PROBABILISTIC APPROACH TO THE DETERMINATION OF SAFETY FACTORS FOR THE BEARING CAPACITY OF COHESIVE SOILS

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There are a number of considerations that influence the factor of safety to be used in the engineering design of a foundation resting on a cohesive soil. Broadly speaking, these may be divided into uncertainty considerations and cost considerations. The former pertain to the evaluation of applied loads, soil properties, and analytical procedures, whereas the latter are concerned with the cost of changing the foundation size and the total cost of failure, if it should occur. In this work, probability functions are used to represent these uncertainties, and a rational procedure is advanced to determine safety factors associated with the bearing capacity of cohesive soils.

•THE uncertainties associated with loading, soil properties, and analytical procedures render very difficult the estimation of a safety factor for the bearing capacity of a cohesive soil. To be sure, the experienced engineer with an appreciation of all three factors intuitively accounts for each when he applies the traditional formulas of soils engineering. However, to make a selection that is economically optimum and at the same time consistent with the various uncertainties involved, a certain number of failures would necessarily have to be experienced. In some engineering situations, failures are sufficiently common that it may be possible over a period of time to develop intuitive guidelines that would enable the selection of an appropriate value for safety factor. Such failures are usually associated with temporary construction procedures, wherein the relationship between cost of failure and savings obtainable by modification of design is concomitant with a somewhat higher than normal permissible frequency of failure. Other cases conducive to a higher frequency of failure occur where failure is not of a catastrophic nature; in such situations the cost of failure is usually relatively low and/or the failure itself is progressive in nature.

On the other hand, because catastrophic failures of permanent structures normally involve extensive loss of property, human life, and professional reputation, as well as the cost of replacement of the structure, a high degree of conservatism is necessarily used. As a result, the frequency of failure is extremely low and, although the experienced designer may be capable of selecting a safety factor that will provide for a safe structure, he may have difficulty in refuting any contention that this safety factor and its related costs are excessive. One example of this situation is the bearing capacity problem for structures on stiff clays. Although it is recognized that settlement may sometimes provide the design criterion in such cases, this work is concerned with the case in which bearing capacity is the critical mode of failure. Because bearing capacity failures under these conditions are extremely rare, it may be argued that safety factors in current use are perhaps too large, and money is being wasted. Of even greater significance, it is desirable for the designer to have some means of determining

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quantitatively how the factor of safety varies with cost of foundation, cost of ultimate failure, degree of knowledge about the subsurface soil, uncertainty of the loading, and analytical procedures used. The individual evaluation of each of these factors poses a difficult problem; in some cases, a reasonably accurate determination is possible, whereas for other cases, only a subjective estimate may be made. However, even within these limitations, if values can be determined and properly analyzed as a group, the resulting procedure for determining a safety factor should be superior to the intuitive approach so commonly employed at present. Accordingly, this work will suggest one possible procedure for determining in a rational manner the bearing capacity of a cohesive soil.

BEARING CAPACITY FORMULATION IN PROBABILISTIC TERMS

The bearing capacity, q_d, of a shallow footing in or on a cohesive soil has been given by Skempton (1) as

$$q_d = 5c\left(1 + 0.2\frac{D_f}{B}\right)\left(1 + 0.2\frac{B}{L}\right) \tag{1}$$

For a 10-ft square footing embedded 5 ft, Eq. 1 would simplify to

$$q_d = 6.6c \tag{2}$$

or

$$A = \frac{L}{6.6c} = \frac{L_D + L_L}{6.6c} \tag{3}$$

where L is the total maximum load on the footing and A is the footing area. Traditionally, the footing area would be determined from the right-hand side of Eq. 3 by dividing the total load, which is composed of a dead load, L_D, and a live load, L_L, by a soil strength, c, which is determined by selecting a conservative value from the results of laboratory tests; then, the result so obtained is multiplied by a safety factor.

However, instead of evaluating Eq. 3 in the conventional deterministic manner just described, the load and strength parameters may be treated in a probabilistic manner, and f(A), a probability distribution function for A, may be determined; the input functions would indicate the probability that strengths or loads may deviate from specified mean values, and the output function, f(A), would indicate the probability that any given area represents the one that is critical from a stability point of view. Provided the input distribution functions are representative of the uncertainties associated with loading and soil strength and provided the theoretical formulation describes precisely the physical phenomenon being investigated, the area distribution function, f(A), will be realistic. Although the uncertainties associated with loading can be represented directly, those associated with the ultimate soil strength on the failure plane combine various distributions, one of which may be obtained from laboratory tests, while the others are determined on a subjective basis. For example, if c represents the undrained shear strength measured in a laboratory test, it is a relatively simple matter to obtain an associated distribution function; however, there are a number of additional factors that must be considered when the ultimate shear strength on the failure plane in the field is desired. These include (a) sample disturbance, (b) the tendency for progressive failure, (c) the drainage conditions in the field and in the laboratory test, (d) the rates of loading in the field and in the laboratory test, and (e) the extent to which the specimens tested represent the actual soil on the failure plane.

