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PRECISE DETERMINATION OF EQUILIBRIUM IN TRAVEL FORECASTING PROBLEMS USING NUMERICAL OPTIMIZATION TECHNIQUES

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In recent years, the techniques for planning improvements to transportation systems as well as for analyzing innovative new systems have been given increased attention in the professional literature. Travel forecasting is an important aspect of the planning process because it is necessary to forecast the pattern and magnitudes of traffic flows in the proposed system so that one can analyze the benefits and costs that will accrue to the users and operators of the system. If it is assumed that the number of trips to be made within a region is dependent on the level of service delivered by the transportation system, the problem of determining equilibrium between supply and demand for a given region and transportation system is fundamental to travel forecasting. In this paper, 2 new algorithms are presented for the precise determination of equilibrium in the travel-forecasting problem. A functional of the demand and service variables associated with a transportation system is introduced, and it is shown that the maximum of this functional occurs at equilibrium. Both a constrained gradient and a modified Newton-Raphson algorithm are then used to determine the network flows that maximize this functional, i.e., the equilibrium flows. Two simple examples are considered to demonstrate the use of the algorithms. The advantage of the algorithms presented over present techniques is that equilibrium is obtained precisely rather than approximately and computation of minimum paths is not required in the iterative process.

•IN RECENT YEARS, the techniques for planning improvements to transportation systems as well as for analyzing innovative new systems have been given increased attention in the professional literature (1, 2, 3, 4, 5, 6, 7). Existing techniques have been criticized and new ones proposed (1, 2, 3, 4, 7). Travel forecasting is an important aspect of the planning process. To analyze the benefits and costs that will accrue to the users and operators of a proposed transportation system or to the government and society as a whole requires that a forecast be made of the pattern and magnitudes of traffic flows in the proposed system. The traffic flows in a transportation network result from the interaction between the demand for transportation services in a region and the service characteristics of the transportation system. It has been pointed out by Kraft and Wohl (1), and earlier by Beckman, McGuire, and Winsten (8), that the number of trips that will be made within a region is not independent of the level of service delivered by the transportation system. Thus, the problem of determining the equilibrium between supply and demand for a given region and transportation system is fundamental to the transportation planning process.

For clarity, it is important to briefly review the concepts of equilibrium between supply and demand as applied to transportation networks. The discussion follows

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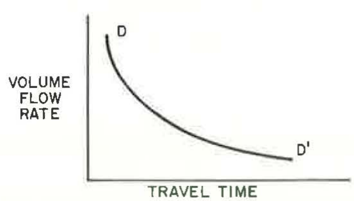
Beckman, McGuire, and Winsten (8). Consider first an isolated transportation link along which the demand for travel as a function of the travel time on the link is as shown in Figure 1a, and the travel time along the link is as shown in Figure 1b. In this case, the equilibrium flow rate and travel time on the link are obtained, by using fundamental microeconomic theory (9), as the intersection of the supply and demand curves as shown in Figure 1c. At this point, the number of trips made, v_e, leads to a travel time on the link t_e, which would in turn imply the number of trips to be made would be v_e.

Generalization of this fundamental idea of equilibrium for an isolated link to a transportation network requires some care. The fundamental assumption on which network equilibrium concepts are based is that all travelers choose to travel along their personal minimum-cost paths through the network from their origins to their destinations. This is Wardrop's first principle (10). It follows that, at equilibrium, if more than 1 path is used by travelers from a given origin to a given destination, the costs along the alternative paths must be the same. Furthermore, the number of trips generated per unit time between all origins and destinations corresponds to the equilibrium network service conditions between the origins and destinations.

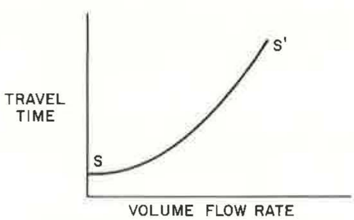
One must be careful when trying to mathematically define equilibrium. Beckman, McGuire, and Winsten (8) have precisely defined equilibrium in terms of the demand and service functions associated with a region and its transportation system. However, their proposed technique for determining the network equilibrium essentially ignores their mathematical definition and is approximate. Other approximate techniques for determining network equilibrium are the standard Federal Highway Administration (FHWA) assignment package (11) and the transportation network analysis software package, DODOTRANS, developed at M.I.T. (12). Beckman, McGuire, and Winsten (8), noted the difficulty in getting their iterative approach to converge to an approximate equilibrium, and one can only guess how close to the true equilibrium the approximation would be. Similar convergence problems exist with the FHWA approach. On the other hand, DODOTRANS will always reach an approximation to equilibrium, but again, one can validly ask whether the approximation achieved is even close to true equilibrium.

In a previous paper (13), numerical techniques of functional maximization were used to find the equilibrium flows and levels of service for a transportation system under the assumption that all interzonal trips are assigned along the unloaded minimum time paths through the network. Because of this assumption, the equilibrium found by that technique differs from the equilibrium defined by Beckman, McGuire, and Winsten (8). In this paper, numerical techniques of functional maximization are used to develop new algorithms for finding equilibrium as defined elsewhere (8).

