigma_{y_{I,J}} - \sigma_{y_{I-1,J+1}}}{\lambda} = 0$$

By substitution, the equilibrium equations for point i, j can be expressed in terms of the deflections of point i, j and the eight surrounding points. These equations are presented in the form of computational molecules as shown in Figure 2. A portion of the equilibrium equation in the x direction would be

$$\Sigma F_x = (C_{I-1,J})\,u_{i-1,j-1} + (B_{I-1,J} + G_{I-1,J+1})\,v_{i-1,j} + \cdots$$

$$+ (C_{I,J+1})\,u_{i+1,j+1} = 0$$

Boundary Conditions

To apply the model to typical pavement structures it is necessary to define consistent boundary conditions for the model (5). Three types of boundary conditions must be evaluated: (a) the boundary conditions at the pavement surface, including a means for applying external forces; (b) boundary conditions for the remainder of the perimeter; and (c) the boundary conditions for the interfaces between layers.

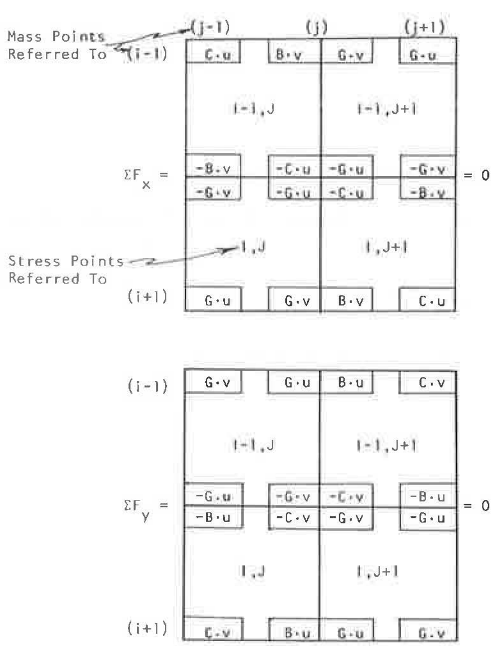


Figure 2. Computational molecules representing the equilibrium equations for an interior mass point in terms of deflections.

The equations defining the surface boundary condition can be determined by defining the equilibrium of the surface mass points. The equilibrium equations can be developed in terms of the deflection of the mass points at the surface. A typical surface mass point with the stresses acting on the point is shown in Figure 3.

The computational molecules for the equilibrium equations of the surface mass points are shown in Figure 4. Note that the computational molecule is the same as the lower half of its corresponding molecule describing the equilibrium of an internal mass point but that the top half has been replaced by the function of applied stresses and λ shown on the right side of the equation.

The boundaries around the remainder of the perimeter are assumed to be immoveable, and values for u and v for mass points along the perimeter are set at zero.

The boundary condition between layers is assumed to be perfectly rough; that is, there can be no relative movement between any point on the bottom of the upper layer and the corresponding point on the top of the lower layer. This condition is satisfied for the numerical model by defining the boundary through a row of mass points and defining the material properties for

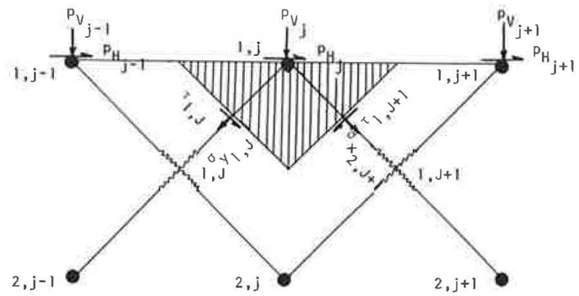


Figure 3. A typical section of the surface of the model.

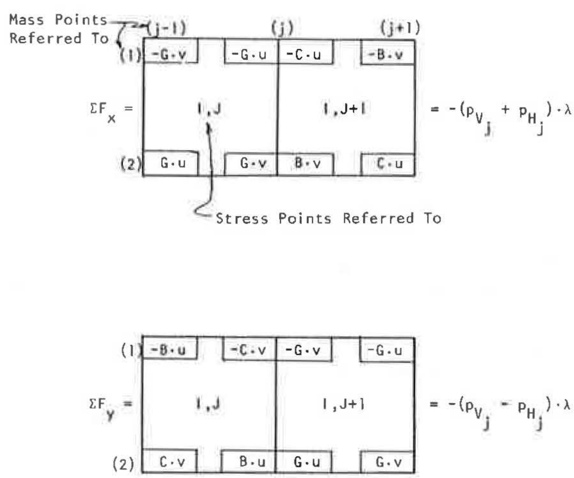


Figure 4. Computational molecules representing the equilibrium equations for a surface mass point in terms of deflections.

the stress points above the boundary to correspond with the material in the upper layer and the properties below the boundary to correspond to the material below the boundary.

Solving the Equilibrium Equations for Unknown Deflections, Strains, and Stresses

The equilibrium equations must be solved for the deflections at each mass point in the model. A modification of the Gauss elimination procedure was used to solve the equations for this study. The form of the unmodified coefficient matrix is shown in Figure 5.

