EXPERIMENTAL VALIDATION OF MODIFIED BOLTZMANN TYPE OF MODEL AND SHIFT MODEL FOR MULTILANE TRAFFIC FLOW

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The Boltzmann type of statistical models proposed by Prigogine et al. for time-independent, space-homogeneous, multilane traffic assumes that traffic flow is described by 3 processes—interaction, relaxation, and adjustment—and has been shown earlier to produce poor validation results. In this paper we propose a new model that uses a shifting process $f^*(v) = \gamma f^0(\gamma v)$, where γ is a concentration-dependent parameter that replaces the desired speed-density $f^0(v)$ in the relaxation process. The resulting modified Boltzmann type of model is shown to have many desirable properties and fits experimental speed data within the data error margin. A second model, called the shift model, uses only the shifting process and neglects the interaction, relaxation, and adjustment processes of the Boltzmann type of model, gives results only slightly less optimal than the result of the best possible fit obtained by the modified Boltzmann type of model, and is far better than the original Boltzmann type of models. Some applications of the shift model are discussed.

•A RECENT paper (1) critically examined the Boltzmann type of statistical models for multilane traffic flow developed by Prigogine et al. (2, 3) and was followed by an experimental validation study (4) of the Prigogine models for time-independent, space-homogeneous, 2-lane, unidirectional traffic flow. It was concluded (4) that the models proposed by Prigogine et al. were incapable of producing a realistic description of the traffic flow behavior and gave poor agreement between the experimental and the computed speed distribution functions even under low-to-moderate (i.e., 49.5 and 88.4 vehicles/mile for a 2-lane highway) traffic conditions. This validation study also indicated that the addition of an adjustment term of the form suggested by Prigogine et al. led to no significant improvement in the realism of the model.

From the validation study it seems that Prigogine et al. were unable to incorporate dominant factors of traffic dynamics into their models. When traffic concentration increases, the Prigogine models predict a "pileup" (through the interaction process) behind the vehicles moving at the minimum desired speed (4), while it is evident from observations that many vehicles are now traveling below minimum desired speed.

In the basic Boltzmann type of model (2), the change of speed was assumed to be due to 2 processes: the interaction process and the relaxation process. Although the interaction process was derived analytically, it was argued (1) that it overestimated the interaction effect.

The generalized Boltzmann type of model (3) with an additional adjustment term seems of little importance from the earlier validation study (4). The adjustment term relaxes some vehicles to the average speed in the relaxation process rather than to their desired speeds. That is, the speed-density shows a δ function (physically, we mean that the number of vehicles traveling in a specific speed range, say 3 to 5 ft/sec, is substantially higher than those immediately outside of that range) at $v = \bar{v}$, the average speed. However, the fact that the validation study (4) shows no significant improvement can be interpreted to mean that the assumption of being relaxed to the average speed is physically ungrounded. Furthermore, in no way does it solve the pileup problem.

The Boltzmann type of models shows even worse results in high concentration, in that the speed-density becomes negative for certain ranges of speeds when concentration increases to a certain extent, particularly when the average speed becomes less than the minimum desired speed. Under such circumstances, Prigogine et al. (3) were forced to assume that a fraction of the vehicles had to have 0 speed. The Appendix gives a more detailed description of the basic and generalized Boltzmann type of models.

A clearly unsatisfactory property of the Boltzmann type of models is that the actual minimum and maximum speeds are unchanged from the desired minimum and maximum speeds. This property is reflected in the relaxation process, in that the vehicles are relaxed to their original desired speeds after the interaction with other vehicles.

Our first attempt to modify the Boltzmann type of models is to incorporate a shifting process in the relaxation process. This shifting process shifts the desired speed at higher concentration toward lower speeds, uniformly through a shift parameter γ. We shall call this shifted speed-density the relaxed speed-density. When the desired speed-density in the relaxation process of the basic Boltzmann type of model is replaced by the relaxed speed-density, we have obtained the modified Boltzmann type of model. The parameter γ depends on the overall vehicle concentration rather than on the local concentration. That is, we ignore the spatial correlations among vehicles that exist in real traffic in a manner consistent with that in which the interaction process is handled in a Boltzmann type of model. By doing this, we achieve smaller variation in speeds and lower maximum and minimum speeds when vehicle concentration increases. These properties are desirable.

Our second model uses the shifting process alone and neglects the interaction, relaxation, and adjustment processes in the original Boltzmann type of models. We shall call this simple model the shift model. Validation results of this shift model will give indications of the importance of the shifting process concept, compared to other Boltzmann processes.

As in the previous validation study (4), only the time-independent, space-homogeneous case is considered. Available Federal Highway Administration data are most suitable for this type of validation, and the Boltzmann type of integrodifferential equation can easily be solved for this case.

MODIFIED BOLTZMANN AND SHIFT MODELS

The basic objective of the Boltzmann type of model is to predict the speed-density function f(x, v, t) at any vehicle concentration from a knowledge of the desired speed or free speed density function $f^0(x, v, t)$. It is assumed that this desired speed-density is realized in the limit of very dilute vehicle concentrations.

We define f(x, v, t) = speed-density function of vehicles at time t and point x whose actual speed is v considered with respect to space.

The corresponding desired speed-density function is $f^0(x, v^0, t)$ = speed-density function of vehicles at time t and point x whose drivers have desired speed v^0 considered with respect to space.

In addition, relaxed speed-density is defined as $f^*(x, v^*, t)$ = speed-density function of vehicles at time t and point x whose drivers have relaxed speed v^* considered with respect to space.

We also define v_{max}, v_{min} = upper and lower limits respectively of the actual speeds, and define v^0_{max}, v^0_{min}, v^*_{max}, v^*_{min} similarly.

The space-mean speed can now be defined as

$$\overline{v}(x, t) = \int_0^\infty v f(x, v, t) dv$$

The desired space-mean speed, $\overline{v}^0(x, t)$, and the relaxed space-mean speed, $\overline{v}^*(x, t)$, are defined similarly.

Available Federal Highway Administration traffic analyzer data were measured according to time (4). A transformation between space and time data (5),

$$f_s(v) = (\overline{v}/v) f_t(v)$$

where the subscripts t and s denote time and space respectively, was used to obtain spacewise data. This paper considers only the analysis and validation of speed-density in space; the case of speed-density in time can be similarly carried out.

The relaxed speed-density f^* is hypothesized, being obtained through a shifting process of the desired speed-density.

$$f^*(x, v^*, t) = \gamma f^0(x, v^0, t) \tag{1}$$

and

$$v^0 = \gamma v^* \tag{2}$$

We shall call γ the shift parameter in the sequel. This shift parameter γ is assumed to be a function of concentration c with the following properties

$$\gamma \geq 1, \lim_{c \to 0} \gamma = 1, \text{ and } \lim_{c \to c_j} \gamma = \infty$$

where c_j is the jam concentration.

It is easy to see that $f^*(x, v^*, t)$ is indeed a probability density function as

$$\int_0^\infty f^*(x, v^*, t)dv^* = \int_0^\infty \gamma f^0(x, v^0, t)\ (dv^0/\gamma) = 1$$

since $f^0(x, v^0, t)$ is a probability density function.

