GRAVITY FLOW TO EXCAVATIONS AND DRAINAGE TRENCHES IN LAYERED AQUIFERS

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The finite-element method is used to analyze the problem of steady-state gravity flow to typical excavations and drainage trenches in layered aquifers. Dimensionless flow quantities and information on the location of the phreatic surface are presented as functions of the relative material permeabilities and the geometric configuration of the soil profile and excavation size or drainage trench position. These results are then applied to an example problem to select a satisfactory configuration of subsurface drains for an actual depressed highway profile.

•ALTHOUGH the topic of plane flow through layered systems has been investigated extensively by workers in the field of agricultural drainage (12), a large portion of the effort has been directed toward the solution of confined flow problems (9, 12, 13, 14). Relatively little attention has been given to gravity flow systems (such as flow to excavations and drainage trenches), which are commonly encountered in civil engineering practice in general and in highway engineering in particular. Accordingly, the multiple aquifer systems shown in Figure 1 were studied to gain insight regarding the interaction effects of the various layers when subjected to conditions of steady-state gravity flow. The single excavation without drains (Fig. 1a) is typical of a general system with wide application on construction sites or in highway cuts, whereas the configuration with drains (Fig. 1b), although broad in use, is limited for illustrative purposes to a particular combination of material permeabilities that are representative of those found in a portion of a large highway project. The latter more specialized situation is used to test the concepts developed during the more general study of the single excavation. Because the combination of a permeable boundary at the trench side walls, the free surface, and the layered materials (with the resulting complex configuration of the free surface) makes these problems very difficult and generally tedious to solve by ordinary methods, the finite-element method (4, 6, 17, 20, 21, 22) was chosen for use in this investigation.

EVALUATION OF FINITE-ELEMENT METHOD

The advantages and disadvantages, as well as the accuracy, of the finite-element method has been well documented by workers in the area of structural analysis. Some typical studies include the consideration of various formulations (5, 8) and the description of various structural systems by different types of discrete elements (15, 21). Much has been published about the methods for obtaining solutions to the associated system of simultaneous equations, the conditioning thereof, and accuracy and error analysis of the procedures and answers. Results obtained by the finite-element method, as applied to the "quasi-harmonic" problem (which includes seepage), have been shown (19, 20, 21) to be comparable with those obtained by finite-difference methods and closed-form solutions. Also, a comparison has been made (6, 17) between results obtained by a finite-element analysis and those obtained by Casagrande