The modification of the Gauss elimination procedure consists of operating on only the non-zero coefficients in the matrix. Two large groups of zero coefficients are located above and below the band of non-zero coefficients shown in Figure 6. Other groups are located in the cross-hatched areas also shown in Figure 6. The modified form of the coefficient matrix is shown in Figure 7.

The solutions to the equilibrium equations are obtained by operating on the modified matrix using a bookkeeping system that relates the location of the coefficients in the modified matrix to the corresponding coefficients in the unmodified matrix. The storage requirement for the modified matrix and constant vector is $4M^2N + 6MN$ for a model with $M \times N$ mass points and where $N \geq M$.

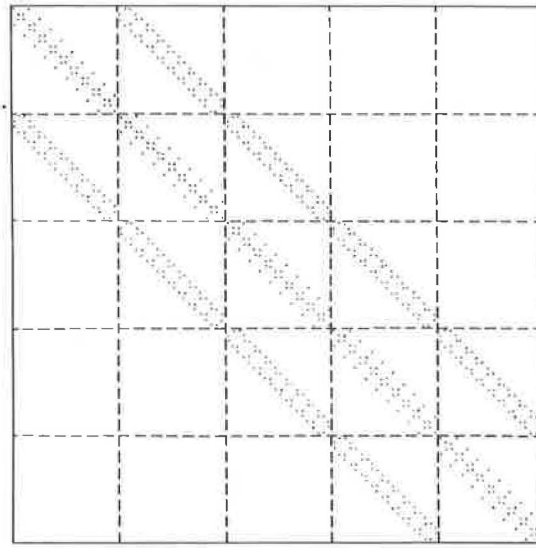


Figure 5. Form of the coefficient matrix for the equilibrium equations for a problem 10 points deep and 5 points wide.

Generation and Assignment of Random Variables

A random number generator was used to assign the material properties to stress points in the model. The method used is based on the central limit theorem of statistics. To obtain a standard normal, pseudo-random variable this procedure, a series of 12 uniformly distributed pseudo-random numbers were generated, normalized, and summed. Since the mean of the sum was 6, with a standard deviation of 1.0, subtracting 6 from the sum resulted in a standard normal random variable; that is, a variable with a mean of 0 and standard deviation of 1.0. A frequency distribution of 100,000 pseudo-normal, random numbers was generated in this manner to test the validity of the procedure. This frequency distribution is shown in Figure 8.

After the random material properties have been generated, they must be assigned to the stress points. The most straightforward method of assigning the values of a

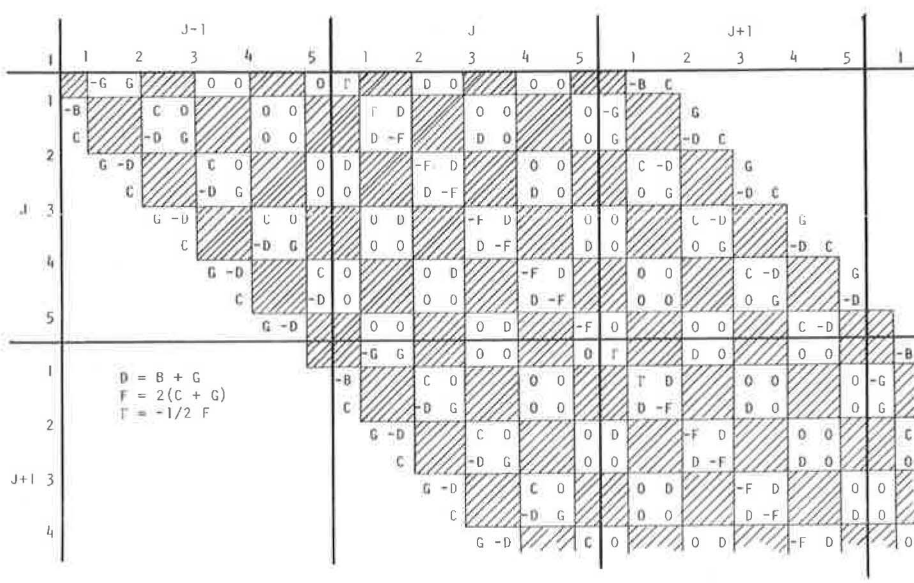


Figure 6. A section of the coefficient matrix of the equilibrium equations of a problem 5 points deep before the elimination of unneeded zeros.

random material property to the model is to generate a series of normal random values with the appropriate mean value and standard deviation and to assign each in turn to the stress points of the layer being considered. Typical assignments thus developed are shown in Figure 9.