As in the previous work (Appendix), the rate of change of f(x, v, t) with time is expressed as

$$[df(x, v, t)]/dt = \{[\partial f(x, v, t)]/\partial t\} + \{\partial f(x, v, t)]/\partial x\} \cdot (dx/dt) \tag{3}$$

and the change of f within time is assumed here through 2 processes: the interaction process and the relaxation process. Here we have neglected the adjustment term because it was shown earlier (1) that the relaxation and adjustment terms could be combined as

$$(\partial f/\partial t)_{rel} + (\partial f/\partial t)_{adj} = - [(f - \tilde{f})/T']$$

where

$$\tilde{f} = \eta f^0 + (1 - \eta)\ \delta(v - \overline{v}),$$
$$\eta = T'/T,$$
$$T' = T/(1 + \lambda T),$$

and T and λ are the relaxation time and the adjustment constant respectively (Appendix). This simply means that the relaxation process relaxes speed back to $\tilde{f}$ rather than f^0. More precisely, η fraction of vehicles relaxes back to f^0, and $(1 - \eta)$ fraction relaxes back to $\overline{v}$. Therefore, we can write Eq. 3 as

$$df/dt = (\partial f/\partial t)_{int} + (\partial f/\partial t)_{rel} \tag{4}$$

$$(\partial f/\partial t)_{int} = (1 - P)(\overline{v} - v)f \tag{5}$$

$$(\partial f/\partial t)_{rel} = - [(f - \tilde{f})/T'] \tag{6}$$

where P = probability of passing. The contribution to the time rate of change of f due to the interaction process is analogously the same as before (1).

The relaxation process in the present model assumes f relaxes back toward f^* rather than toward f^0. That is,

$$(\partial f/\partial t)_{rel} = - [(f - f^*)/T] \tag{7}$$

instead of Eq. 6 in the generalized Boltzmann type of model.

Finally, the interaction term in Eq. 5 and the relaxation term in Eq. 7 are added together to obtain the modified Boltzmann type of equation for traffic flow.

$$df/dt = (1 - P)(\overline{v} - v)f - [(f - f^*)/T] \tag{8}$$

For time-independent, space-homogeneous traffic flow, Eq. 8 reduces to

$$f(v) = [f^*(v)]/[1 - \beta(\overline{v} - v)] = [\gamma f^0(\gamma v)]/[1 - \beta(\overline{v} - v)] \tag{9}$$

where $\beta \equiv (1 - P)T$.

The incorporation of an adjustment term by Prigogine et al. (3) has achieved qualitatively the effect of follow-the-leader type of traffic theory. Because the decrease in speed dispersion (variance) is a major consequence of this adjustment, we can see that it is also achieved through the shifting process, i.e., $\mathrm{var}(v^*) = 1/\gamma^2\,\mathrm{var}(v^0)$ (for proof, see an earlier paper, 6) in the modified model. Furthermore, the vehicle concentration can be divided into 3 sections, corresponding to cases A, B, and C (Appendix), and the basic and generalized Boltzmann type of models produce realistic speed-densities only for case A. The introduction of γ gives realistic speed-densities for the entire range of vehicle concentration by shifting both maximum and minimum speeds toward lower values, that is, $v_{max} = v^*_{max} = 1/\gamma\, v^0_{max}$ and $v_{min} = v^*_{min} = 1/\gamma\, v^0_{min}$. Therefore, we have achieved the objective of extending the model so that it is applicable over a wider range of vehicle concentrations. The available experimental data support our argument that maximum and minimum speeds decrease at higher concentration. Thus, our proposed modified Boltzmann type of model retains the general Prigogine form but relaxes speed toward f^* rather than f^0. By doing this, we have retained all desirable properties and deleted the shortcomings.

We shall focus our discussions and validations on the modified model (Eq. 9).

Some interesting properties of γ and β are illustrated below.

$$\gamma = \overline{v^0}/\overline{v^*}$$

$$\gamma^n = \overline{(v^0)^n}/\overline{(v^*)^n} \tag{10}$$

for positive integer n.

$$\sigma^2_{v^*} = (1/\gamma^2)\,\sigma^2_{v^0} \tag{11}$$

where $\sigma^2_{v^*}$ and $\sigma^2_{v^0}$ are the variances of v^* and v^0 respectively.

$$\gamma_{max} = \overline{v}^0/\overline{v} \tag{12}$$

and γ takes its maximum value, γ_{max}, when $\beta = 0$.

$$\beta = (\bar{v}^* - \bar{v})/\sigma_v^2 \tag{13}$$

where σ_v^2 is the variance of v.

$$\beta < 1/(\bar{v} - v_{min}^*) \tag{14}$$

The proofs are given in an earlier paper (6).

Some of the important and interesting properties of the Boltzmann type of models have been described in the Appendix. In the following discussion, we shall see that the modified model retains all promising properties and avoids the shortcomings.

The property that $\gamma \geq 1$ implies that $\bar{v}^* \leq \bar{v}^0$, $v_{min}^* \leq v_{min}^0$, and $v_{max}^* \leq v_{max}^0$; and we are now free of the restriction that the actual maximum and minimum speeds do not change from the desired maximum and minimum at very dilute concentration. These differences do exist, as we shall see later in validation. Empirically, we should have $\bar{v} \to 0$ when $c \to c_j$ and $v \to v^0$ when $c \to 0$. This means that in the former case $v_{min} = v_{max} = 0$, an obvious change from the desired minimum and maximum. The modified model gives $\gamma \to \infty$ as $c \to c_j$, or $v_{min}^* = v_{max}^* = 0$, a satisfactory result. In the latter case, the modified model gives $\gamma \to 1$ as $c \to 0$, or $v_{min}^* = v_{min}^0$, $v_{max}^* = v_{max}^0$, again a satisfactory result.

The property $\sigma_{v^*}^2 = (1/\gamma^2)\,\sigma_{v^0}^2$ implies that the variance (dispersion) of v^*, and consequently of v, decreases when traffic concentration increases, a phenomenon observed in the experimental data. This also retains the major merit of the adjustment term in the generalized Boltzmann type of model without the physically unsound condition that a finite percentage of vehicles have to travel exactly at the speed $\bar{v}$.

The role of β, as we examine Eq. 9, is to increase the speed-density at speeds $v < \bar{v}$ and decrease the speed-density at $v > \bar{v}$. This is the same as was observed from the Boltzmann type of models. However, here we have imposed a bound (Eq. 14) on β so that the speed-density will never be negative. This enables us to avoid the unrealistic assumptions that f^0 has a pole at $v = 0$ and is 0 between $v = 0$ and $v = v_{min}^0$.

The relations among the desired speed, the relaxed speed, and the actual speed-density functions are shown in Figure 1. Contrary to the Boltzmann type of models (Fig. 11), this modified model represents the entire $\bar{v}$ - c (speed-concentration) range and does not give poles at $v = \bar{v}$, $v = v_{min}$, or $v = 0$ unless the traffic is at jam concentration, in which case $f(v) = \delta(v)$, or unless the desired speed-density specifies poles at certain v. When concentration increases, v_{min} and v_{max} shown in Figure 1 will shift farther left through a larger value of γ and will not go beyond $v = 0$, in which case $\gamma = \infty$.