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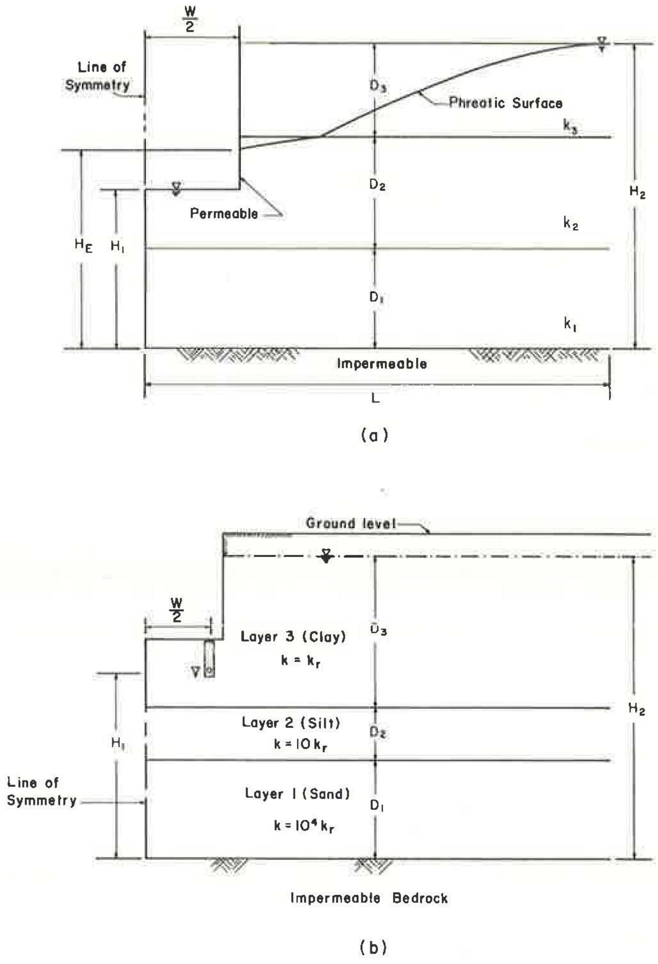
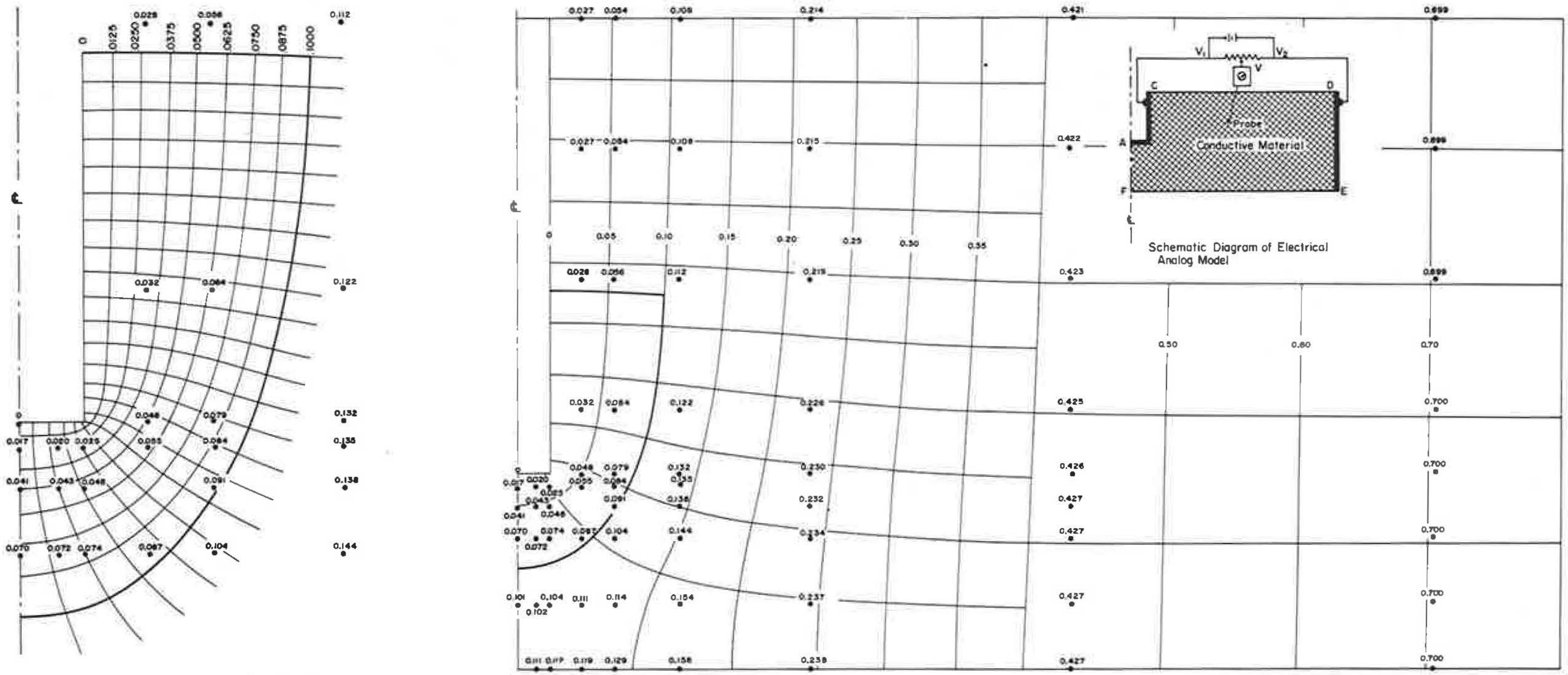


Figure 1. Typical geometric configurations considered.

(2) for the location of the free surface and the exit point for steady-state flow through a dam that rests on an impermeable base. The effect of mesh size on the solution for a geometrically simple, confined-flow problem has also been reported elsewhere (11).

The accuracy of the finite-element program (essentially that described by Taylor and Brown, 17) used in this study was ascertained by making several checks of flow quantities, seepage pressures, and free surface locations for various axisymmetric and plane flow cases. In the first case, an electric analog model of a trench, which partially penetrates a homogeneous, isotropic aquifer, was used (18) to check the flow quantity and potential distribution obtained by the finite-element method. The results shown in Figure 2 indicate very good agreement. In another comparison, 3 cases of a well, which partially penetrates an unconfined, homogeneous, isotropic aquifer underlaid by an impermeable stratum, were analyzed by the finite-element method. Results in all cases differed by less than 10 percent from those obtained (1) by relaxation and the methods of Kozeny and Forchheimer.

The problem of gravity flow through a dam on an impermeable base has been studied by relaxation (16), by the hodograph method (13), by an iteration scheme (7), and by use of a flow model (3). Hence, this problem affords an excellent opportunity for comparison of solutions.