When the material for a given pavement layer is produced by a batch type of process, a random step function must be used to represent the batch-to-batch variability. The level of each step in the step function is determined by generating normal random numbers using the mean value for the layer and the batch-to-batch standard deviation as parameters. The width of each step, except the first, is related to the area covered for a given batch size. In order to prevent a possible bias, the size of the first step is determined so that it will be a uniformly distributed random value that will be less than the size of the other steps in the layer. The within-batch variability of the material is superimposed on the random step function representing the batch means. By applying this procedure, material properties characterized by a batch-wise variability can be assigned to the stress points of any layer in the pavement. Figure 10 shows an example of two typical assignments.

		A													B
J	I \ K	1	2	3	4	5	6	7	8	9	10	11	12	13	
	1	X	X	X	X	X	X	Γ	D	0	0	0	-B	C	$-p_x$
		X	X	X	X	X	X	Γ	D	0	0	0	-G	G	$-p_y$
	2	X	X	X	X	X	D	-F	D	0	0	G	-D	C	0
		X	X	X	X	X	D	-F	D	0	0	C	-D	G	0
1	3	X	X	X	X	0	D	-F	D	0	0	G	-D	C	0
		X	X	X	X	0	D	-F	D	0	0	C	-D	G	0
	4	X	X	X	0	0	D	-F	D	0	0	G	-D	C	0
		X	X	X	0	0	D	-F	D	0	0	C	-D	G	0
	5	X	X	0	0	0	D	-F	0	0	0	G	-D	0	0
		X	X	0	0	0	D	-F	0	0	0	C	-D	0	0
	1	X	-G	G	0	0	0	Γ	D	0	0	0	-B	C	$-p_x$
		X	-B	C	0	0	0	Γ	D	0	0	0	-G	G	$-p_y$
	2	C	-D	G	0	0	D	-F	D	0	0	G	-D	C	0
		G	-D	C	0	0	D	-F	D	0	0	C	-D	G	0
2	3	C	-D	G	0	0	D	-F	D	0	0	G	-D	C	0
		G	-D	C	0	0	D	-F	D	0	0	C	-D	G	0
	4	C	-D	G	0	0	D	-F	D	0	0	G	-D	C	0
		G	-D	C	0	0	D	-F	D	0	0	C	-D	G	0
	5	C	-D	0	0	0	D	-F	0	0	0	G	-D	0	0
		G	-D	0	0	0	D	-F	0	0	0	C	-D	0	0
	1	0	-G	G	0	0	0	Γ	D	0	0	0	-B	C	$-p_x$
		0	-B	C	0	0	0	Γ	D	0	0	0	-G	G	$-p_y$
	2	C	-D	G	0	0	D	-F	D	0	0	G	-D	C	0
		G	-D	C	0	0	D	-F	D	0	0	C	-D	G	0
3	3	C	-D	G	0	0	D	-F	D	0	0	G	-D	C	0
		G	-D	C	0	0	D	-F	D	0	0	C	-D	G	0
	4	C	-D	G	0	0	D	-F	D	0	0	G	-D	C	0
		G	-D	C	0	0	D	-F	D	0	0	C	-D	G	0

X = Locations not existing in the actual coefficient matrix

Figure 7. A section of the coefficient matrix and constant vector for a problem 5 points deep after the elimination of unneeded zeros.

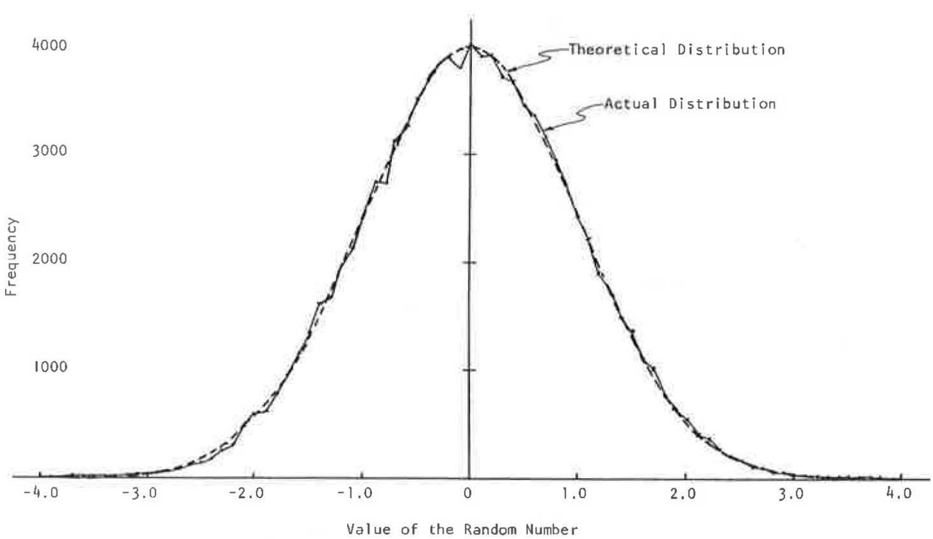


Figure 8. Frequency distribution of 100,000 normally distributed pseudo-random numbers.