A second model, called the shift model, that uses only the shifting process to describe traffic flow and ignores all other Boltzmann type of terms (i.e., interaction, relaxation, and adjustment) is also proposed. This model has the following form:

$$\begin{aligned} f(v) &= \gamma_{max} f^0(\gamma_{max} v) \\ &= (\bar{v}^0/\bar{v})\, f^0[(\bar{v}^0/\bar{v})\, v] \end{aligned} \tag{15}$$

Although the shift model is simple in structure, the desired properties of decreasing minimum and maximum speeds and the speed variance at higher concentrations are achieved. We summarize the properties of the shift model as follows:

1. $\gamma_{max} = \bar{v}^0/\bar{v} \geq \bar{v}^0/\bar{v}^* = \gamma$. $\gamma_{max} \to 1$ when $c \to 0$ $(\bar{v} \to v^0)$ and $\gamma_{max} \to \infty$ when $c \to c_j$ $(\bar{v} \to 0)$.
2. $\sigma_v^2 = (1/\gamma_{max}^2)\,\sigma_{v^0}^2$. The variance of computed speed by the shift model is $1/\gamma_{max}^2$ of the variance of the desired speed. We have, therefore, achieved the decrease in speed variance.
3. $v_{min} = (1/\gamma_{max})\, v_{min}^0$ and $v_{max} = (1/\gamma_{max})\, v_{max}^0$. The minimum and maximum speeds from the shift model are reduced to $1/\gamma_{max}$ of the desired minimum and maximum respectively.

In the next section we shall see that the shift model, which uses only the shifting process (Eq. 15), produces good validation results. On the contrary, the basic and generalized models that use interaction, relaxation, and adjustment processes without the incorporation of the shifting process produce very poor validation results. This suggests that the shifting process alone plays a much more dominant role in describing traffic flow compared to the interaction, relaxation, and adjustment terms in the Boltzmann type of models.

VALIDATION PROCEDURE AND RESULTS

We attempt to validate the modified Boltzmann type of model and the shift model and compare them with the basic and generalized Boltzmann type of models for time-independent, space-homogeneous conditions and for low-to-moderate vehicle concentrations because (a) the closed-form solution is available in Eq. 9, (b) the available traffic analyzer data represent these conditions, and (c) the basic and generalized Boltzmann equations are promising only for concentration ranges given by case A (Appendix). We are also planning to use aerial photographic data for similar studies so that these models are tested over a variety of sites and vehicle concentrations.

Equations 9 and 15 represent relations between the actual speed-densities at any traffic concentration for the modified Boltzmann and the shift models and the desired speed-density or free speed-density realized at very dilute traffic concentrations. The basic approach is to validate the relation between f^0 and f given by Eqs. 9 and 15. For this purpose, the desired speed-density f^0 is measured, and the other parameters of Eq. 9 are determined; the actual speed-density f can be computed from f^0 for any concentration. To evaluate the validity of the model requires that such computed speed-densities be compared with speed-densities that have been measured at different concentrations.

The validation for Eq. 15 is straightforward, as there is no parameter estimation involved. However, the parameters γ and β in Eq. 9 need to be determined from experimental data. Although β has an analytical expression (Eq. 13)

$$\beta = (\overline{v}^* - \overline{v})/\sigma_v^2$$

it is difficult to obtain its value without knowing $\overline{v}^*$, or equivalently γ, and σ_v^2. Furthermore, the calculation of σ_v^2 requires far more information than the average speed $\overline{v}$. Therefore, we proceed as follows.

Given the model

$$f(v) = \{[f^*(v)]/[1 - \beta(\overline{v} - v)]\} + \epsilon = \{[\gamma f^0(\gamma v)]/[1 - \beta(\overline{v} - v)]\} + \epsilon \tag{16}$$

Find γ and β such that

$$1 \le \gamma \le (\overline{v}^0/\overline{v})$$

$$\int_0^\infty \{[\gamma f^0(\gamma v)]/[1 - \beta(\overline{v} - v)]\}\, dv = 1 \tag{17}$$

and

$$d^2 = \int_0^\infty \left(f(v) - \{[\gamma f^0(\gamma v)]/[1 - \beta(\overline{v} - v)]\}\right)^2 dv \tag{18}$$

is minimized. We note that there is one, and only one, value of β besides 0 satisfying the normalization of Eq. 17. Proof is given in an earlier paper (6). The variables

$f(v)$, $\bar{v}$, and $f^0(v)$ are respectively the measured speed-density function, its space-mean speed, and desired speed-density function realized at very dilute concentration. ϵ is introduced here to represent errors caused by measurements, traffic disturbances, and so on. The minimization of Eq. 18 is the standard least squares minimization, and d^2 is a measure of goodness of fit.

The desired speed-density at very dilute concentration f^0, the actual speed-density f, and the space-mean speed corresponding to f are computed from the available traffic analyzer data. There are 5 data sets considered as 5 constant flow levels; these are represented by the 5 circles shown in Figure 2. The flows and concentrations are the sums of 2-lane traffic. The 3 lowest flow levels are used to estimate $f^0(v)$ and the 2 highest flow levels are used for experimental validation.

We first make the following computations:

1. $\hat{f}^0$, estimate of desired speed-density function in space, calculated by $\hat{f}^0 = (f_1 + f_2 + f_3)/3$, where f_1, f_2, and f_3 are the speed-density functions of the lowest 3 flow levels shown in Figure 2. The estimate $\hat{f}^0$ is shown in Figure 3.
2. $\hat{f}$, estimate of speed-density function at a flow above free-flow level. The highest 2 flow levels shown in Figure 2 represent this case. They are 2,345 cars/hour and 3,695 cars/hour corresponding to concentration 49.5 cars/mile and 88.4 cars/mile respectively (for 2-lane). Figure 4 shows an example of $\hat{f}$ for the 2,345 cars/hour flow.
3. $\hat{\bar{v}}$, estimate of space-mean speed corresponding to $\hat{f}$ and computed as the harmonic mean, $1/(1/n \sum 1/v_i)$, of the speeds $v_1, v_2, \ldots, v_n$ of the successive cars passing the observation point.

The value found for $\hat{\bar{v}}^0$, the estimated desired mean speed, was 48.18 mph, and the estimates $\hat{\bar{v}}$ for the 2 flow levels of 2,345 and 3,695 cars/hour were 47.43 and 41.92 mph respectively. Next, various trial values for γ were selected from the range $1 \leq \gamma \leq \bar{v}^0/\bar{v}$, and a corresponding value of β was calculated from Eq. 17.

Each pair of γ and β, along with the estimates given above, $\hat{f}^0$, and so on were then used in the modified Boltzmann type of Eq. 9 to calculate estimated speed-density $\hat{\hat{f}}$ for each of the 2 flow levels. The number d^2 in Eq. 18 was computed for each choice of parameters γ and β in which we use $\hat{f}$ for f and $\hat{\hat{f}}$ for $[\gamma f^0(\gamma v)]/[1 - \beta(\bar{v} - v)]$, and the "best" fit was then determined to be given by the values of γ and β that resulted in minimum d^2.