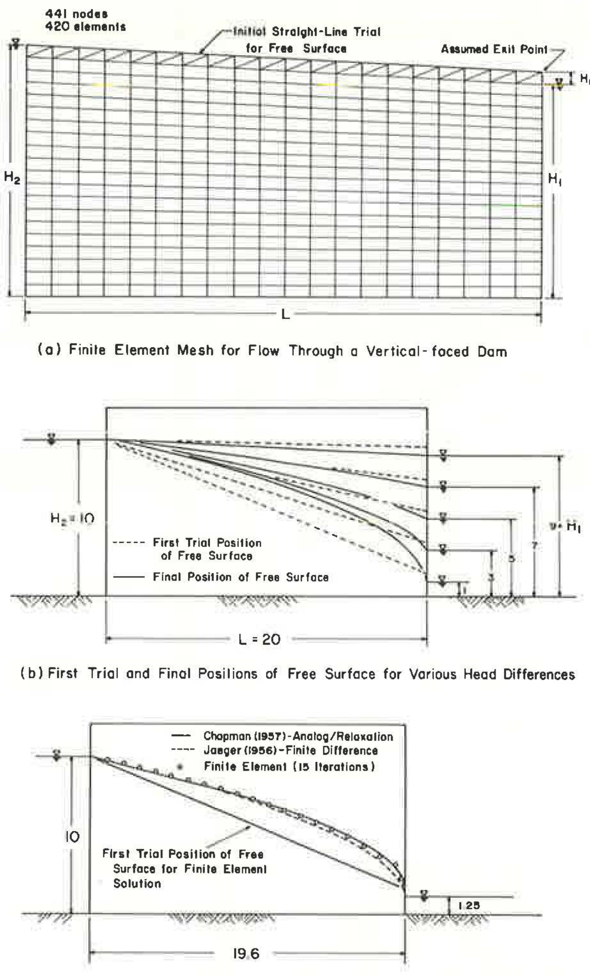


(a) Detailed Portion of General Model

(b) General Model

Figure 2. Comparison of potential values obtained for an electric analog and the finite-element method.

Figure 3a shows the finite-element mesh that was constructed to describe this problem, and Table 1 gives the geometric information and the resulting flow and pressure characteristics for the specific cases considered. The maximum number of iterations allowed for each case was 15, and, if completed, they required about 200 sec of central processor time on a CDC 6400 computer. The specified beta quantity is used in the program as an under-relaxation factor for free-surface adjustments between iterations, and the maximum pressure variation (associated with particular values of beta and tolerance) for the nodes along the free surface is given in Table 1. Also shown are the dimensionless flow parameters calculated from the finite-element analysis and the Dupuit assumption, which, according to Muskat (13), yields values almost identical to those obtained by the hodograph method. The first trial and final locations for the free surface in each case are shown in Figure 3b, and Figure 3c shows the results for a problem whose geometry was chosen to compare with that of the experimental model investigated by Chapman (3). The solution obtained (10) by the method of finite differences is also shown. Because of the excessive computer time required, the finite-element solution was not allowed to attain the best possible free-surface location, nor was a better first-trial free surface tried; however, all points that lie between the solution curves were moving upward when the solution was terminated. The single point lying above the Chapman curve near the tail water was observed to be essentially stable during the last 2 iterations.



(a) Finite Element Mesh for Flow Through a Vertical-faced Dam

(b) First Trial and Final Positions of Free Surface for Various Head Differences

(c) Comparison of Free Surface Positions as Caculated by Various Methods

Figure 3. Flow through a dam with vertical faces.

For the cases considered here, the convergence of the solution determined by the finite-element idealization and the convergence of the free-surface iterative procedure are of concern. The latter may be evaluated for each specific solution by observing the residual pressures calculated for the free-surface nodes, whereas the former may be assessed by comparing results obtained from several different idealizations for any given problem. In general, as the size of the elements becomes small and the number of elements becomes large, convergence is ensured for most conditions (21). For the several hundred computer solutions performed during the course of this and associated studies, satisfactory convergence was obtained through the exercise of reasonable care and judgment in the apportionment and placement of elements, even with seemingly rather coarse configurations. Although no comprehensive rules can be established with regard to the size and number of elements used in any given case, satisfactory convergence was obtained for the problems considered here by idealizations ranging from 125 nodes and

TABLE 1

COMPARISON OF FLOW PARAMETERS FOR SEEPAGE THROUGH A RECTANGULAR DAM WITH VERTICAL FACES

H_1/H_2	L	Number of Iterations	Maximum Variation in Free Surface Pressure	H_S/H_1		Q/kH_2	
				FEM	Polubarinova-Kochina	FEM	Dupuit
0.100	20.0	15	$+7 \times 10^{-3}$	0.1680	0.1143	0.2475	0.2475
0.125	19.6	15	$+6 \times 10^{-3}$	0.2821	—	0.2511	0.2511
0.300	20.0	14	$+3 \times 10^{-4}$	0.0002	0.0003	0.2275	0.2275
0.500	20.0	8	$\pm 7 \times 10^{-5}$	0	0	0.1875	0.1875
0.700	20.0	6	$\pm 7 \times 10^{-5}$	0	0	0.1275	0.1275
0.900	20.0	5	$+2 \times 10^{-6}$	0	0	0.0475	0.0475

Note: H_2 = 10.0; k = 1.0; maximum iterations = 15; beta = 0.9; and tolerance = 0.001.

101 elements to 461 nodes and 447 elements, and the number of iterations required to locate the free surface varied from 3 to 10.