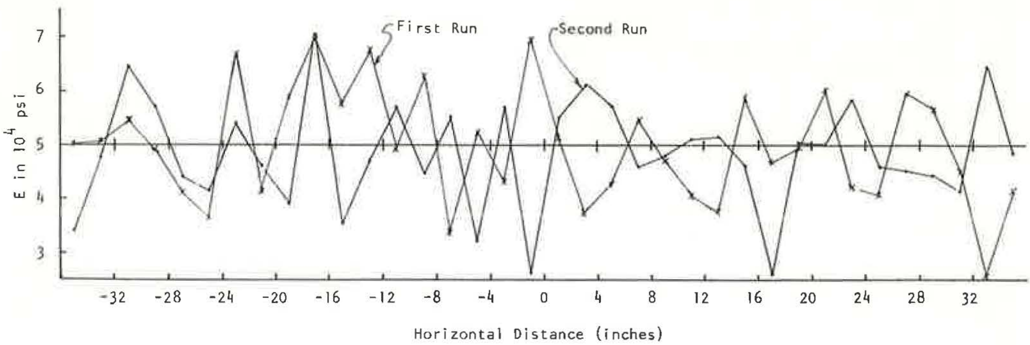


Figure 9. Assignments of normal random numbers (mean = 5 x 10^4 psi, coefficient of variation = 20) assuming that each value is independent of all other values.

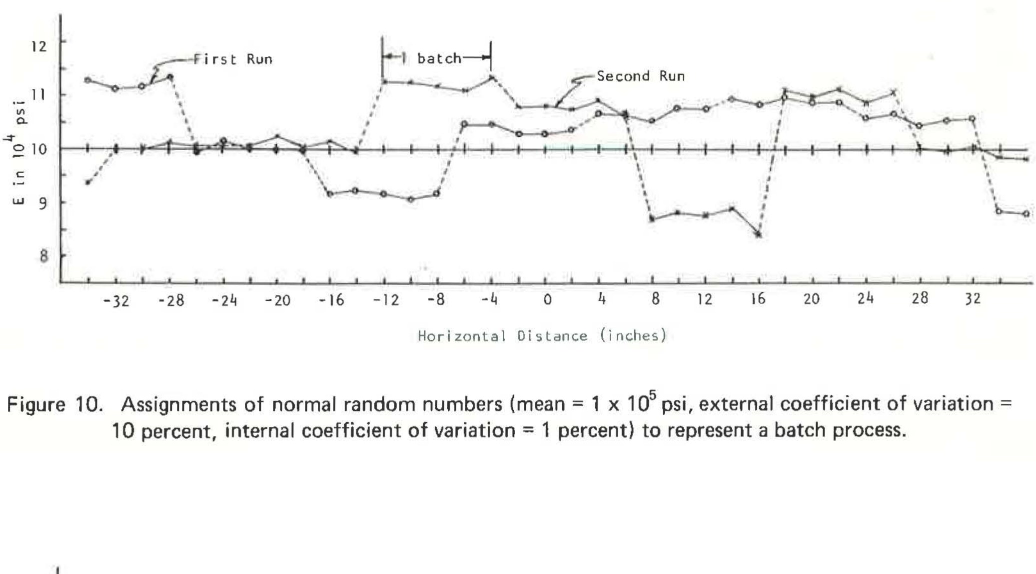


Figure 10. Assignments of normal random numbers (mean = 1 x 10^5 psi, external coefficient of variation = 10 percent, internal coefficient of variation = 1 percent) to represent a batch process.

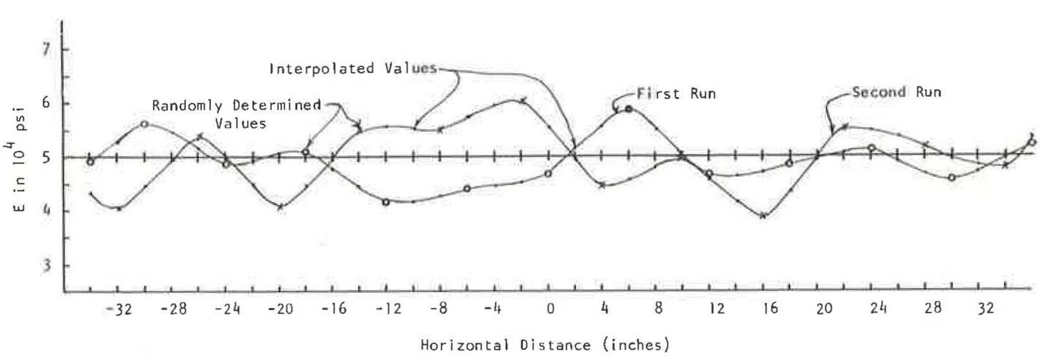


Figure 11. Assignments of normal random numbers (mean = 5 x 10^4 psi, external coefficient of variation = 10 percent, internal coefficient of variation = 0 percent) to represent a continuous process.