Before discussing these results, let us first examine the criterion on which one can decide whether a particular calculated speed-density is a "good" fit to the experimental density. We first need some measure of the scatter (i.e., fluctuations) in the experimental speed-densities. Because our estimate of the desired speed-density f^0 was obtained by averaging 3 low flow-concentration periods, the sum of the squares of the differences between these 3 low speed-densities (taken pairwise) would be a reasonable measure of the fluctuations of the experimental data. These 3 densities are based on observations of 674, 1,090, and 945 vehicles respectively. The sum of squares, d^2, of density differences was found to be 0.0026 (between 1 and 2), 0.0028 (between 1 and 3), and 0.0035 (between 2 and 3). Next, because the 2 higher flow-concentration periods against which we validated the model are based on observations of 984 and 1,369 vehicles (i.e., numbers of vehicles similar to those in the low-concentration periods), we argue that any model prediction of a speed-density that results in a sum of squares of differences (when compared to the corresponding experimental distribution) of approximately 0.003 or less would be acceptable.

Table 1 gives our findings. First, it is clear that the lower density case is relatively uninteresting. All of the sum of squares of differences are similar and come close to our criterion for significance. Here it makes little difference whether we use the basic Boltzmann model with $\gamma = 1$ and β determined by Eq. 17 or the generalized Boltzmann model (it happened that $\lambda = 0$ gave the best fit) or the modified Boltzmann model with γ and β chosen to give least squares or even the simple shift model that uses γ_{max}. These results for the lower density case are hardly surprising when one refers to data shown in Figure 2. There is very little deviation from a flow versus concentration linear relation (i.e., $\hat{\bar{v}}$ is only 0.75 mph less than $\hat{\bar{v}}^0$). Thus, all of the various models make very minor adjustments on their inputs, $\hat{f}^0$, and return density, $\hat{\hat{f}}$, that compares some-

Figure 1. Actual, f, desired, f°, and relaxed, f*, speed-density functions.

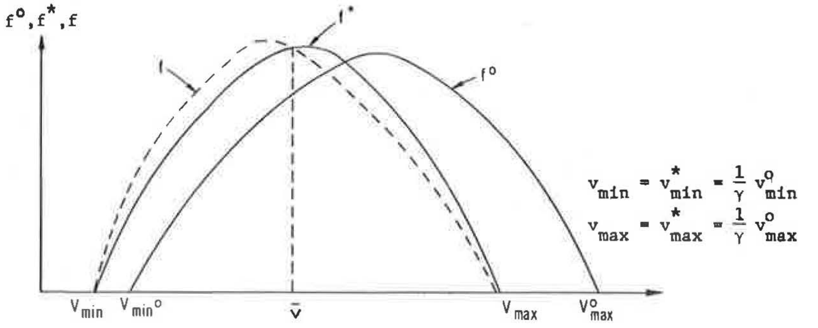


Figure 2. Measured flow and concentration.

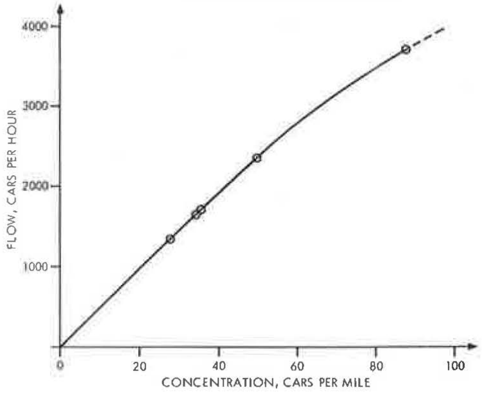


Table 1. Sum of squares of various models.

Model and Number of Vehicles	γ	β	d^{2a}
Basic Boltzmann			
2,345 vehicles/hour	—[b]	0.0175	0.0036
3,695 vehicles/hour	—[b]	0.0861	0.0495
Generalized Boltzmann			
2,345 vehicles/hour .	—[b]	0.0175	0.0036
3,695 vehicles/hour	—[b]	0.091[c]	0.0467
Modified basic Boltzmann			
2,345 vehicles/hour	1.010 to 1,016	0.005 to 0	0.0033
3,695 vehicles/hour	1.116	0.036	0.0012
Shift			
49.4 vehicles/mile	1.016	—[d]	0.0033
88.4 vehicles/mile	1.149	—[d]	0.0015

[a]Avg between 3 lowest concentration speed-densities is 0.003.
[b]Does not apply; equivalently, $\gamma = 1$.
[c]$\lambda = 0.622$.
[d]Does not apply; equivalently, $\beta = 0$.

Figure 3. Measured desired speed-density.

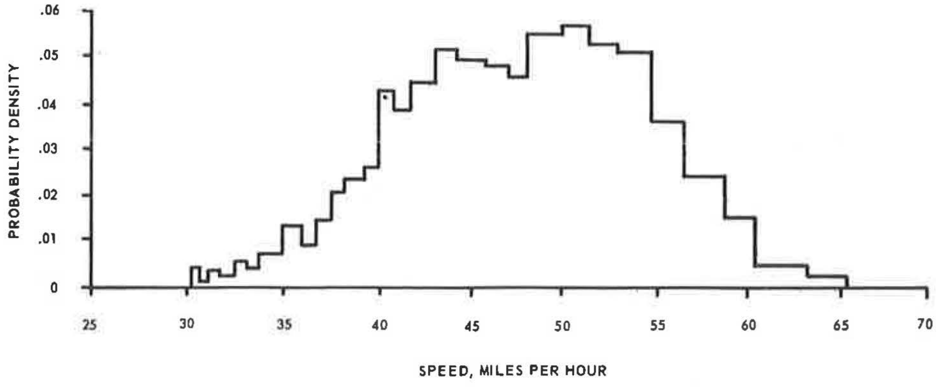


Figure 4. Measured and computed speed-densities by shift model.

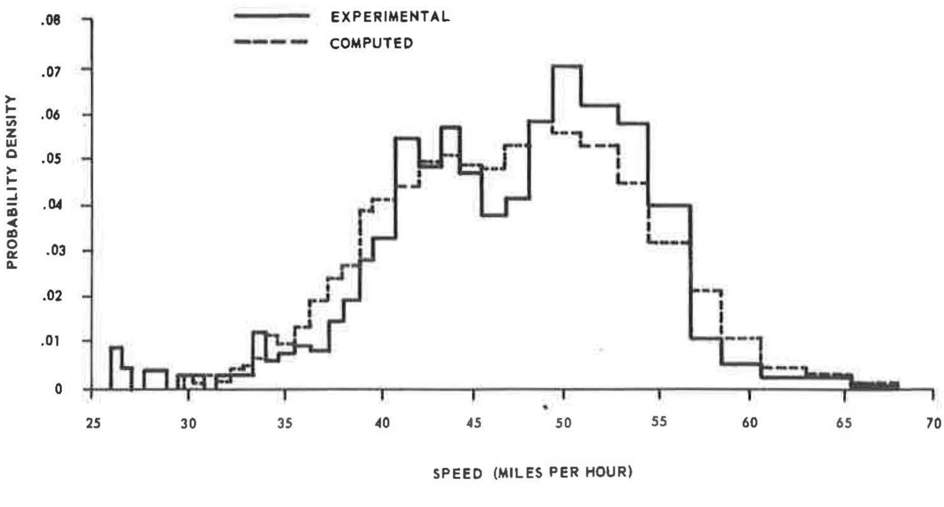


Figure 5. Measured and computed speed-densities by basic Boltzmann.