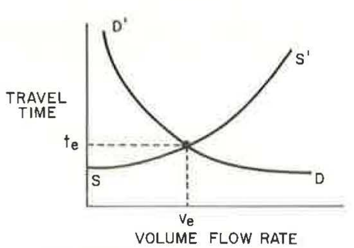
In the next sections, a mathematical definition of equilibrium is presented based on Beckman's work. A function of the demand and service variables associated with a transportation system is introduced such that the maximum of this function occurs at equilibrium. Algorithms are then developed to maximize this function by iterative adjustment of the network flows. Both a gradient (steepest



(a) DEMAND FUNCTION FOR AN ISOLATED LINK



(b) SERVICE CHARACTERISTIC FOR AN ISOLATED LINK



(c) EQUILIBRIUM DETERMINATION

Figure 1. Equilibrium for an isolated link.

descent) method and a Newton-Raphson algorithm are considered, and various examples are presented to exhibit the characteristics of the algorithms.

MATHEMATICAL STATEMENT OF EQUILIBRIUM

The usual means of representing travel in a region is assumed. Namely, the region is partitioned into a set of disjoint zones. The trips from (or to) any zone are assumed to originate (or terminate) at the zone centroid. The transportation network by which interzonal trips are made is represented by a series of links connecting the zones as shown in Figure 2. In general, the links in a network are characterized by a vector of service parameters that may depend on the volume on the link. In the following discussions, the only service variable used to characterize the links is travel time.

Let link ij denote a one-way link directly connecting nodes i and j, and on which persons may travel from node i to node j. (Note that zones and nodes will be used interchangeably through the rest of the paper. There is often a distinction made between a zone being a representation of an area where trips originate or terminate in a region and a node as being a junction of two or more links in a network. This distinction is not made here.) Let $v_{i,j}$ denote the volume flow rate on link ij, and let $v^k_{i,j}$ denote the portion, if any, of the trips $v_{i,j}$ having node k as ultimate destination. Summing the trips on a link over all possible destinations yields the relation

$$v_{i,j} = \sum_k v^k_{i,j} \tag{1}$$

By definition, all $v^k_{i,j}$ and $v_{i,j}$ are non-negative.

Let v^k_i represent the number of trips per unit time originating at node i and having destination k. Then the dependence of the number of trips v^k_i on the level of service in the transportation network is expressed by the demand function

$$v^k_i = d^k_i\,(t^k_i) \text{ if } t^k_i \geq 0 \tag{2}$$

where t^k_i is the travel time through the network from zone i to zone k. The demand function d^k_i is assumed to be a monotone nonincreasing function for $t^k_i \geq 0$. In all subsequent developments, it is also assumed that the inverse of the demand function in Eq. 2 exists; i.e., a function g^k_i exists such that

$$t^k_i = \{d^k_i\}^{-1}\,(v^k_i) = g^k_i\,(v^k_i) \tag{3}$$

The case when g^k_i does not exist is discussed in an example.

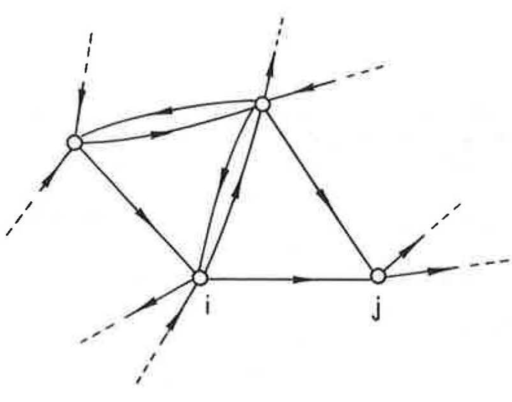


Figure 2. Network representation of a transportation system.

For each node i, let $\{a_i\}$ denote the set of all nodes directly connected to i by a link carrying traffic away from i, and let $\{b_i\}$ denote the set of all nodes directly connected to i by a link carrying traffic toward i. Thus, in Figure 3, $\{a_1\} = \{3,5\}$ $\{b_1\} = \{2,3,4\}$ $\{b_3\} = \{1,2,4\}$, and so forth. Then, the number of trips from node i to node k, v^k_i, obviously equals the flow with destination k away from origin node i minus the flow with destination k toward i. That is,

$$v^k_i = \sum_{\{a_i\}} v^k_{i,a_i} - \sum_{\{b_i\}} v^k_{b_i,i} \tag{4}$$

If $\{a_i\} = \{3,4,7\}$, then define

$$\sum_{\{a_i\}} v^k_{i,a_i} = v^k_{i,3} + v^k_{i,4} + v^k_{i,7}$$

To relate the level of service in the network to the trips using links, requires that supply functions be introduced. Let $t_{i,j}$ denote the time required to travel from node i to node j along the link ij. The relationship between the volume of traffic on the link ij and travel time on the link ij is expressed by the supply function

$$t_{i,j} = h_{i,j}(v_{i,j}) \tag{5}$$

where $h_{i,j}$ is a monotone nondecreasing function.

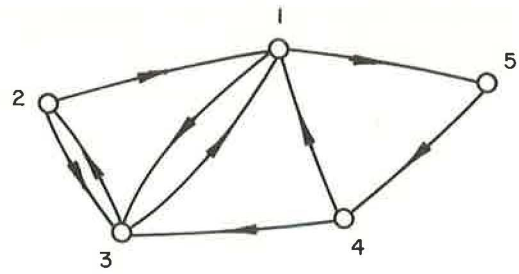


Figure 3. An illustrative network.

As previously explained, the trips from 1 zone to another at equilibrium use a minimum time path between those zones. Hence, $v^k_{i,j}$ is greater than zero at equilibrium if and only if link ij is part of a minimum time path from node i to node k. (Note that there is not necessarily a unique minimum time path between zones at equilibrium.) Therefore, the equilibrium travel time from zone i to zone k can be expressed in terms of the travel time on any link ij for which $v^k_{i,j} > 0$ and the equilibrium travel time from j to k is

$$t^k_i = t_{i,j} + t^k_j \text{ if } v^k_{i,j} > 0 \tag{6}$$

This statement is equivalent to the well-known principle of optimality stated by Bellman (14).