When the material for a given layer is produced by a continuous type of process, a continuous random function can be used to represent the external variability. This function is generated by passing an interpolation polynomial through equally spaced normal random variables, using the layer mean value and the external standard deviation as parameters. The spacing of the points is the minimum distance needed so that the values representing the material property are independent random variables from point to point. Again, the function can be translated a random distance horizontally so that bias due to maximum and minimum values occurring in the same place for successive solutions can be eliminated. The within-batch variability can then be superimposed upon this random function to provide values for all of the stress points in the layer. Examples are shown in Figure 11.

LIMITATIONS

Three possible sources of error that might decrease the value of the model for evaluating pavement systems are (a) the constraints due to the fixed boundaries of the model, (b) the spacing of the mass points, and (c) roundoff error. Sources a and b can be minimized if the computer system is large enough to handle very large problems with many mass points. Unfortunately, most computer systems available for general use do not have the storage capacity for handling large problems. Thus, the size of the model must be carefully chosen to give the maximum accuracy within the limits of the computer capability. The effect of each of the sources of error is examined in the following.

Boundary Constraints

The magnitude of the error involved by imposing finite boundaries on the layered half plane was determined by varying the size of the model while keeping the loading and mass point spacing constant. Values calculated from the model were compared with the corresponding values calculated from an analytical solution of the layered semi-infinite half plane developed by Iyengar and Alwar (8).

A study of the results reveals the following trends. The vertical stresses calculated in the model are in good agreement with those obtained by the theoretical solution for widths exceeding 60 in. The horizontal stresses and vertical deflections obtained using the model, however, are not in good agreement with those from the theoretical solution for practical model sizes. The disagreements are most likely caused by the zero deflection boundary conditions on the sides and bottom of the model. An increase of the modular ratio increases the error in the vertical stress. The rigid side boundaries are probably responsible for this increase.

One side boundary condition that was not evaluated but that would seem to eliminate most of the boundary condition errors is one using the boundary deflections obtained from the theoretical solution for homogeneous, two-dimensional layers rather than the zero deformations discussed above. This approach would eliminate much of the bridging of the fully restrained boundary and would provide for the deformation of the lower boundary. Also, since all theoretical deflection values used at the boundaries would appear in the constant term of each equation, use of this method would not increase the computer storage requirement.

Grid Spacing Errors

The grid spacing error results from the approximation of the differential equations in the theory of elasticity with finite difference equations. To estimate the magnitude of this error, a model of constant size representing a single layer of homogeneous material supported on a rigid base was used. To reduce the effect of the boundary condition error, the theoretical solution used for comparison consisted of two layers, the upper layer having an E and thickness the same as in the model, and the lower layer having a very high E value to represent the lower rigid boundary used in the model. The number of points in the model was varied to relate the grid spacing to the gird spacing error.

The magnitude of the grid spacing error in the vertical stress under the loaded area was on the order of 1 percent of the applied pressure for a grid spacing of 2 in. This error increased to about 10 percent of the applied pressure for a grid spacing of 8 in. Thus the grid spacing errors can be significant and should be considered when using this type of a model.

Roundoff Errors

The magnitude of error caused by roundoff during the solution of the equations is estimated by utilizing the fact that, for a layered half plane composed of homogeneous layers and loaded symmetrically, the deflection in the horizontal direction must be zero.

The results indicate that as the number of equations is doubled, the grid spacing error is increased by a factor of 6.5. The magnitude of the roundoff error in the horizontal deflection is fairly small, on the order of 0.6×10^{-6} for the largest problems that have been solved, but these errors propagate to the strain and then to the stresses. Roundoff errors in the stress and strain terms can be estimated by their deviation from symmetry. An estimate of this type revealed that the roundoff error in the stresses for the problem involving 1,400 equations was on the order of 0.01 psi, which is of little consequence.

A more complete discussion of all errors is presented by Levey (9).

TYPICAL SOLUTIONS AND RESULTS

The solution procedure described above was applied to the layered system shown in Figure 12 in order to demonstrate the ability of the procedure to determine the statistical nature of the response of such a system. For this analysis, the material is assumed to be elastic, and only the moduli of elasticity of the materials was assumed to vary. Coefficients of variation and the types of variability considered for the analysis are shown on the appropriate figures with the results.

The mean values and standard deviations for the vertical deflection, vertical stresses, horizontal stresses, and shear stresses respectively are shown in Figures 13 through 16. In these figures, the

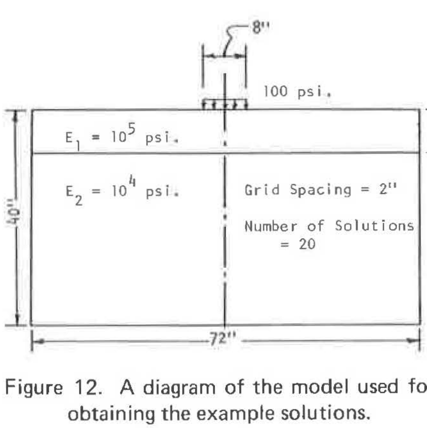


Figure 12. A diagram of the model used for obtaining the example solutions.