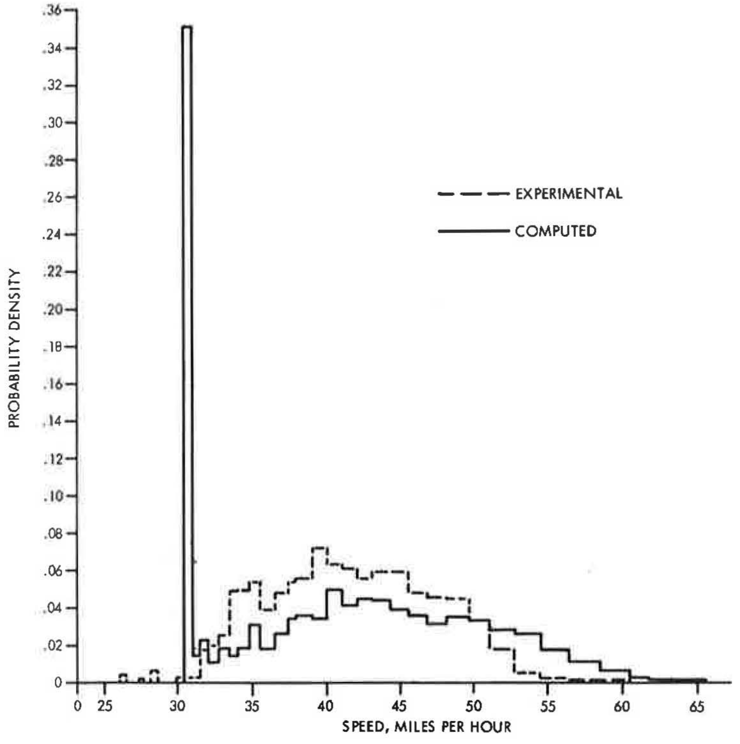


Figure 6. Measured and computed speed-densities by generalized Boltzmann.

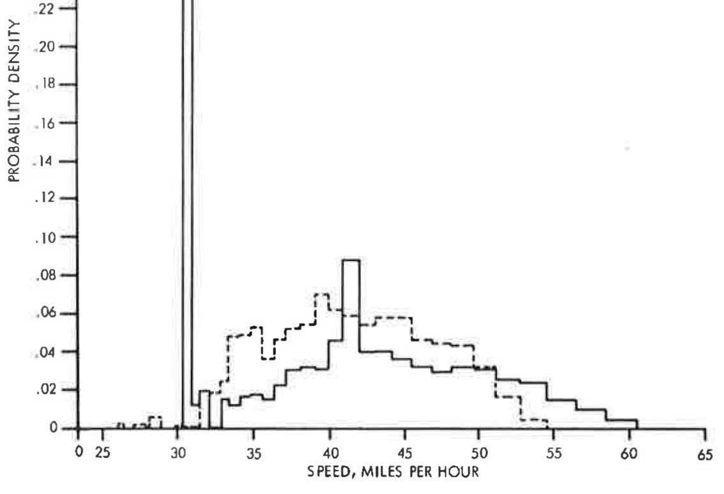


Figure 7. Measured and computed speed-densities by modified Boltzmann.

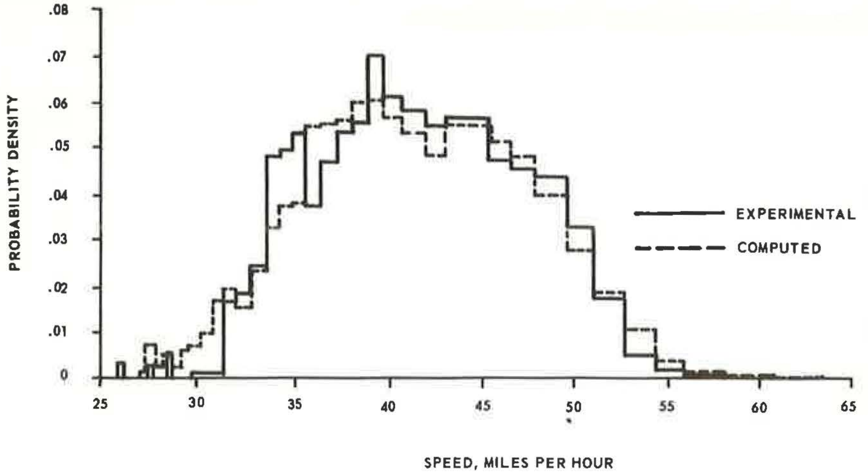


Figure 8. Measured and computed speed-densities by shift model.

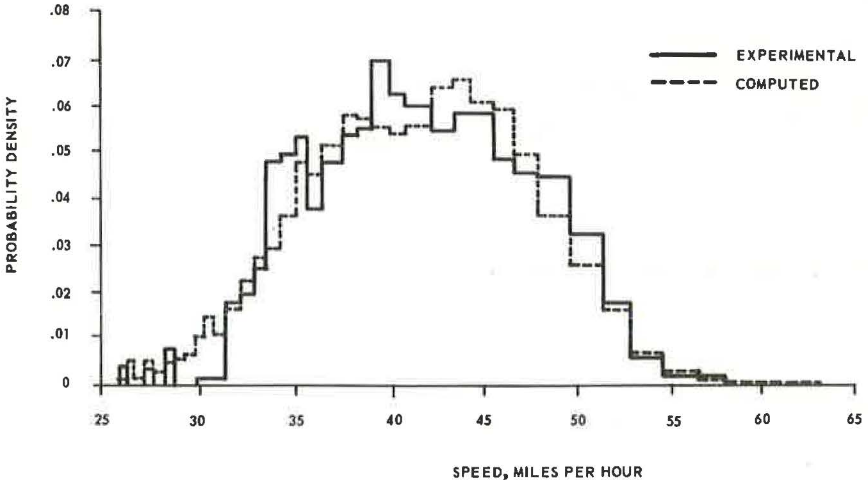
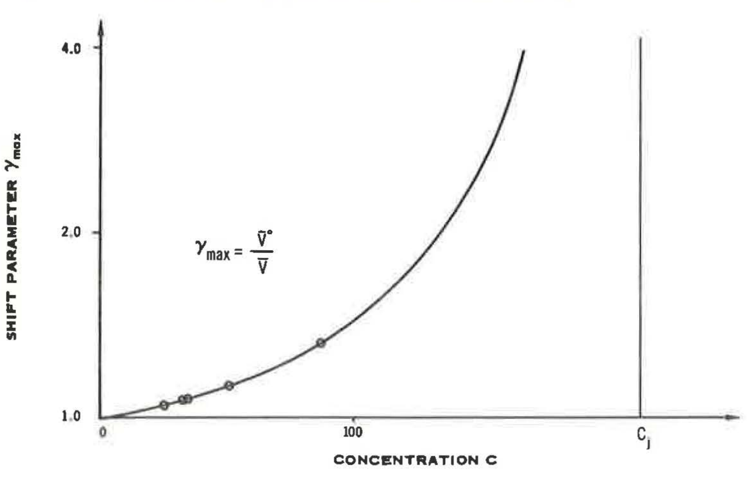


Figure 9. Shift parameter γ_{max} as function of concentration.



what favorably with $\hat{f}$; therefore, in the low-concentration case, these models do not produce significantly different results. We also notice that all of the trial values of γ and β within these ranges gave roughly the same sum of squares, d^2, and that $\gamma = \gamma_{max}$ and $\beta = 0$ yield an acceptable fit even though we are then dealing with only the shifting process. Figure 4 shows the computed speed-density obtained from this shift model. Results obtained by using other models are very similar, and there is no need to show them graphically.