FLOW TO AN EXCAVATION IN A LAYERED AQUIFER SYSTEM

For the case of flow to the type of excavation shown in Figure 1a, the following boundary conditions were used with the finite-element idealization. Zero normal flow conditions were imposed along the line of symmetry, the phreatic surface, and the impermeable lower boundary; the boundary nodes at distance L were subjected to a hydrostatic pressure distribution; zero pressure was specified at the nodes that describe the lower boundary of the excavation; and flow was allowed to occur through the excavation side wall. Finite-element meshes ranged in size from 125 nodes and 101 elements for the narrow trench with deep penetration to 236 nodes and 215 elements for the wide trench with shallow penetration. Relatively small, nearly square elements (2.5:2 units) were used in areas of rapidly changing pressures, whereas larger rectangular elements (60:30 units) were used in areas with small pressure changes. Various sizes of rectangular elements were used in the transition regions. Because of the complex free-surface configurations that resulted from the various layered systems and permeability ratios, 15 different meshes were used for this problem.

The number of variables needed to characterize the excavation problem shown in Figure 1a was reduced as follows: (a) the layer thicknesses D, were assigned a value of $H_2/3$ equal to 30 units; (b) the effective length L of the domain was taken to be 1,000 units; (c) the depth of penetration of the trench $(H_2 - H_1)/H_2$ was chosen to be $H_2/6$, $H_2/2$, and $5H_2/6$; (d) the width W of the excavation was given values of 80 units and 4 units to obtain information on width effects; (e) the excavation was assumed to be dewatered and to have permeable side walls; and (f) the relative permeability values (ratio of variable permeability k_v to reference permeability k_r) were taken to be 0.01, 0.1, 1, 10, and 100. Each layer, in turn, was considered to be the variable layer (whereas the permeability of the other 2 layers was held constant), and its relative permeability was allowed to traverse the full range of assumed values, thus yielding a 2- or 3-layer system composed of 2 materials. This was done for each combination of excavation width and penetration, and information regarding the exit point on the seepage face and the flow quantity is given in Table 2.

Interpretation of Results

The location of the exit point varies as a function of the confined nature of the system, the exit point being higher when the uppermost layer (or layers) is less permeable than the underlying layer (or layers). This effect is present, to a degree, whether or not the excavation extends into the more pervious underlying layer, and it may also be observed when the layer that contains the bottom of the excavation is least permeable and the overlying layers are more permeable. In addition, for these same situations,

TABLE 2

SUMMARY OF SEEPAGE CHARACTERISTICS FOR AN EXCAVATION

Penetration Ratio	Relative Permeability of Layer			Exit Point of Phreatic Surface		Flow Parameter	
	k_1/k_r	k_2/k_r	k_3/k_r	4-Unit Width	80-Unit Width	4-Unit Width	80-Unit Width
1/6	1.00	1.00	1.00	0.83	0.83	0.169	0.184
	1.00	1.00	0.01	0.90	0.87	0.016	0.046
	1.00	1.00	0.10	0.85	0.83	0.076	0.116
	1.00	1.00	10.00	0.83	0.83	0.643	0.654
	1.00	1.00	100.00	0.83	0.83	5.288	5.336
	1.00	0.01	1.00	0.83	0.83	0.089	0.094
	1.00	0.10	1.00	0.83	0.83	0.111	0.118
	1.00	10.00	1.00	0.84	0.83	0.541	0.724
	1.00	100.00	1.00	0.89	0.85	1.436	3.450
	0.01	1.00	1.00	0.83	0.83	0.115	0.120
	0.10	1.00	1.00	0.83	0.83	0.120	0.126
	10.00	1.00	1.00	0.84	0.83	0.475	0.647
	100.00	1.00	1.00	0.86	0.84	0.995	2.080
1/2	1.00	1.00	1.00	0.50	0.50	0.149	0.156
	1.00	1.00	0.01	0.50	0.50	0.126	0.131
	1.00	1.00	0.10	0.50	0.50	0.128	0.133
	1.00	1.00	10.00	0.51	0.50	0.314	0.341
	1.00	1.00	100.00	0.89	0.85	1.404	1.721
	1.00	0.01	1.00	0.93	0.83	0.019	0.045
	1.00	0.10	1.00	0.53	0.50	0.071	0.089
	1.00	10.00	1.00	0.50	0.50	0.720	0.728
	1.00	100.00	1.00	0.50	0.50	6.416	6.466
	0.01	1.00	1.00	0.50	0.50	0.086	0.086
	0.10	1.00	1.00	0.50	0.50	0.092	0.093
	10.00	1.00	1.00	0.58	0.51	0.530	0.702
	100.00	1.00	1.00	0.77	0.63	1.534	3.327
5/6	1.00	1.00	1.00	0.17	0.17	0.119	0.119
	1.00	1.00	0.01	0.17	0.17	0.105	0.105
	1.00	1.00	0.10	0.17	0.17	0.106	0.107
	1.00	1.00	10.00	0.17	0.17	0.196	0.198
	1.00	1.00	100.00	0.86	0.85	0.996	1.071
	1.00	0.01	1.00	0.17	0.17	0.077	0.078
	1.00	0.10	1.00	0.17	0.17	0.082	0.083
	1.00	10.00	1.00	0.31	0.30	0.442	0.445
	1.00	100.00	1.00	0.84	0.83	1.372	1.517
	0.01	1.00	1.00	0.91	0.90	0.015	0.017
	0.10	1.00	1.00	0.45	0.42	0.060	0.062
	10.00	1.00	1.00	0.17	0.17	0.714	0.714
	100.00	1.00	1.00	0.17	0.17	6.650	6.654