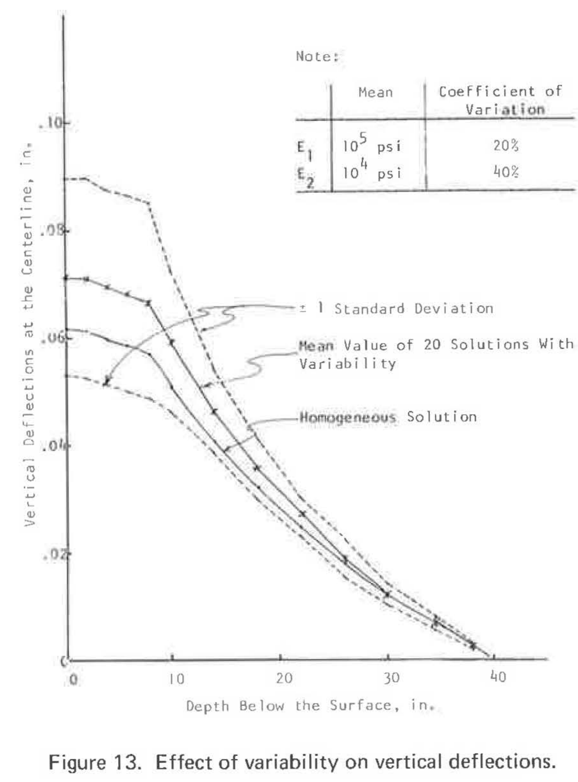


Figure 13. Effect of variability on vertical deflections.

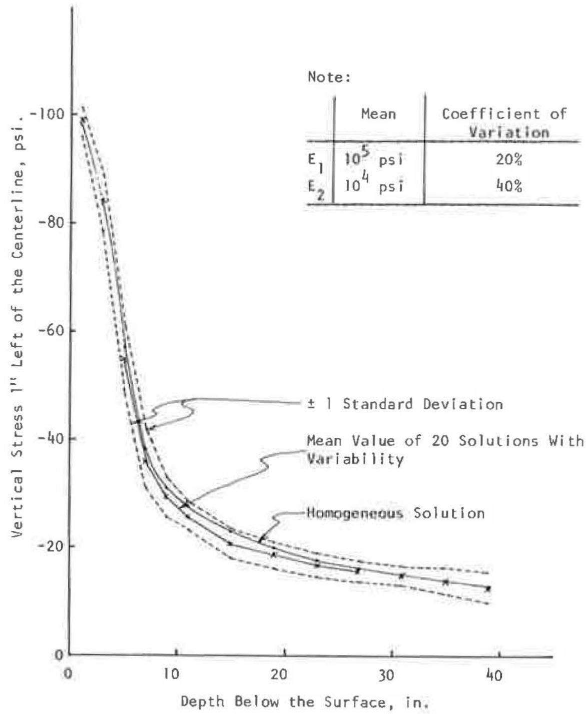


Figure 14. Effect of variability on vertical stresses.

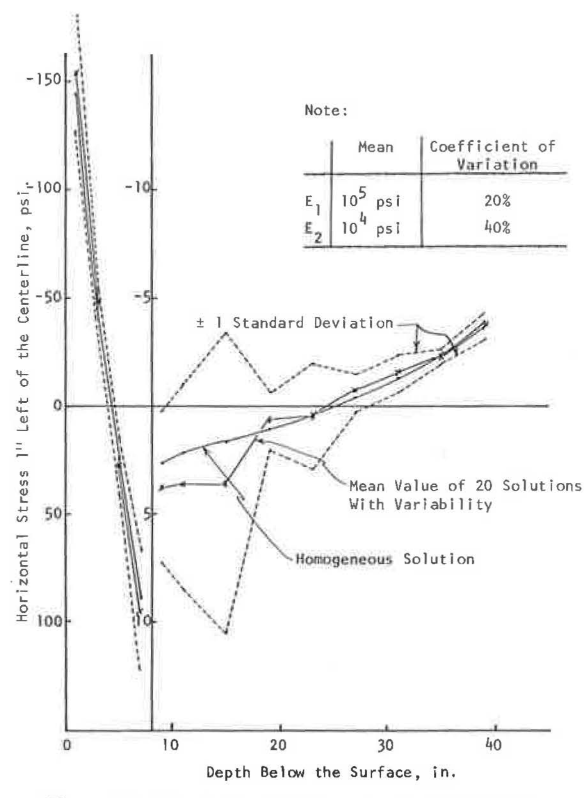
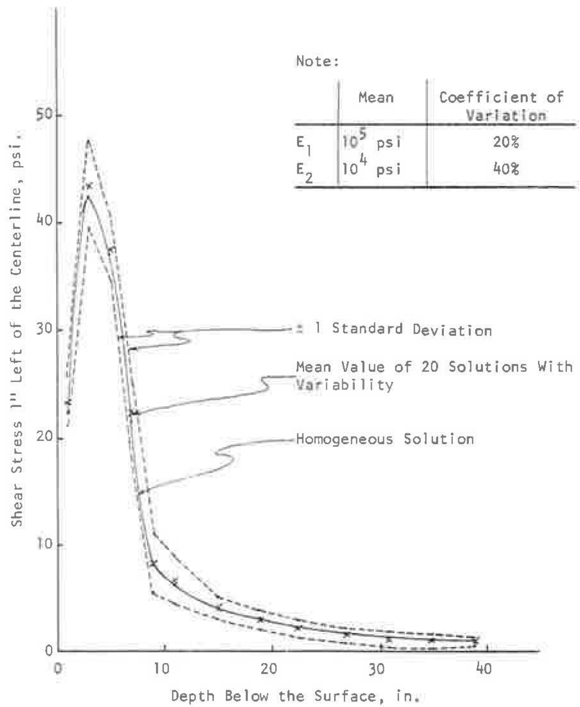