Now let us examine the results given in Table 1 for the higher density and flow case, 88.4 vehicles/mile and 3,695 vehicles/hour (2 lanes). For this case, $\hat{\bar{v}} = 41.92$ mph compared to $\hat{\bar{v}}^0 = 48.18$ mph. So, there is a 15 percent drop off in space-mean speed (i.e., $\gamma_{max} = 1.149$). The basic Boltzmann type of model with $\beta = 0.0861$ yields poor predictions of the speed-density. It gives a d^2 of 0.0495, which is outside of our criterion (0.003) by more than an order of magnitude. The generalized Boltzmann type of model with $\beta = 0.091$ and $\lambda = 0.622$ gives a d^2 of 0.0467, which does not show significant improvement. These values of β and λ were the best possible fit values (the ones that gave the least d^2) one could determine by trying all the possible pairs of β and λ. The failure of the Boltzmann type of models at this density has been reported previously (4) and is partly due to the large pileup of cars around v^0_{min}. Figures 5 and 6 show the speed-density obtained from the basic and generalized Boltzmann models respectively.

This problem of pileup of vehicles at low speeds is relieved by the introduction of the shift parameter γ in the modified Boltzmann type of model (Fig. 7). The best fit values of γ and β, 1.116 and 0.036 respectively, for the modified basic Boltzmann model gives a d^2 of only 0.0012, which is well within our criterion. Thus, the introduction of a shifting process has resulted in a reduction of d^2 by a factor of 40. Data given in Table 1 further show that the results of using the shift model ($\beta = 0$, $\gamma = \gamma_{max} = \hat{\bar{v}}^0/\hat{\bar{v}} = 1.149$) give a d^2 of 0.0015, which is only slightly less optimal (and still well within the criterion) than the best possible γ and β fit case for the modified basic Boltzmann model (Fig. 8).

CONCLUSIONS AND APPLICATIONS

The results of this study have shown that the Boltzmann type of statistical models put forward by Prigogine et al. can be improved significantly by the incorporation of a shifting process, which results in a modified Boltzmann type of model when the shifting process is included in the basic Boltzmann model. A shift model that uses this shifting process alone has also been studied and has shown good results. The important observations of the 2 new models, as well as the basic and generalized Boltzmann type of models, are summarized below.

1. The basic Boltzmann type of model is described by the interaction and relaxation processes. The main objective of this model and of the other 3 models is to predict speed-density function at any vehicle concentration based on the knowledge of the desired speed-density function. This model says

$$f(v) = [f^0(v)]/[1 + \beta(v - \bar{v})]$$

It can be shown that there is only one value of β that normalizes f (6) for given knowledge of $f^0(v)$ and the average speed $\bar{v}$ at that concentration. Therefore, given the $\bar{v}$ - c relation and the knowledge of $f^0(v)$, one can easily compute the speed-density function for any concentration.

It can also be shown that there is only one value of average speed $\bar{v}$ that normalizes f (6) for given knowledge of $f^0(v)$ and the value of β at that concentration. Therefore, given the β - c relation and the knowledge of $f^0(v)$, one can easily compute the $\bar{v}$ - c relation for the freeway and its corresponding speed-density functions for every concentration. It should be mentioned here that there is no need to know P and T $[\beta = (1 - P)T]$ in the computations given above. However, the fact that this model gives poor validation results indicates that it has little application value.

2. In the generalized Boltzmann type of model,

$$f(v) = [f^0(v) + \lambda\beta\delta(v - \bar{v})]/[1 + \lambda\beta + \beta(v - \bar{v})]$$

an adjustment process is added to the interaction and relaxation processes. In this case, 2 parameters, λ and β, are to be determined. However, β can no longer be determined by merely normalizing f(v). A direct way to obtain β is to estimate P and T as functions of concentration, for which no relations are available at present, and then to obtain λ by normalization. An indirect way is to use (4)

$$\beta = (\bar{v}^0 - \bar{v})/\sigma_v^2$$

and determine λ by normalization. In this case, we need the relation of σ_v^2 with traffic concentration.

Therefore, given the β - c (or σ_v^2 - c) and $\bar{v}$ - c relations and the knowledge of $f^0(v)$, one can compute the speed-density function for any concentration by selecting a value of λ that meets the normalization requirement of the speed-density function. Or, given the β - c and λ - c relations and the knowledge of $f^0(v)$, one can compute the $\bar{v}$ - c relation for the freeway and its corresponding speed-density function for any concentration. However, at present there is no easy way to determine the β - c or λ - c relation for a freeway. In addition to these difficulties, the validation results show that the generalized Boltzmann type of model produces poor results for all possible combinations of β and λ. Therefore, we again argue that this model is of little importance in terms of application.

3. The modified Boltzmann type of model was obtained by replacing the desired speed-density in the relaxation process of the basic Boltzmann type of model by a relaxed speed-density, produced by the shifting process. This results in

$$f(v) = [\gamma f^0(\gamma v)]/[1 + \beta(v - \bar{v})]$$

The shift parameter γ is concentration-dependent. Although β is determined if P and T are determined, it is not independent of γ because, when the value of γ is assigned, there is only one β that normalizes f(v).

Furthermore, our validation results suggested that the shifting process is far more important than the other processes. Therefore, we should determine γ first and obtain β by normalizing f(v). The validations show good results for the best possible value of γ, indicating a potential application of the modified Boltzmann type of model. To make use of this model, however, requires that a direct relation between γ and concentration be determined.

Therefore, given the γ - c and $\bar{v}$ - c relations and the knowledge of $f^0(v)$, one can compute the speed-density function for any concentration by selecting a value of β that meets the normalization requirement of the speed-density function. Or, given the γ - c and β - c relations and the knowledge of $f^0(v)$, one can compute the $\bar{v}$ - c relation for the freeway and its corresponding speed-density function for any concentration. However, at present there is no easy way to determine the γ - c and β - c relations for a freeway, and the resulting calculations will require the need of a computer program to compute f(v) by meeting the normalization requirements of the density function.

4. The shift model is obtained by using the shifting process alone and neglecting the interaction, relaxation, and adjustment processes. The resulting model

$$f(v) = \gamma_{max} f^0(\gamma_{max} v) = (\bar{v}^0/\bar{v})\, f^0[(\bar{v}^0/\bar{v})\, v]$$

was shown in the validation to be only slightly less optimal than the best possible modified Boltzmann type of model and still much better than the basic and generalized Boltzmann type of models. This suggests that the concept of the shifting process by itself plays a much more dominant role to describe traffic than when it is compared with the interaction, relaxation, and adjustment processes in the Boltzmann type of models. The shift model has great application value, for we need only to know—in addition to the desired speed-density—the mean speed at any vehicle concentration ($\overline{v}$ - c relation).