the case with the narrower excavation width indicates consistently higher values for the exit point elevation.

For a penetration ratio, $(H_2 - H_1)/H_2$, of 1/6, the results of which are shown in Figures 4a and 4b, the relative permeability of layer 3 exerts the most influence on the flow quantity, and little variation in flow quantity is caused by changes in k_1 and k_2 when their relative permeability values are approximately less than 1. In addition, the effect of the excavation width can be evaluated by comparing the 2 sets of curves. In the case of the wide excavation, there is very little variation in flow quantity with a variation in any of the layer relative permeabilities, except for the very high or very low relative permeability values; whereas, in the case of the narrow excavation, the influence of layer 3 manifests itself more readily. Another quantitative appraisal of excavation width can be obtained by comparing the flow quantities for a given set of permeability conditions; the difference between any two such values essentially represents the additional quantity of flow that is passing through the bottom of the excavation. The curves for layer 1 indicate that it exerts the least influence on the flow characteristics of the system.

As the depth of the excavation extends into layer 2, that layer exerts the largest influence on the flow in the system, and these effects are shown in Figures 4c and 4d for the different excavation widths. Figure 4c shows that the underlying layer (layer 1) has a dominant influence in the relative permeability range from 1 to 10, even though

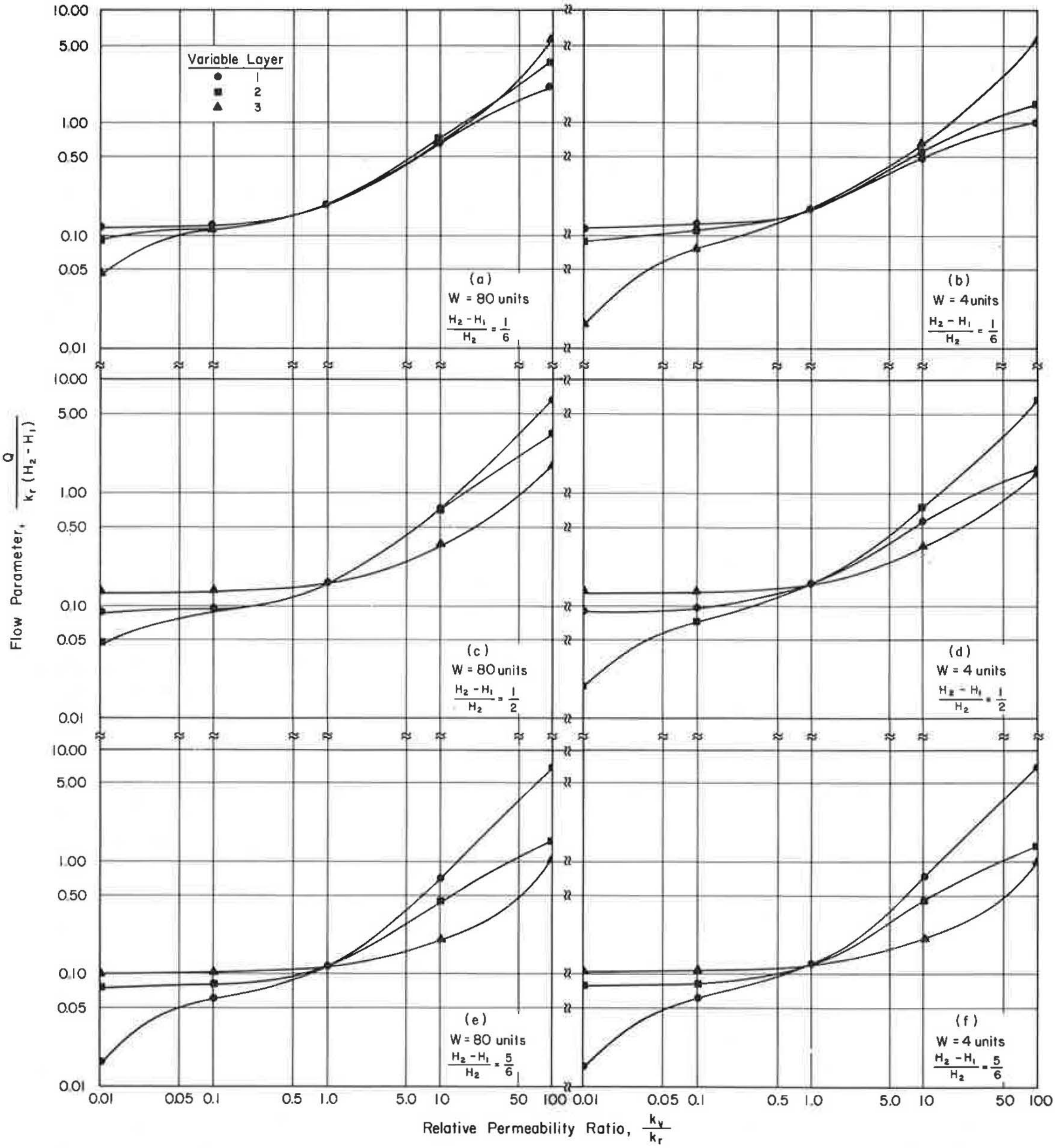


Figure 4. Flow parameter versus relative permeability ratio for various conditions of excavation width and penetration depth.

it has not been penetrated. Also, a reduction in the influence of layers 1 and 3 may be observed for relative permeabilities of more than 1, and an increase in the influence of layer 2 for relative permeabilities of less than 1 is apparent; these changes are attributable to excavation width.

Figures 4e and 4f show the results for the case where the excavation penetrates the bottom layer. For this situation, layer 1 exerts the major effect on the flow quantity. A comparison of these results shows very little change in flow due to excavation width because the underlying layer is impermeable.

Conclusions

Several qualitative conclusions may be drawn from the results of flow studies. For the geometrical configuration studied, it is apparent that the layer containing the bottom of the excavation manifests the dominant influence on the flow quantity. Where the excavation is wide and an underlying layer is relatively permeable, the underlying layer may, for a certain range of relative permeabilities, contribute more to the flow quantity than the layer that contains the bottom of the excavation. The exit point of the phreatic surface may be expected to assume a location that is a function of the degree to which the system is confined. In particular, there are certain commonly encountered field conditions that make it difficult to lower the free surface.