Sample disturbance will normally decrease the soil strength, and a multiplying factor, D, is introduced to account for this effect; because of the uncertainty involved with determining the value of D, an associated probability function will be used. Similarly, progressive failure will normally lead to a reduction in ultimate strength, and a factor, P, having an associated probability function, is introduced. This treatment applies specifically to the $\phi = 0$ design approach, and it presumes that strength values are obtained from undrained laboratory tests conducted at conventional rates of strain (on the order of 1 percent per minute). Although these conditions may represent reasonably well the case where a load is applied rapidly to a poorly draining soil, the slower rates of loading usually associated with building construction will make this approach somewhat conservative, owing to consolidation and the related increase in effective stress. Even though the undrained strength is generally less at lower rates of strain, this effect is usually overshadowed by the consequences of consolidation, and no account is taken of these effects in this study. As a consequence of the foregoing factors, it is extremely difficult, if not impossible, to evaluate the correctness of the bearing capacity formulation given by Eq. 1; therefore, it is necessary to provide for some uncertainty in this formulation, and a factor, T, having an associated probability function is included. Subject to the above modifications, Eq. 3 now becomes

$$A = \frac{L_D + L_L}{6.6c \times D \times P \times T} \tag{4}$$

EVALUATION OF AREA DISTRIBUTION FUNCTION

Although other distribution forms have been proposed, Wu and Kraft (2) have suggested that live loading may be represented by a normal distribution function; dead load may be considered as deterministic or, alternatively, also as normally distributed. Such assumptions permit the combination of loads to form a single load function on the basis of a total load, L, because normal distribution functions, when summed, produce a normal distribution having a mean value equal to the sum of the individual means and a variance equal to the sum of the individual variances. The relatively small significance of load distribution in the final result indicates that some inaccuracy in the assumptions of normal distributions for loading is not important.

The number of specimens tested and the degree to which these specimens represent the soil on the failure plane require careful consideration. Any evidence of geological factors that may provide a lower strength in the vicinity of the failure plane may dictate that the strength distribution be based on these specimens alone. In general, test specimens should be distributed throughout the entire region that encloses the theoretical failure zone. If the number of specimens is larger (30 or more), the distribution may be represented as normal with a mean, $\bar{x}$, equal to the sample mean and a standard deviation, S, given by

$$S = \sqrt{\frac{\Sigma(\bar{x} - x)^2}{n - 1}} \tag{5}$$

where x is the test value and n is the number of specimens. When there is a smaller number of specimens, an approximate approach may be taken if the coefficient of variation, V (ratio of standard deviation to the mean), is estimated for the region. If we consider a normal distribution with a mean, $\bar{x}$, equal to the sample mean, the standard deviation, S, may be expressed as

$$S = \frac{\bar{x}\,V}{\sqrt{n}} \tag{6}$$

Hooper and Butler (3) have reported consistent values of V for the strength of soil specimens taken from various sites, even though the mean strengths were different. Accordingly, a conservative value of V, based on previous work, may very well provide a more realistic estimate of the mean than the t distribution, another alternative for smaller sample numbers. The details of this approach, based on the coefficients of variation V, have been discussed by Kay and Krizek (4).

For the functions representing sample disturbance, progressive failure, and theory uncertainty, a purely subjective judgment of functional relationships must be made at this time. Accordingly, normal distribution functions, characterized as given in Table 1

TABLE 1

FIXED FUNCTIONS ASSIGNED

Function	Mean Value	Standard Deviation
Sample disturbance	1.33	0.15
Progressive failure	0.67	0.07
Theory precision	1.00	0.10

and discussed subsequently, have been assigned to each phenomenon, and the respective assigned fixed functions have been used throughout this work.