Figure 15. Effect of variability on horizontal stresses.



	Mean	Coefficient of Variation
E_1	10^5 psi	20%
E_2	10^4 psi	40%

Figure 16. Effect of variability on shear stresses.

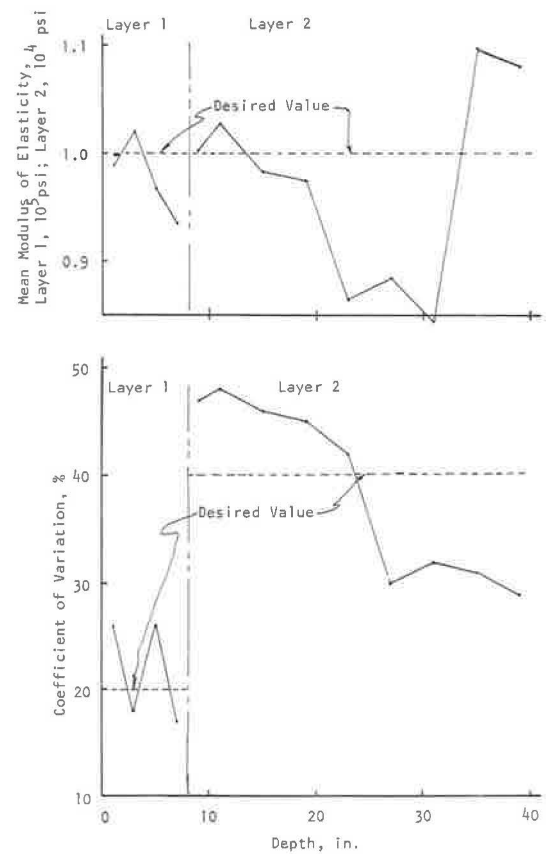


Figure 17. Mean values and coefficients of variation of the modulus of elasticity at the given depth and 1 in. from the centerline.

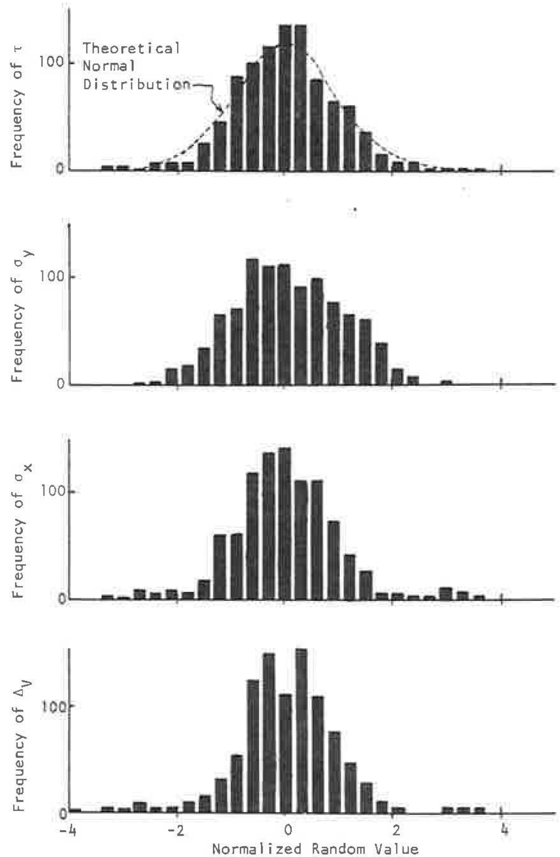


Figure 18. Frequency distributions of the stresses and vertical deflection for the upper layer.