On the other hand, if $v^k_{i,j} = 0$, link ij is not part of a minimum time path from i to k. It follows that the equilibrium travel time from j to k plus the equilibrium travel time on link ij must be greater than (or possibly equal to) the time to travel from i to k at equilibrium along a path that does not include link ij; that is,

$$t^k_j + t_{i,j} \geq t^k_i \text{ if } v^k_{i,j} = 0 \tag{7}$$

Equations 6 and 7 give a precise mathematical statement of equilibrium.

Substitution of the inverse demand functions from Eq. 3 and the service functions from Eq. 5 into Eqs. 6 and 7 yields the following alternate statement of equilibrium in terms of the network flows:

$$g^k_i(v^k_i) - g^k_j(v^k_j) = h_{i,j}(v_{i,j}) \text{ if } v^k_{i,j} > 0 \tag{8}$$

$$g^k_i(v^k_i) - g^k_j(v^k_j) \leq h_{i,j}(v_{i,j}) \text{ if } v^k_{i,j} = 0 \tag{9}$$

Thus, the problem of finding equilibrium in a travel-forecasting problem is equivalent to determining the values of the non-negative flows $v^k_{i,j}$ that satisfy Eqs. 1, 4, 8 and 9. However, determining the solution of these simultaneous nonlinear equations, without knowing what the minimum time paths between zones are at equilibrium, is a difficult task and has led to the use of approximate iterative techniques to determine equilibrium.

In the next section, the statement of equilibrium as the maximum of a functional of the network flows is developed, following Beckmann, McGuire, and Winsten (8).

EQUILIBRIUM AS A MAXIMIZATION PROBLEM

For convenience, define a vector $\underline{v}$ that has all $v^k_{i,j}$ as its components; i.e., $\underline{v}$ transpose is given by

$$\underline{v}^T = [v^2_{1,j}, \ldots v^k_{i,j}, \ldots] \tag{10}$$

Consider the functional

$$H(\underline{v}) = \sum_m \sum_\ell \int_0^{v_m^\ell} g_m^\ell(x)\,dx - \sum_m \sum_n \int_0^{v_{m,n}} h_{m,n}(x)\,dx \tag{11}$$

where each summation is over all node numbers. It is shown in the Appendix that the gradient components of $H(\underline{v})$, $\partial H/\partial v_{i,j}^k$, are given by

$$\frac{\partial H}{\partial v_{i,j}^k} = g_i^k(v_i^k) - g_j^k(v_j^k) - h_{i,j}(v_{i,j}) \tag{12}$$

Now, Eq. 8 and Eq. 12 together imply that, at equilibrium,

$$\frac{\partial H}{\partial v_{i,j}^k} = 0 \text{ if } v_{i,j}^k > 0 \tag{13}$$

whereas Eq. 9 and Eq. 12 imply that at equilibrium

$$\frac{\partial H}{\partial v_{i,j}^k} \leq 0 \text{ if } v_{i,j}^k = 0 \tag{14}$$

Equations 13 and 14 are by definition the necessary conditions for the functional H to have a maximum at a point $\underline{v}$ under the constraint that all $v_{i,j}^k$ be non-negative. The fact that Eqs. 13 and 14 are satisfied at equilibrium implies that finding equilibrium is equivalent to the problem of maximizing H subject to the constraint that $v_{i,j}^k \geq 0$ for all i, j, k. It is shown in the work by Beckman, McGuire, and Winsten (8) that the solution to the problem of maximizing H is unique (in the sense that the equilibrium flows $v_{i,j}$ on all roads are unique) whenever the inverse demand functions g_i^k are strictly decreasing functions of v_i^k and the supply functions $h_{i,j}$ are strictly increasing functions of $v_{i,j}$.

The question is, Does the statement of equilibrium as a functional maximization make the problem of finding equilibrium easier to solve? It is the authors' opinion that the answer to this question is yes, because numerical techniques of functional maximization can readily be used to determine the equilibrium network flows that maximize $H(\underline{v})$. Such numerical techniques are used effectively in parameter optimization problems associated with control system design (15), and it is shown in the following sections that these techniques can be used in the solution of the problem being considered.

ALGORITHMS FOR DETERMINING EQUILIBRIUM

Two algorithms that use iterative numerical techniques to find the values $v_{i,j}^k$, $\forall$ i, j, k, which maximize H (i.e., to find equilibrium), are presented in this section. Both algorithms utilize the vector $\nabla_v H$, the gradient of H with respect to $\underline{v}$. By definition

$$\nabla_{\underline{v}} H^T = \left[\frac{\partial H}{\partial v_{1,j}^2} \cdots \frac{\partial H}{\partial v_{i,j}^k} \cdots \right] \tag{15}$$

and thus the components of $\nabla_v H$ are obtained by using Eq. 12.

The algorithms presented are a constrained gradient algorithm and a modified Newton-Raphson procedure. The rationale for these algorithms is not discussed here but can be found in any standard reference for optimization techniques (16).