In the case of sample disturbance, the data in Table 1 imply, for example, that there is a 50 percent chance that the measured strength should be increased by 33 percent to account for this phenomenon, a 16 percent chance that the strength should be increased by 18 percent, and a 5 percent chance that the strength should be increased by only 8 percent. The 33 percent value of mean probable strength increase is a somewhat subjective evaluation, but it is equivalent to the effects of sample disturbance found by Terzaghi (5) on the Chicago subway by use of careful sampling procedures. The spread of the distribution is purely subjective, but it seems reasonable.

For progressive failure, the tabulated values imply a 50 percent chance that progressive failure will reduce the soil strength by a factor of 0.67, a 10 percent chance that it will be reduced by 0.58, and a 5 percent chance that it will be reduced by 0.55. Again, these figures appear to indicate a reasonable choice of distribution form.

There is no real basis for knowing the accuracy with which the Prandtl solution predicts the bearing capacity of cohesive soils because it is difficult to evaluate the effects of progressive failure and measurement of soil properties. The selected distribution indicates a 16 percent chance that the required foundation area should be 10 percent larger than indicated by Prandtl, a 10 percent chance that it should be 15 percent larger, and a 5 percent chance that it should be 20 percent larger.

The selection of normal distributions for these cases is also purely subjective, and it is difficult to defend these selections except to point out that an analysis of the implications of such distributions has been made and the results have been judged subjectively to be reasonable.

The equation from which the area distribution function, f(A), may be determined is now of the form

$$A = \frac{L}{6.6c \times D \times P \times T} \tag{7}$$

and each of the probability distribution functions associated with the variables, L and c, may have either its mean or its standard deviation varied. For a particular set of values, the Monte Carlo technique (6) may be used to obtain f(A). A digital computer may be used to generate sets of random numbers sufficient in number (perhaps 50,000) to obtain a reliable functional form in the significant part of the function (Fig. 1). This function indicates the probability that any given area will be required, based on the probability distributions of the input parameters.

OPTIMIZATION OF FOUNDATION AREA

To determine the optimum design area for the foundation, two cost values are required; one is the estimated cost of a catastrophic failure of the foundation, and the other is the unit cost of variation in the footing size. In addition, the concept of including in the total cost an appropriate portion of the cost of failure of the foundation is required; this may be considered analogous to, or may in fact be, the appropriate insurance premium. The total cost of the structure, T, may be written in the form

$$T = U_{CF} \cdot A + C_F \cdot p(F) + B \tag{8}$$

where U_{CF} is the unit cost of the foundation, A is the foundation area, C_F is the total cost of failure of the foundation, p(F) is the probability of failure for a given area A (obtained from the area distribution function), and B is the total remaining cost of the

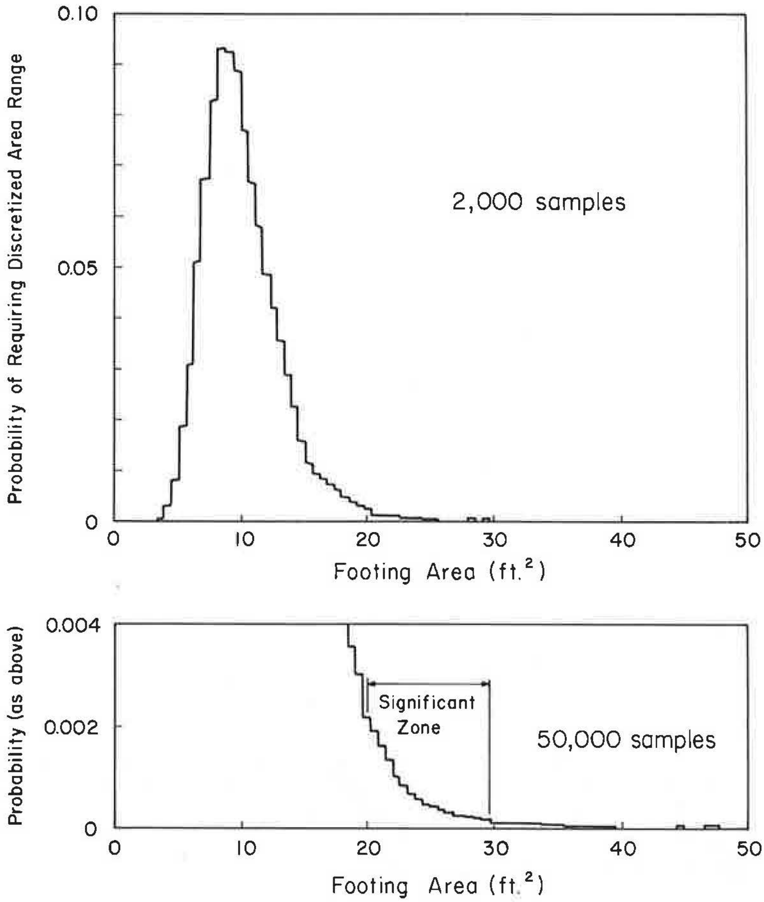

Figure 1. Area distribution diagram.