Therefore, given the $\overline{v}$ - c relation (or, equivalently, γ_{max} - c relation, because $\gamma_{max} = \overline{v}^0/\overline{v}$) and the knowledge of $f^0(v)$, one can compute the speed-density function f(v) for any concentration. The relation of $\overline{v}$ - c and $f^0(v)$ for any freeway can be obtained relatively with less difficulty as compared to the γ - c or β - c relation.

Many theories have been developed to obtain the $\overline{v}$ - c relation. For example, Pipes (7) suggests

$$\overline{v} = \overline{v}^0[1 - (c/c_j)]^n, \quad n > 0 \tag{19}$$

Using $\gamma_{max} = \overline{v}^0/\overline{v}$, γ_{max} can be plotted easily. Figure 9 shows the γ_{max} - c relation based on our experimental data. If $\gamma_{max}(c)$ is known , the shift model is able to predict the speed-density at any concentration through

$$f(v) = \gamma_{max} f^0(\gamma_{max} v) = [\overline{v}^0/\overline{v}(c)]\, f^0\{[\overline{v}^0/\overline{v}(c)]\, v\} \tag{20}$$

If the $\overline{v}$ - c relation is that specified in Eq. 20, then

$$\gamma_{max} = [1 - (c/c_j)]^{-n}, \quad n > 0 \tag{21}$$

Clearly, $\gamma_{max} \geq 1$, $\gamma_{max} = 1$ when c = 0, and $\gamma_{max} = \infty$ when $c = c_j$.

The space-speed-density is definitely related to space-headway distributions and, consequently, influences the lane-change and car-following behavior. We expect that the more accurate speed-density prediction will eventually help to develop better freeway control strategy.

REFERENCES

1. Munjal, P. K., and Pahl, J. An Analysis of the Boltzmann-Type Statistical Models for Multi-Lane Traffic Flow. Transportation Research, Vol. 3, 1969, pp. 151-163.
2. Prigogine, I., and Andrews, F. C. A Boltzmann-Like Approach for Traffic Flow. Op. Res., Vol. 8, 1960, pp. 789-797.
3. Prigogine, I., Resibois, P., Herman, R., and Anderson R. L. On a Generalized Boltzmann-Like Approach for Traffic Flow. Bull. Acad. r. Belg. Cl. Sci., Vol. 48, 1962, pp. 805-814.
4. Gafarian, A. V., Munjal, P. K., and Pahl, J. An Experimental Validation of Two Boltzmann-Type Statistical Models for Multi-Lane Traffic Flow. Transportation Research, Vol. 5, 1971, pp. 211-224.
5. Breiman, L. Space-Time Relationships in One-Way Traffic Flow. Transportation Research, Vol. 3, 1969, pp. 365-376.
6. Munjal, P. K., Hsu, Y. S., and Carpenter, R. Experimental Validation of Modified Boltzmann-Type and Shift Models for Multi-Lane Traffic Flow. System Development Corp., Rept. TM-4638/007/01, 1971.
7. Pipes, L. A. Car Following Models and the Fundamental Diagram of Road Traffic. Transportation Research, Vol. 1, 1967, pp. 21-29.

APPENDIX

BASIC AND GENERALIZED BOLTZMANN TYPE OF MODELS

The speed-density function f(x, v, t) was assumed by Prigogine et al. (2, 3) to change with time only as a result of 3 traffic processes: relaxation, interaction, and adjustment; i.e.,

$$df/dt = (\partial f/\partial t) + v(\partial f/\partial t) = (\partial f/\partial t)_{rel} + (\partial f/\partial t)_{int} + (\partial f/\partial t)_{adj} \quad (22)$$

and

$$(\partial f/\partial t)_{rel} = - [(f - f^0)/T] \quad (23)$$

$$(\partial f/\partial t)_{int} = (1 - P)(\overline{v} - v)f \quad (24)$$

$$(\partial f/\partial t)_{adj} = \lambda(1 - P)[\delta(v - \overline{v}) - f] \quad (25)$$

where

- P = probability of passing = $1 - (c/c_j)$;
- T = relaxation time = $[\tau(1 - P)]/P$;
- λ = unknown parameter for adjustment;
- c_j = jam concentration; and
- τ = proportionality constant.

The parameters T, P, and λ are all functions of concentration.

For time-independent, space-homogeneous traffic flow, the solution of Eq. 22 is

$$f[1 + \lambda\beta + \beta(v - \overline{v})] = f^0 + \lambda\beta\delta(v - \overline{v}) \quad (26)$$

where $\beta = (1 - P)T$; or

$$f(v) = [f^0(v) + \lambda\beta\delta(v - \overline{v})]/[1 + \lambda\beta + \beta(v - \overline{v})] \quad (27)$$

Equation 27 is called the generalized Boltzmann type of model for $\lambda \neq 0$ and the basic Boltzmann type of model for $\lambda = 0$.

We note that $c = 0$ implies that $P = 1$ and $T = 0$; thus, $\beta = 0$. In this case, Eq. 27 reduces to

$$f = f^0$$

When $c = c_j$, we have $P = 0$ and $T = \infty$; thus, $\beta = \infty$. In this case, Eq. 27 reduces to

$$f = \delta(v - \overline{v})$$

Both extreme cases represent physical realism. It is also interesting to see how λ and β play the role in the relation between f and f^0 for arbitrary c, where $0 < c < c_j$.

For the basic Boltzmann type of model we have $\lambda = 0$ and

$$f(v) = [f^0(v)]/[1 + \beta(v - \overline{v})] \quad (28)$$

It is easy to see that for $v > \overline{v}$, $1/[1 + \beta(v - \overline{v})] < 1$; for $v = \overline{v}$, $1/[1 + \beta(v - \overline{v})] = 1$; and for $v < \overline{v}$, $1/[1 + \beta(v - \overline{v})] > 1$. The role of β is thus to skew the speed-density

Figure 10. Average speed-concentration relation for basic and generalized Boltzmann type of models (2-lane).