In the constrained gradient algorithm, the change in each volume at the end of the r th iteration is given by

$$\Delta v_{i,j}^{k}\Big|_{r} = v_{i,j}^{k}\Big|_{r+1} - v_{i,j}^{k}\Big|_{r} = \begin{cases} 0 & \text{if } v_{i,j}^{k}\Big|_{r} = 0 \text{ and } \dfrac{\partial H}{\partial v_{i,j}^{k}}\Big|_{r} < 0 \\ \alpha \dfrac{\partial H}{\partial v_{i,j}^{k}}\Big|_{r} & \text{if otherwise} \end{cases} \tag{16}$$

In Eq. 16, α is a positive scalar constant that is chosen to guarantee that none of the flows is adjusted to be negative or to locally maximize the functional H in the direction defined by the gradient. A flow chart for the algorithm is given in Figure 4.

The iteration procedure continues until at some step Eqs. 13 and 14 are satisfied to within the desired degree of accuracy. Typically, this means that iteration continues until

$$-\epsilon < \frac{\partial H}{\partial v_{i,j}^{k}}\Big|_{r} < \epsilon \tag{17}$$

for all $v_{i,j}^{k}|_r > 0$, where ϵ is a small scalar constant (e.g., 10^{-3}). Results using this algorithm to determine equilibrium are given in the examples.

A second approach to finding the non-negative $v_{i,j}^{k}$'s that maximize H is to use a modified Newton-Raphson algorithm. If the $v_{i,j}^{k}$'s were not constrained to be non-negative, then the change in the volumes at the end of the r th iteration, $\Delta v|_r$, using the Newton-Raphson procedure would be given by

$$\Delta \underline{v}\Big|_{r} = -\left[\frac{\partial \nabla_{\underline{v}} H}{\partial \underline{v}}\Big|_{r}\right]^{-1} \nabla_{\underline{v}} H\Big|_{r} \tag{18}$$

(Thus, the second partial derivatives of H with respect to all $v_{i,j}^{k}$'s must be computed and the matrix of these terms inverted in the Newton-Raphson procedure.) However, because of the constraint that all of the components of $\underline{v}$ must be non-negative, a modified Newton-Raphson algorithm must be used in the problem being considered.

Basically, the idea of the algorithm is to eliminate from consideration, at step r of the iteration procedure, all components of $\underline{v}$ that satisfy Eq. 14 at step r. Therefore, at step r, determine the number c of components of $\underline{v}|_r$ that do not satisfy Eq. 14. Define $\underline{w}^r$ as a vector of dimension c and denote its value at step r as $\underline{w}^r|_r$. (The dimension of $\underline{w}^r$ may be different at each step.) Every component of $\underline{v}|_r$ that does not satisfy Eq. 14 is included sequentially as a component of $\underline{w}^r|_r$.

Then, similar to the iteration procedure given in Eq. 18, a vector $\Delta\underline{w}^r|_r$ is calculated by

$$\Delta \underline{w}^{r}\Big|_{r} = -\left[\frac{\partial \nabla_{\underline{w}^r} H}{\partial \underline{w}^{r}}\Big|_{r}\right]^{-1} \nabla_{\underline{w}^r} H\Big|_{r} \tag{19}$$

Then, the components of the c-dimensional vector $\underline{w}^r|_{r+1}$ are computed by the relations

$$w_{\ell}^{r}\Big|_{r+1} = \max\left\{w_{\ell}^{r}\Big|_{r} + \Delta w_{\ell}^{r}\Big|_{r}, 0\right\}, \quad \ell = 1, \ldots, c \tag{20}$$

Finally, the components of $\underline{v}|_{r+1}$ are obtained in the following way: The components of $\underline{v}$ that at step r are not members of $\underline{w}^r$ are set to zero in $\underline{v}|_{r+1}$. The remaining