structure (which, for present purposes, may be considered constant). The total cost may be minimized by equating to zero the partial derivative of Eq. 8 with respect to area,

$$\frac{\partial T}{\partial A} = U_{CF} + C_F \cdot \frac{\partial p(F)}{\partial A} = 0 \tag{9}$$

and solving for R,

$$R = \frac{C_F}{A \cdot U_{CF}} = -\frac{1}{\frac{\partial p(f)}{\partial A} \cdot A} \tag{10}$$

where R is the ratio of the unit cost of the foundation failure to the unit cost of the foundation. Equation 10 is written specifically to show that the costs can be studied in the form of the dimensionless parameter R. Because the area distribution function formulation is not known, the optimum value for the area is obtained on the basis of Eq. 8; the computer is used to test progressively each discretized area value until the minimum total cost value is found.

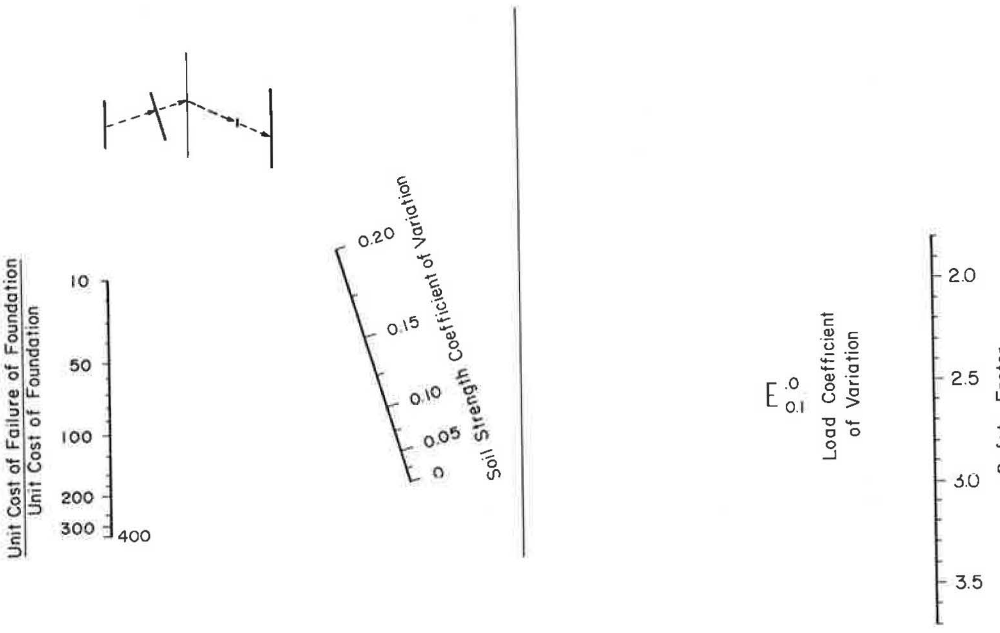

Figure 2. Safety factor nomograph for bearing capacity of cohesive soils.

Although the unit cost of the foundation, U_{CF}, is considered independent of the foundation area, A, in the preceding treatment, this may not be the case in many instances. However, the dependency will usually be slight, and a single repetition of the design process, using a unit cost compatible with the initially computed area, should be sufficient to account for such variations.

SAFETY FACTORS

With an optimum value for the foundation area thus obtained, we may now ask what value of safety factor would have been required in the conventional design procedure to produce the same design area. This value may easily be determined by dividing the area so obtained by the area obtained by substituting the mean values for load and soil strength into Eq. 3. Thus, a safety factor is determined in terms of one dimensionless parameter, the cost ratio R, and four other dimensioned parameters, the means and standard deviations for the load and soil strength respectively. One logical dimensionless parameter for these variables is the coefficient of variation, which is defined as the ratio of the standard deviation to the mean. In fact, several trials have indicated that the safety factor, computed as indicated earlier, does depend only on the respective values of the coefficient of variation of these parameters and not on the means. Dividing by the deterministic value of area removes any dimensional dependence, as may be expected. Similarly, any dependence on the coefficient (6.6 in Eq. 3) is removed, and the computed safety factor is independent of foundation shape and embedment depth (with the possible reservation that some allowance should be made for the inability to determine accurately the soil unit weight associated with the latter).