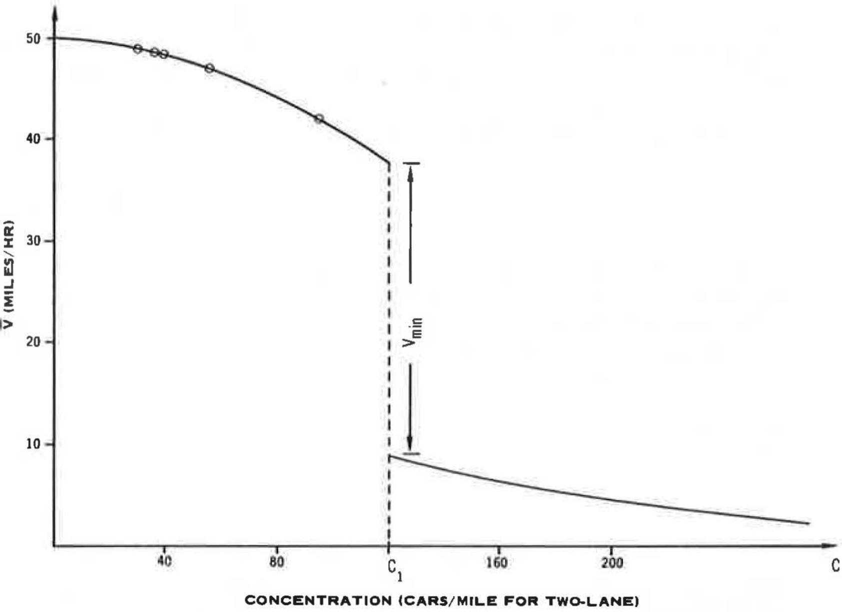
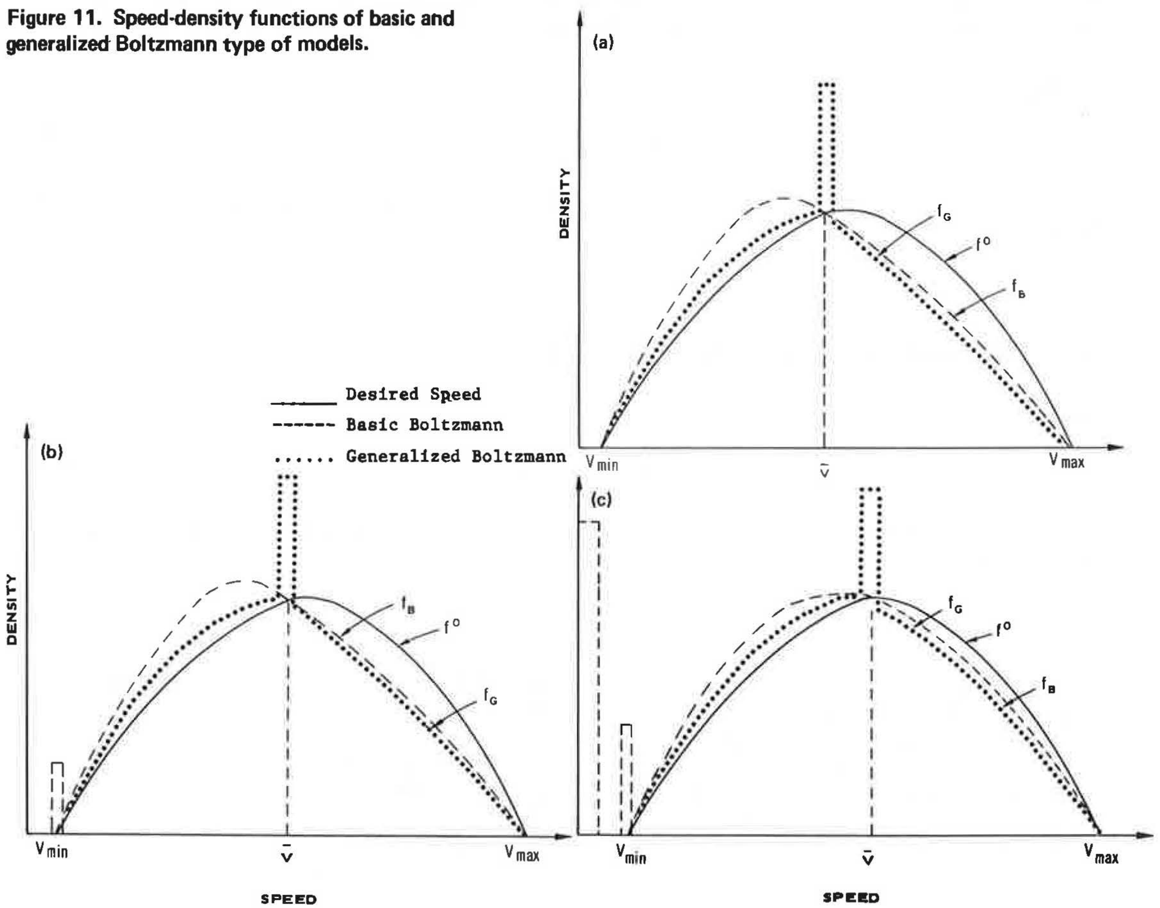


Figure 11. Speed-density functions of basic and generalized Boltzmann type of models.



toward lower values and subsequently to produce a $\overline{v}$ smaller than $\overline{v}^0$. In other words, the speed-density f decreases to a smaller value from the desired speed-density f^0 when speeds are higher than the space-mean speed of f, remains the same when equal, and increases when lower.

Furthermore, with the adjustment term $\lambda \neq 0$, we have

$$1/[1 + \beta(v - \overline{v}) + \beta\lambda] < 1/[1 + \beta(v - \overline{v})]$$

and the resulting generalized Boltzmann speed-density decreases everywhere between v_{min} and v_{max} except at $v = \overline{v}$, where there is an increase of density represented by the δ-function of magnitude $\lambda\beta/(1 + \beta\lambda)$.

The Boltzmann type of models are also influenced by vehicle concentrations. That is, we have these 3 cases discussed below.

Case A

$$1 + \lambda\beta - \beta(\overline{v} - v_{min}) > 0 \tag{29}$$

This is the most realistic case, and a previous validation study (4) concentrated on this case. Figure 10 shows the speed-concentration ($\overline{v}$ - c) curve corresponding to the Boltzmann type of models. The 5 circles are the same 5 flow levels shown in Figure 2. Case A corresponds to the range $0 \le c < c_1$, case B corresponds to $c = c_1$, and case C corresponds to $c_1 < c \le c_j$. The Boltzmann type of models produces a drop in $\overline{v}$ at $c = c_1$ by an amount of v_{min}. We shall explain this point in the discussion of cases B and C.

Case B

$$1 + \lambda\beta - \beta(\overline{v} - v_{min}) = 0 \tag{30}$$

This case corresponds to the situation when $c = c_1$. The average speed is determined from Eq. 30 as

$$\overline{v} = \lambda + v_{min} + (1/\beta)$$

It can be seem from Eq. 27 that f has a pole at $v = v_{min}$.

Case C

$$1 + \lambda\beta - \beta(\overline{v} - v_{min}) < 0 \tag{31}$$

This case corresponds to the situation when $c_1 < c \le c_j$. In this case there exists a $v_1 > v_{min}$ such that

$$1 + \lambda\beta - \beta(\overline{v} - v_1) = 0$$

This means that f is singular at v_1 and negative for $v_{min} \le v < v_1$, which is absurd from a physical point of view. To compensate for this, Prigogine et al. (3) had to assume that some vehicles were traveling at 0 speed. Comparing case C to case B shows that there is a reduction of average speed of v_{min} or a decreasing flow, $\Delta q = c_1 v_{min}$, at $c = c_1$. That is, the Boltzmann type of models produce a discontinuity in the $\overline{v}$ - c (speed-concentration) curve as shown in Figure 10. Speed-density functions of the basic and generalized Boltzmann type of models corresponding to the 3 concentration ranges, $0 \le c < c_1$, $c = c_1$, $c_1 < c \le c_j$ (cases A, B, and C respectively), are shown in Figure 11. Only case A is promising in terms of physical reality.