SUBDRAINS IN LAYERED AQUIFERS

In a more specific application of the preceding concept, the cross sections shown in Figure 1b were considered (18), and the effect of varying the thickness of layer 2 on the flow characteristics for several subdrain penetration depths was investigated. Although this case represents a subdrain system that is quite general in nature, the particular combination of soil profile, varying layer thicknesses, and relative permeabilities is felt to typify a portion of the proposed Crosstown Expressway in Chicago. Before the specific results obtained in this phase of the study are considered, several deductions may be made from the preceding results. First of all, for the particular combination of relative permeabilities shown, it should prove difficult to lower the phreatic surface sufficiently unless the silt-sand layer is tapped directly by the drains. Second, changes in the spacing of the subdrains will probably have little effect on the resulting flow quantities (analogous to the width effect for the excavation) because most practical spacings for highway drains will approximate a wide excavation. Third, changes in the thickness of the silt layer will probably not have a significant effect on the quantity of flow until the subdrains tap the silt layer or until it becomes very thick (i.e., the sand layer will maintain an almost constant pressure on the lower interface of the silt layer).

With the foregoing thoughts in mind, we obtained solutions for drain spacing widths W of 50 units and 200 units and for penetration ratios of 1/6, 1/3, 1/2, and 1, where the water level in the drains was assumed to be at the penetration depth. Portions of the resulting data are given in Table 3 and shown in Figure 5. Table 3 gives information that indicates the position of the phreatic surface (offset values from the original water table are shown for various points along the effective length L of the domain). The value for L was 10 times the penetration ratio (H_2 - H_1), each drain was 2 units wide, the thickness D_2 of the silt layer was set at 0, 1/3, 2/3, and 1 times D_3 whereas D_3 was a constant 30 units, and H_2 was held constant at 60 units.

For the purpose of this study, the absolute magnitudes of the flow quantities were considered to be of little concern. The major interest centered around the effect of the thickness of layer 2 on the relative flow quantities and the location of the free surface. Thus, the case where D_2 equals zero was selected as a reference case for each different drain penetration, and the flow quantity Q for various values of D_2 is divided by the flow quantity Q_r to form a dimensionless flow parameter Q/Q_r, which is shown in Figure 5 as a function of the thickness ratio D_2/D_3 and the depth of penetration parameter $(H_2 - H_1)/H_2$. When D_2/D_3 is zero, the bottom layer is all sand, and the corresponding value of Q/Q_r is 1 for all cases. When D_2/D_3 equals 1, the bottom layer is all silt, and appropriate values for Q/Q_r are shown in Figure 5.