By evaluating the safety factor for a range of values (expected to include those found in practice) for each of the independent parameters (cost ratio, R, load coefficient of variation, L, and soil strength coefficient of variation, C), a large number of observations may be obtained, and multiple regression techniques may be used to describe the observations by an empirical equation. For example, the following empirical relationship was obtained on the basis of 296 observations:

$$S.F. = 2.03 + 1.58L + 0.226e^{10C} - 0.28e^{-0.01R} - 0.147e^{10C-0.01R} \quad (11)$$

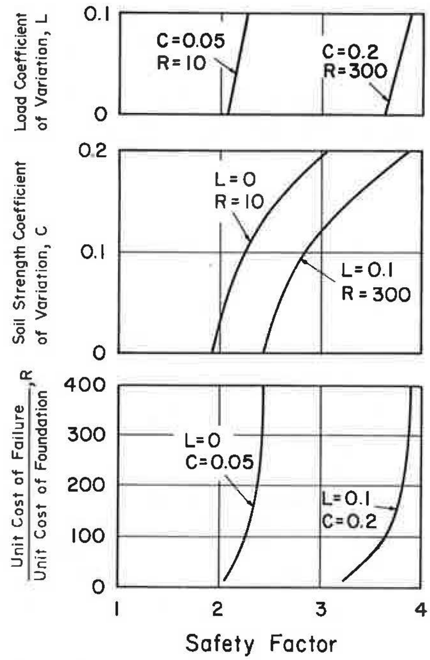

Figure 3. Influence of various parameters on safety factor.

Once determined, all of the original independent variables can be substituted into the equation to compare the observed and computed values of safety factor. Of the 296 observations, four deviated by 0.2 to 0.23, 51 deviated by 0.1 to 0.2, and the remainder were within 0.1. For convenience in using the results of this analysis, a nomograph based on an equation considered to be a good representation of the data has been constructed and is shown in Figure 2.

To illustrate the independent influence of L, C, and R on safety factor, the typical curves shown in Figure 3 relate each of the variables in turn to safety factors while holding the other two constant. Both the load coefficient of variation, L, and the cost ratio, R, are shown to have only a small effect, the latter being more influential at lower values. However, the effect of the soil strength coefficient of variation, C, is considerable, particularly for values above 0.1.

DISCUSSION OF RESULTS

Although very appealing from many points of view, the foregoing analysis has some shortcomings. Review of Figure 1 will indicate that the significant zone of the area distribution diagram is in the tail of the diagram. This tail is highly dependent on the assumed tails of the input distribution functions, and, provided these assumptions are correct, any desired accuracy of the output function may be obtained, subject only to restrictions imposed by cost of computer time. Accordingly, the accuracy of the input functions is a major consideration. In this regard, Hooper and Butler (3) and Lumb (7) have shown that shear strength test values for cohesive soils closely approximate a normal distribution. Furthermore, provided that the number of samples is large enough, the distribution of the estimate of the mean strength of the population closely approximates a normal distribution, even if the distribution of the population departs from the normal to some extent, so long as it is approximately bell-shaped and not skewed (8). The assumption of normality for the estimate of mean shear strength, therefore, appears justified. In the case of the subjective assumptions of normal distributions for sample disturbance, progressive failure, and theory accuracy, no similar justification can be made. The selection of a normal distribution, as well as a standard deviation, is purely "engineering judgment." Furthermore, the shapes of the tails of these distributions have significance similar to that of strength.

It is emphasized that full consideration must be given to the geological aspects of site investigation when selecting the shear strength parameters. The foregoing development is based on the assumption of a random distribution of soil strength in the tested zone and a similar random distribution on the failure plane. Such an idealization will be frequently violated, and the proposed method in no way precludes the necessity for careful localized investigations where possible weak zones are suspected. A judicious increase in the soil strength coefficient of variation used to select safety factor may be advisable under certain conditions.

CONCLUSIONS

A probabilistic approach for determining in a rational manner the bearing capacity of a cohesive soil has been presented and illustrated. Despite the shortcomings that are inherent in the procedure at this stage of development, it is considered that this method for the selection of a safety factor offers a number of advantages. Primarily, the design engineer, who is charged with the responsibility of selecting a safety factor, will be forced to consider the relevant parameters on which safety factor depends instead of exercising a gross intuitive judgment. In addition, some guidance is provided for determining the economic relevance of the various aspects of the bearing capacity problem.

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