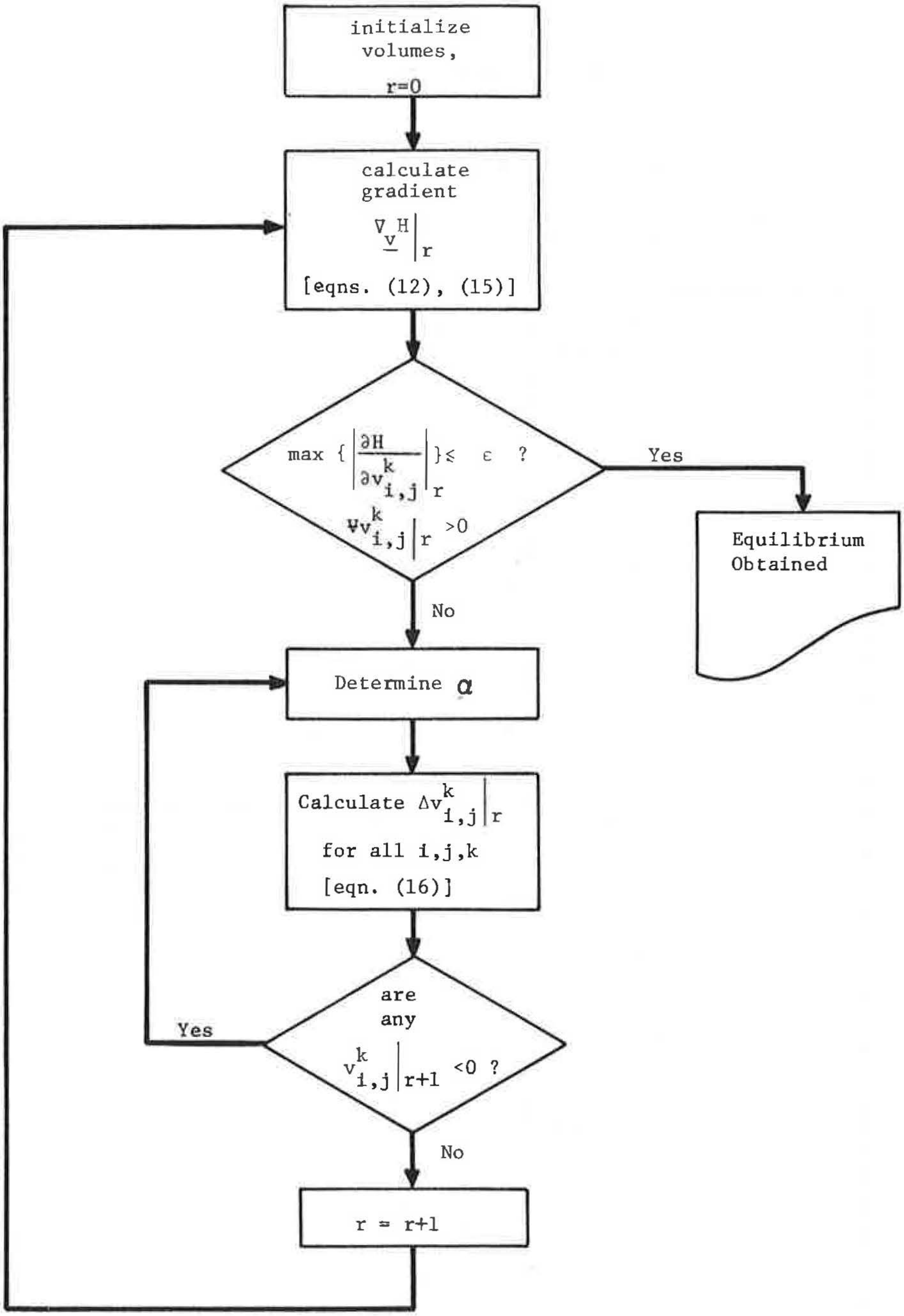


Figure 4. Flow chart for constrained gradient algorithm.

components of $\underline{v}\,|_{r+1}$ are obtained sequentially from $\underline{w}^r|_{r+1}$. The iteration procedure continues until Eq. 17 is satisfied. Figure 5 shows a flow chart for the modified Newton-Raphson algorithm.

The algorithm just presented requires the calculation of $\partial\nabla_{w^r} H/\partial \underline{w}^r$, the matrix of second partial derivatives of H with respect to the $v^k_{i,j}$'s that are components of $\underline{w}^r$. Instead of an expression for this matrix being obtained separately for each iteration

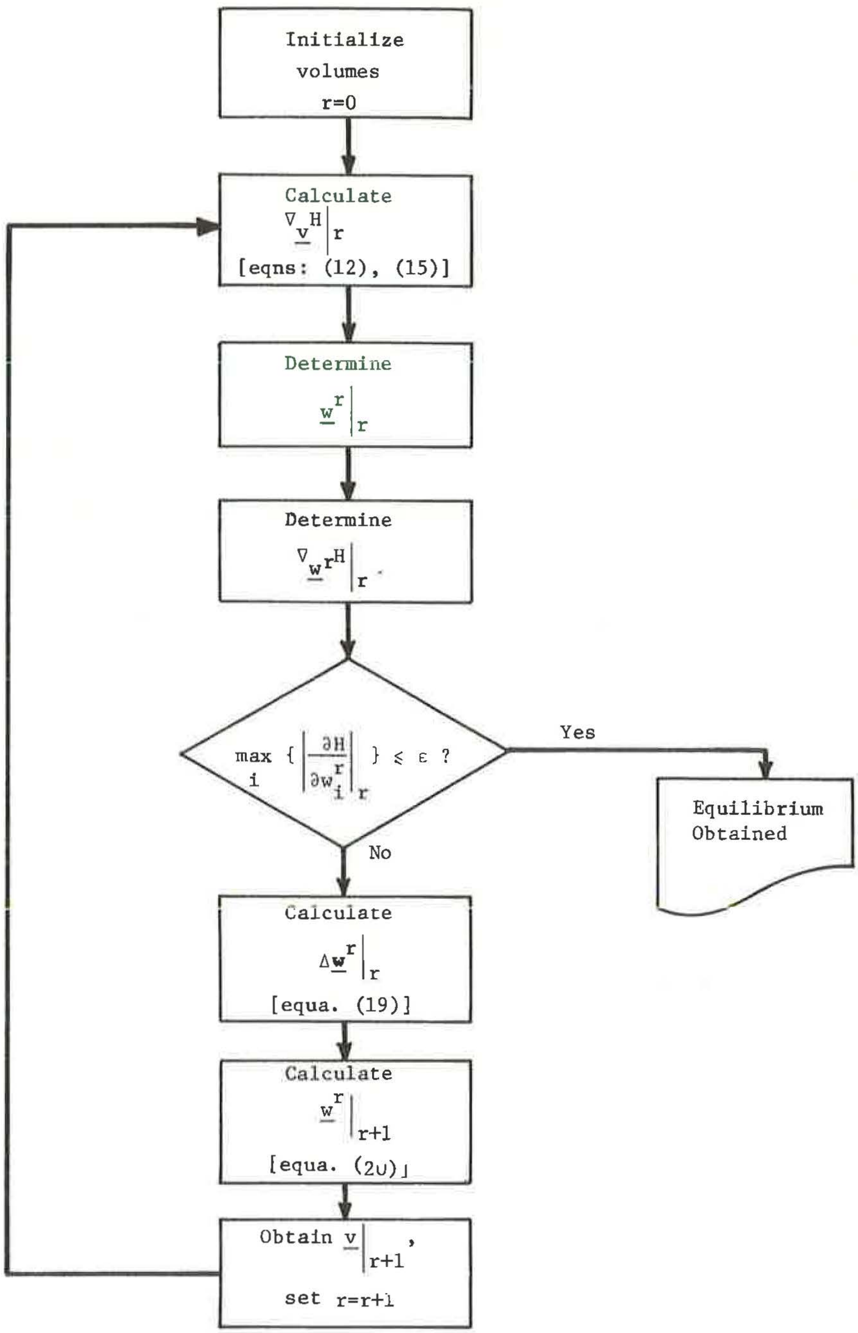


Figure 5. Flow chart for modified Newton-Raphson algorithm.