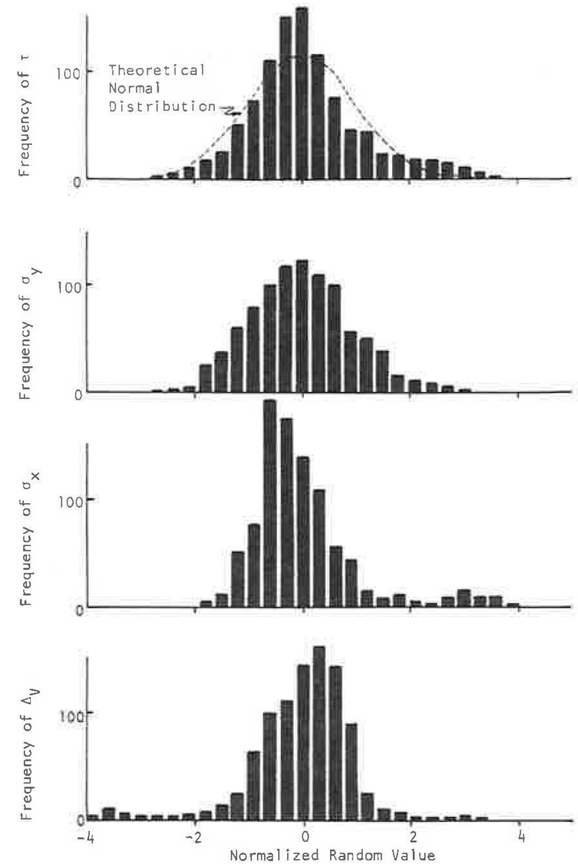


Figure 19. Frequency distributions of the stresses and vertical deflection for the top 8 in. of the lower layer.

smooth curves represent the values of stresses and vertical deflections in the model when homogeneous materials are used, i. e., the coefficient of variation is zero. The broken curves connecting the x's represent the mean values of the stresses and vertical deflections in the model when variable materials are used. The dashed curves connect points that are one standard deviation above and below the corresponding means. Figure 17 shows the mean values and overall coefficients of variation of the moduli of elasticity that were generated and used in the 20 solutions. These values correspond to the same stress points as the vertical and horizontal stresses shown in Figures 14 and 15.

In all cases, the values of stress and deflections that were determined with homogeneous materials are within one standard deviation of the corresponding mean values determined with variable materials. However, the hypothesis that the mean values for heterogeneous materials approach the corresponding values for homogeneous materials, which implies that the value obtained with the homogeneous material is within the limit of accuracy of the mean, cannot always be accepted at the 5 percent level ($\alpha = 0.05$). The deviation between the means for the heterogeneous materials and their corresponding values for homogeneous materials may be due to a difference between the mean modulus of elasticity that was actually generated at each point and the true mean.

The presentation of the data generated by the model in the format of the frequency distribution is not as straightforward as it might appear. If a frequency distribution were obtained for each stress, strain, and deflection at every mass point in the demonstration problem, 5,600 frequency distributions would be produced, and each one would consist of only 20 values. To get a better indication of the frequency distributions of the variables, the frequency distributions of the variables for points in the same region of the model can be lumped together. To do this, each value used in the composite frequency distribution must be normalized by subtracting its mean value and dividing by its standard deviation. A composite frequency distribution can then be compiled from all of the normalized values of the property within the desired region of the model.

Composite frequency distributions for the vertical deflection and the stresses are shown in Figures 18 and 19. The distributions in Figure 18 are for the area that extends 12 in. on either side of the centerline and the full depth of the upper layer. The distributions in Figure 19 are for the area of the model that extends 12 in. on either side of the centerline and includes the top 8 in. of the lower layer; 960 values are represented in each frequency distribution.

Most of the frequency diagrams shown in Figures 18 and 19 appear to resemble the normal distribution, although in some cases—the vertical deflection in particular—the tails of the distributions tend to extend farther than would be expected for a normal distribution. The only distribution that departs radically from a normal distribution is the distribution of the horizontal stress in the lower layer. Here a skewed distribution is produced by a random value that is physically restricted. In this case the horizontal stresses in the top of the lower layer must always be positive. Since the mean values of the horizontal stresses are near zero, the variability above the mean (zero on the composite frequency diagram) can be much greater than it can be below the mean; hence the skewed distribution.

CONCLUDING REMARKS

A procedure was developed that can analyze layered systems with nonuniform materials. High amounts of variability in the stresses and strains calculated in the typical problem shown indicate a need for establishing design criteria that consider this variability and also construction control criteria for controlling the material variability within economically acceptable limits.

Very limited work on defect size and degree indicated that the procedure can be used to determine the characteristics of critical defects. This information can be used in establishing statistical sampling plans or in establishing realistic rejection criteria for non-statistical sampling techniques.

It must be emphasized that this research did not produce an analysis of the statistical response of pavements to random material properties or of the defect problem mentioned above. Research using this or similar techniques will be needed to obtain these analyses. The technique mentioned can be used to study the effects of the elastic modulus and Poisson's ratio for most pavement structures if the changes in boundary conditions recommended are made. The model can be modified to analyze materials stressed above their yield points also, and the effects of yield point variability can be studied. When larger computing systems are variable and more efficient models and methods of formulating and solving the equations are developed, three-dimensional problems can be studied. This procedure can be extremely flexible and should be a valuable tool in the repertory of all serious pavement analysis.

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