TABLE 3

LOCATION OF PHREATIC SURFACE FOR FLOW TO DRAINAGE TRENCHES

Penetration Ratio	Layer Thickness Ratio	50-Unit Width					200-Unit Width				
		Line of Symmetry	1 Unit Left	1 Unit Right	0.05 L Right	0.40 L Right	Line of Symmetry	1 Unit Left	1 Unit Right	0.05 L Right	0.40 L Right
1/6	1	5.25	9.80	9.00	7.10	2.27	3.16	8.60	8.70	7.12	2.30
	2/3	3.83	8.05	8.05	6.24	0.96	0.12	8.05	8.20	6.22	0.95
	1/3	3.71	8.00	8.00	6.18	0.89	0.09	8.00	8.10	6.16	0.88
	0	3.58	8.00	8.00	6.11	0.83	0.07	8.00	8.10	6.11	0.83
1/3	1	12.89	17.70	17.20	13.37	6.67	11.27	17.50	17.40	13.41	6.66
	2/3	7.13	13.10	11.60	7.80	0.29	0.26	11.30	11.60	7.79	0.29
	1/3	6.75	12.80	11.50	7.53	0.22	0.17	11.00	11.20	7.54	0.22
	0	6.38	12.10	11.10	7.28	0.18	0.13	10.80	10.80	7.28	0.18
1/2	1	28.44	29.90	29.60	26.97	16.10	28.28	29.80	29.60	26.98	16.51
	2/3	12.77	18.50	17.00	9.35	0.72	1.49	12.00	14.00	9.37	0.34
	1/3	9.05	13.80	13.70	7.02	0.49	0.94	13.50	12.80	7.04	0.49
	0	28.47	30.00	29.50	27.06	16.83	28.31	30.00	30.00	27.07	16.83
1	1	60.00	60.00	57.30	48.22	26.51	60.00	60.00	57.30	48.22	26.51
	2/3	60.00	60.00	57.40	50.04	30.00	60.00	60.00	57.40	50.04	30.00
	1/3	60.00	60.00	57.60	52.51	34.00	60.00	60.00	57.60	52.51	34.00
	0	60.00	60.00	57.10	48.43	27.31	60.00	60.00	57.10	48.43	27.31

The data given in Table 3 show that the spacing of the drains has little effect on the position of that portion of the free surface located outside the drains. However, the extent of the drawdown between the drains is significantly affected when the underlying layers are a combination of sand and silt but not when only 1 material is present for a penetration ratio of 1/2. The effect of drain spacing on flow quantity is negligible for all cases, and the curves shown in Figure 5 represent both values of W. Although the thickness of the silt layer has relatively little effect on flow quantity when the drains do not tap that layer, a large effect is observed when the drains do tap the silt layer. However, full penetration of the drainage trenches reduces the effect of the silt layer, because the sand aquifer contributes the major portion of the flow quantity for all cases.

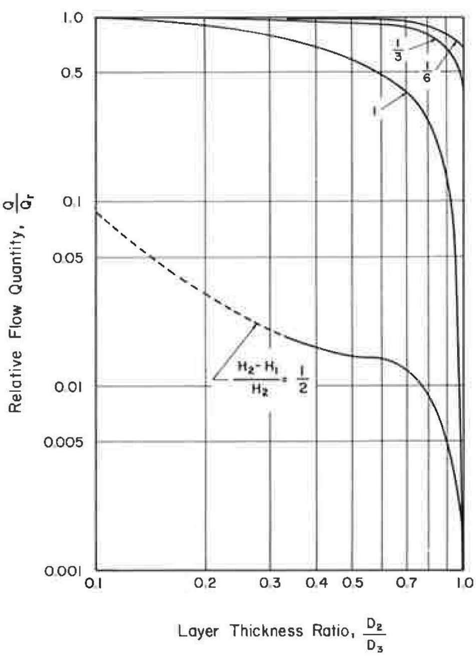


Figure 5. Relative flow quantity versus layer thickness ratio for various drain penetrations.

Results of Subdrain Studies

Although the spacing of drains in an open cut has been shown to have little effect on flow quantities and on the position of that portion of the free surface located outside of the drains, it does exert a substantial influence on the drawdown between the drains. However, under certain conditions the thickness of an intermediate silt layer, typical of a situation encountered along a portion of a proposed highway cut in the Chicago area, was found to affect significantly the seepage characteristics of the system. The results obtained from this latter, more specific study are qualitatively consistent with those that were deduced from the preceding, more general study of flow into an excavation.

APPLICATION OF QUALITATIVE RESULTS TO AN EXAMPLE PROBLEM

The area to be traversed by the proposed Crosstown Expressway in Chicago consists of various surficial soils, lacustrine sands and silts, glacial till sheets, pockets of dense granular soils, and bedrock. Conditions are quite variable, and the water table is approximately 5 to 10 ft beneath the existing ground surface in most places. Accordingly, depressed sections of the roadway will generally be located in soil profiles very similar to those considered in the previous sections of this study. As a particular example, consider the typical soil profile (obtained for the preliminary subdrain study) shown in Figure 6a with its representative permeability values. Because a depressed roadway is to be constructed in this profile, it is necessary to specify the type and placement of permanent drains to maintain satisfactory dewatering and free-surface location.