step, it is easier to obtain an expression for $\partial\nabla_{\underline{v}}H/\partial\underline{v}$, the matrix of second partial derivatives of H with respect to all of the $v^k_{i,j}$'s, and then to recognize that $\partial\nabla_{\underline{w}^r} H/\partial\underline{w}^r$ consists of some of the rows and columns of $\partial\nabla_{\underline{v}}H/\partial\underline{v}$. In addition, it can be shown that

$$\frac{\partial^2 H}{\partial v^\ell_{m,n}\,\partial v^k_{i,j}} = -\dot{h}_{i,j}(v_{i,j})\,\delta_{im}\,\delta_{jn} + \dot{g}^k_i(v^k_i)\,[\delta_{im}\sum_{\{a_i\}}\delta_{n,a_i} - \delta_{in}\sum_{\{b_i\}}\delta_{m,b_i}]\,\delta_{kl} - \dot{g}^k_j(v^k_j)\,[\delta_{jm}\sum_{\{a_j\}}\delta_{n,a_j} - \delta_{jn}\sum_{\{b_j\}}\delta_{m,b_j}]\,\delta_{kl} \tag{21}$$

In Eq. 21, the dot above a function implies differentiation of that function with respect to its argument, $\delta_{ij} \equiv 1$ if $i = j$ and 0 if $i \neq j$; and $\sum_{\{a_i\}}\delta_{n,a_i} \equiv 1$ if n is a member of $\{a_i\}$ and 0 if otherwise.

Thus, all the elements of $\partial\nabla_{\underline{v}}H/\partial\underline{v}$ are available from Eq. 21. Furthermore, if all the inverse demand and supply functions g^k_i and $h_{i,j}$ respectively are linear functions of their arguments, then each component of $\partial\nabla_{\underline{v}}H/\partial\underline{v}$ is a constant.

EXAMPLES

Two relatively simple examples are considered in this section to illustrate the application of the preceding results to equilibrium problems in travel forecasting. The network used in the examples is shown in Figure 6. For simplicity, the demand and service functions in the examples are assumed to be linear. However, this restriction is certainly not necessary, and any nonlinear (or piece-wise linear) functions can readily be used in the preceding algorithms.

Example 1

The interzonal demand and link service functions used in this example are as follows:

$$v^2_1 = d^2_1(t^2_1) = 16.625 - t^2_1 \qquad v^3_1 = 28.25 - t^3_1$$

$$v^4_1 = 21.375 - t^4_1 \qquad v^3_2 = 21.625 - t^3_2$$

$$v^3_4 = 16.875 - t^3_4 \qquad v^3_5 = 28 - t^3_5$$

and

$$t_{1,2} = h_{1,2}(v_{1,2}) = 5 + 0.1\,v_{1,2} \qquad t_{2,3} = 10 + 0.1\,v_{2,3}$$

$$t_{1,4} = 10 + 0.1\,v_{1,4} \qquad t_{4,3} = 5.5 + 0.1\,v_{4,3}$$

$$t_{5,1} = 1 \qquad t_{5,3} = 18$$

Two points to note are that the inverse of every demand functions exists and two of the service functions are independent of the volume flow rate of trips using the link. The relationships between the various volume components can be clarified by considering

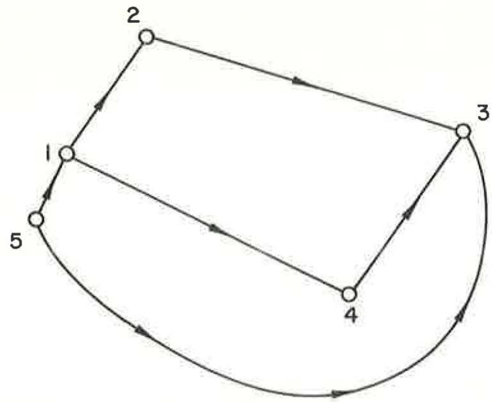


Figure 6. Network for the examples.

the calculation of the gradient component $\partial H/\partial v_{1,2}^3$. From Eq. 12,

$$\frac{\partial H}{\partial v_{1,2}^3} = g_1^3(v_1^3) - g_2^3(v_2^3) - h_{1,2}(v_{1,2}) \tag{24}$$

Inverting the appropriate demand functions given in the preceding yields

$$g_1^3(v_1^3) = 28.25 - v_1^3 \tag{25}$$

$$g_2^3(v_2^3) = 21.625 - v_2^3$$

However, v_1^3 and v_2^3 must be related to the component flows on the various links. Thus, using Eq. 4, we obtain

$$v_1^3 = v_{1,2}^3 + v_{1,4}^3 - v_{5,1}^3 \tag{26}$$

$$v_2^3 = v_{2,3}^3 - v_{1,2}^3$$

If we substitute Eqs. 25 and 26 and the appropriate service function in Eq. 24, it follows that

$$\frac{\partial H}{\partial v_{1,2}^3} = 1.625 - 2.1\, v_{1,2}^3 - v_{1,4}^3 - 0.1\, v_{1,2}^2 + v_{2,3}^3 + v_{5,1}^3 \tag{27}$$

Similar expressions are obtained for all the gradient components $\partial H/\partial v_{i,j}^k$ for use in both the gradient and modified Newton-Raphson algorithm.