Based on the preceding study, the system appears to display "confined flow" characteristics, and hence the free surface will be difficult to lower. Narrow drainage trenches will probably be used, and therefore no increase in flow quantity should be expected for each drain because of width effect. However, the use of multiple trenches will create a pseudo-wide-trench response, and disproportional amounts of water will be drawn from the underlying strata by creating a large region in which vertical flow

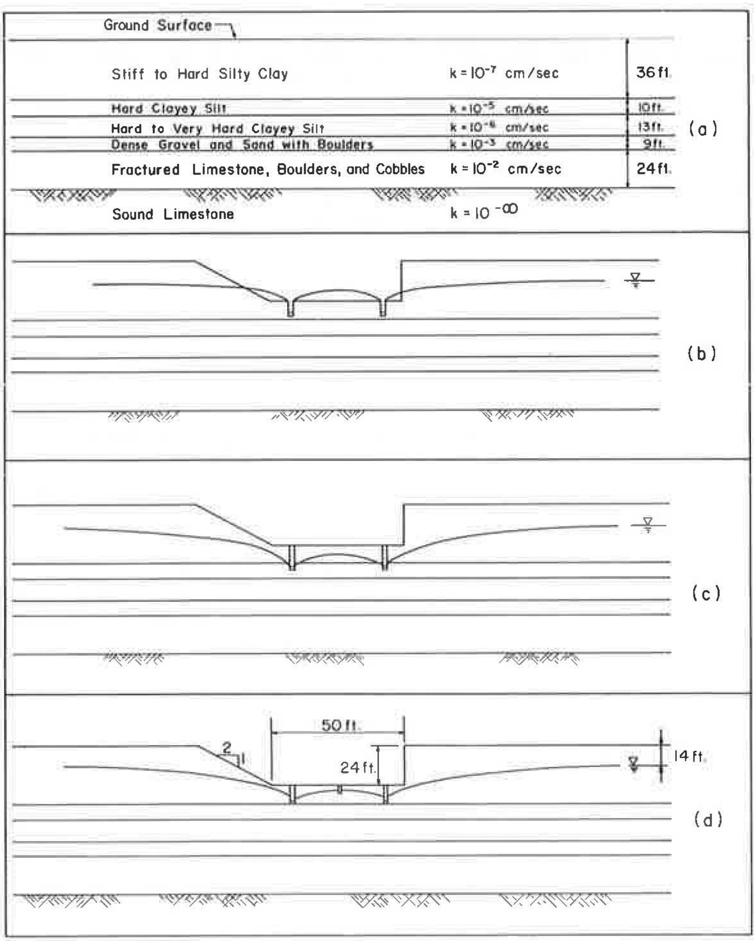


Figure 6. Typical soil profile and drainage systems for a roadway cut in the Chicago Crosstown Expressway.

predominates. Furthermore, it would appear that at least 2 drains are necessary, one on either side of the roadway, and that they must penetrate the layer that has a 10^{-5} cm/sec permeability to lower the free surface. Also, these drains should be spaced as close together as possible to maximize the drawdown between the drains.

The preceding qualitative deductions were checked by studying 4 different drain configurations by means of the finite-element method. Idealizations involving from 236 nodes and 223 elements to 265 nodes and 252 elements were used, and between 3 and 7 iterations were needed to obtain the free-surface location. Although Figure 6b shows what might be a first intuitive selection of drain penetration, the results of the earlier studies presented here have demonstrated that a configuration of this type cannot provide satisfactory drainage. This conclusion was verified by the finite-element solution, which is shown in Figure 6b. As can readily be seen, the phreatic surface is totally unsatisfactory; and problems with slope stability, subbase and retaining wall drainage, and excess pressures on the pavement will be encountered if no supplementary drains are supplied.

Extension of the drains into the more pervious silt layer, the next logical choice in light of the general study, yields the satisfactory solution shown in Figure 6c, as long as a minimum spacing of drains is maintained. According to the preceding study, just tapping the silt layer will give essentially the same position for that portion of the free surface outside the drains and the same flow quantity into the drains. However, the position of the free surface between the drains will need to be controlled by use of a third or centerline drain, as shown in Figure 6d. This latter solution, which was verified by a finite-element computation, has the advantage of requiring smaller quantities of excavation and filter materials.

In each of these cases, no special provision for drainage of the retaining wall was considered. The calculated seepage quantity for the final drainage system shown was less than 2 cu ft per day per foot of length for each drain, with the bottom layer contributing approximately 96 percent of this quantity. Thus, the qualitative predictions, based on the results of previous studies, for the seepage characteristics of a typical depressed roadway in a layered system provided a reasonably accurate description of the results that were obtained from a finite-element analysis.

SUMMARY

The qualitative concepts developed for the general case of an excavation in a layered aquifer are shown to apply to more specific field problems that involve the installation of subdrains. These data aid in understanding the complex interaction that occurs in plane flow through systems of layered aquifers, and they allow a more direct and logical approach to the design of highway drainage systems.

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