Obviously, some initial estimates of the component flows on the various links must be made to use the algorithms discussed earlier. There are various approaches that can be taken in making initial estimates; for example, completely random volumes could be chosen, all zero volumes could be chosen, or volumes corresponding to the unloaded minimum time path could be chosen. The latter 2 cases were tried here. The results for both the gradient and modified Newton-Raphson methods are shown in Table 1. Two sets of stopping criteria, i.e., $\epsilon = 10^{-2}$ and 10^{-3}, were used for the gradient method. The stopping criterion for the modified Newton-Raphson method was $\epsilon = 10^{-7}$. It is seen that both techniques converged to the correct equilibrium. The modified Newton-Raphson method converged in 1 iteration with the zero initial estimates and in 3 iterations with the other estimates. In the latter case, the number of variables considered at each iteration was different. Thus, the modifications in the Newton-Raphson method required by the constraints were indeed exercised. It is interesting to note that

TABLE 1

RESULTS FOR EXAMPLE 1

Function	Initial Guess	Gradient $\epsilon = 10^{-2}$ (196 iterations)	Gradient $\epsilon = 10^{-3}$ (312 iterations)	Initial Guess	Gradient $\epsilon = 10^{-2}$ (135 iterations)	Gradient $\epsilon = 10^{-3}$ (164 iterations)	Modified Newton-Raphson (1 iteration)
$v_{1,2}^2$	11.625	9.989	9.999	0.0	9.998	10.000	10.00
$v_{1,2}^3$	25.25	6.354	6.261	0.0	6.251	6.247	6.25
$v_{1,4}^3$	0.0	3.646	3.739	0.0	3.76	3.753	3.75
$v_{1,4}^4$	11.375	10.010	10.001	0.0	9.998	9.999	10.00
$v_{2,3}^3$	36.875	16.353	16.260	0.0	16.259	16.248	16.25
$v_{4,3}^3$	11.375	13.646	13.739	0.0	13.768	13.753	13.75
$v_{5,1}^3$	12.0	0.0	0.0	0.0	0.0	0.0	0.0
$v_{5,3}^3$	0.0	9.999	9.999	0.0	9.999	9.000	10.0

both the gradient and modified Newton-Raphson algorithms converged more quickly with the zero initial conditions. It is also interesting to note that the unloaded minimum time path from zone 5 to zone 3 is from 5 → 1 → 2 → 3, whereas the minimum time path from 5 to 3 at equilibrium is along the link directly connecting zones 5 and 3. Thus, the first set of initial estimates assigns all the volume from zones 5 to 3 along the wrong path, but the techniques still converge. Perhaps, however, this explains the need for fewer iterations with zero initial estimates for all link volumes. These results are indeed quite good, especially those using the modified Newton-Raphson algorithm.

Example 2

In this example, the case where the inverses of all demand functions do not exist is considered. This corresponds to a fixed demand between zones. This case is of interest because some of the nodes used in representing transportation networks may be "throughpass" nodes where no trips originate or terminate but where alternate routes intersect.

The technique used here is to define a pseudo-demand function for which an inverse function does exist but which is a good approximation to the fixed, zero demand function. The algorithms presented in this paper can then be applied; the question is simply, Will the algorithms converge?

The network shown in Figure 6 is used again. The volumes v_2^3 and v_1^4 are assumed to be zero, and the demand and service functions are as follows:

$$v_1^2 = d_1^2(t_1^2) = 16.25 - t_1^2 \qquad v_1^3 = 19.0 - t_1^3$$

$$v_4^3 = 16.5 - t_4^3 \qquad v_5^3 = 22.5 - t_5^3$$

The zero volume v_2^3 is approximated by

$$v_2^3 = 10^{-3} - 10^{-5}\ t_2^3$$

and

$$t_{1,2} = h_1^2(v_1^2) = 5 + 0.1\ v_{1,2}$$

$$t_{2,3} = 10 + 0.1\ v_{2,3} \qquad t_{1,4} = 10 + 0.1\ v_{1,4}$$

$$t_{4,3} = 5.5 + 0.1\ v_{4,3} \qquad t_{5,1} = 1 + 0.1\ v_{5,1}$$

$$t_{5,3} = 17 + 0.1\ v_{5,3}$$

No pseudo-demand function was defined for v_1^4 because it turns out that it does not enter into any of the gradient components and can thus be completely ignored. The gradient components are expressed in terms of all the component flows $v_{i,j}^k$ as discussed in the preceding example. One of the gradient components that involves a pseudo-demand function is $\partial H/\partial v_{1,2}^3$, and this is found to be

$$\frac{\partial H}{\partial v_{1,2}^3} = -86 - 1.1\ v_{1,2}^3 - v_{1,4}^3 + v_{5,1}^3 + 10^5\ (v_{2,3}^3 - v_{1,2}^3) - 0.1\ v_{1,2}^2 \qquad (28)$$

One would suspect that numerical difficulties would occur when gradient components such as those in Eq. 28 are used. Indeed, the gradient algorithm did not converge for any set of initial estimates tried and for up to 5,000 iterations. However, the modified Newton-Raphson algorithm converged to the equilibrium in two iterations ($\epsilon = 10^{-4}$). This is interesting because the matrix of second partial derivatives that must be inverted contains terms that differ by 5 orders of magnitude. The equilibrium flows in this case are

$$v_{1,2}^2 = 10 \qquad v_{1,2}^3 = 2.5 \qquad v_{1,4}^3 = 0$$

$$v_{2,3}^3 = 2.5 \qquad v_{4,3}^3 = 10 \qquad v_{5,3}^3 = 5 \qquad v_{5,1}^3 = 0$$

SUMMARY AND CONCLUSIONS

The problem of precisely determining the equilibrium between supply and demand was considered in a travel-forecasting problem when travel demand is assumed to depend on the level of service delivered by a transportation system. Based on some results of Beckman, McGuire, and Winsten (8), 2 new algorithms for precisely determining equilibrium were developed by using numerical optimization techniques. The algorithms use a constrained gradient technique and a modified Newton-Raphson procedure to maximize a functional of network flows, and Beckman, McGuire, and Winsten (8) have shown that the flows thus determined are the equilibrium flows in the network. Very good results were obtained in the examples considered by using the algorithms given.

The advantages of the algorithms presented in the paper over techniques like DODOTRANS (12) are that equilibrium is obtained precisely rather than approximately, and computation of minimum paths during the iterative process is not required. Certainly more work remains to be done in applying the techniques to very large networks. However, one can be very optimistic about the application of these techniques to large networks because of known techniques in other fields.

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APPENDIX

The functional H is given in Eq. 11. If the chain rule for differentiation is used, it follows that

$$\frac{\partial H}{\partial v_{i,j}^k} = \sum_m \sum_\ell \frac{d}{dv_m^\ell} \left[\int_0^{v_m^\ell} g_m^\ell (x)\, dx \right] \frac{\partial v_m^\ell}{\partial v_{i,j}^k}$$

$$- \sum_m \sum_n \frac{d}{dv_{m,n}} \left[\int_0^{v_{m,n}} h_{m,n} (x)\, dx \right] \frac{\partial v_{m,n}}{\partial v_{i,j}^k} \qquad (29)$$

Then, using Leibnitz's rule in Eq. 29 yields

$$\frac{\partial H}{\partial v_{i,j}^k} = \sum_m \sum_\ell g_m^\ell (v_m^\ell) \frac{\partial v_m^\ell}{\partial v_{i,j}^k} - \sum_m \sum_n h_{m,n}(v_{m,n}) \frac{\partial v_{m,n}}{\partial v_{i,j}^k} \qquad (30)$$

Now, if v_m^ℓ is expressed in the form of Eq. 4, it can be seen that the partial derivative of v_m^ℓ with respect to $v_{i,j}^k$ is zero unless $\ell = k$. Similarly, if $v_{m,n}$ is expressed in the form of Eq. 1, $\frac{\partial v_{m,n}}{\partial v_{i,j}^k}$ is seen to be zero unless $m = i$ and $n = j$. Hence, Eq. 30 becomes

$$\frac{\partial H}{\partial v_{i,j}^k} = \sum_m g_m^k (v_m^k) \frac{\partial v_m^k}{\partial v_{i,j}^k} - h_{i,j}(v_{i,j}) \frac{\partial v_{i,j}}{\partial v_{i,j}^k} \qquad (31)$$

Again making use of Eq. 4, we obtain

$$\sum_m g_m^k(v_m^k)\frac{\partial v_m^k}{\partial v_{i,j}^k} = \sum_m g_m^k(v_m^k)\frac{\partial}{\partial v_{i,j}^k}\left[\sum_{\{a_m\}} v_{m,a_m}^k\right] - \sum_m g_m^k(v_m^k)\frac{\partial}{\partial v_{i,j}^k}\left[\sum_{\{b_m\}} v_{b_m,n}^k\right] \quad (32)$$

$$g_i^k(v_i^k)\frac{\partial}{\partial v_{i,j}^k}\left[\sum_{\{a_i\}} v_{i,a_i}^k\right] - g_j^k(v_j^k)\frac{\partial}{\partial v_{i,j}^k}\left[\sum_{\{b_j\}} v_{b_j,j}^k\right] \quad (33)$$

Equation 33 was obtained by recognizing that the partial derivatives involved in the first and second terms on the right of Eq. 32 are respectively equal to zero when $m \neq n$ and $m \neq j$. The fact that $v_{i,j}^k$ exists as a variable implies that link ij exists to carry traffic from i to j. By definition, then, one of the members of $\{a_i\}$ is j, and one of the members of $\{b_j\}$ is i. Hence, $v_{i,j}^k$ appears once in each of the summations, $\sum_{\{a_i\}} v_{i,a_i}^k$ and $\sum_{\{b_j\}} v_{b_j,j}^k$; and the partial derivative with respect to $v_{i,j}^k$ of each of these summations thus equals 1. Equation 33 then becomes

$$\sum_m g_m^k(v_m^k)\frac{\partial v_m^k}{\partial v_{i,j}^k} = g_i^k(v_j^k) \quad (34)$$

Using Eq. 1, we have

$$\frac{\partial v_{i,j}}{\partial v_{i,j}^k} = 1 \quad (35)$$

so that substitution of Eq. 34 into Eq. 31 and use of Eq. 35 finally yields

$$\frac{\partial H}{\partial v_{i,j}^k} = g_i^k(v_i^k) - g_j^k(v_j^k) - h_{i,j}(v_{i,j}) \quad (36)$$