USE OF TOPAZ FOR GENERATING ALTERNATE LAND USE SCHEMES

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A technique for the optimum placement of activities in zones was developed initially for use in Melbourne, Australia. The intent of the technique was to generate land use allocation schemes that were optimal according to some preset objectives. In this paper, the technique is employed for Blacksburg, Virginia, both to test a proposed land use scheme and to use this scheme as a basis for finding better arrangement patterns. Capital cost data for water, sewerage, local streets, electricity, and individual building units for each zone in the study area were used in conjunction with travel cost and land value information to derive overall cost minus benefit values for the schemes indicated above. Results seem to suggest that the technique is capable of creating worthwhile alternatives to the proposed schemes, although additional effort needs to be devoted to improving the variety of output information that can be generated by the technique.

•THE DEVELOPMENT of land use plans for an urban area usually is a time-consuming and an expensive process. As a result, the planner often is limited to investigating only a few alternate land use development schemes, and these investigations generally are rather quick and rough. In many instances, the best that can be done is to draw a few sketch plans and determine their probable impacts subjectively. Moreover, the planner is almost always working with the anxiety that more time spent on broad-scale plan development means less time available for the arduous task of completing the final plan in detail.

The planner would be greatly aided if he had a fairly rapid technique that, with a given set of data, would generate and determine some of the consequences of various land use schemes. In those cases where it is possible for him to be more specific about his objectives, he would also be aided by a technique that would generate schemes that were fairly close to optimal in terms of these objectives. Quite obviously, though, the complexity of most urban areas would hinder the development of techniques that would provide anything but first-order approximations of consequences. But then, first-order approximations may be more than adequate for initial sketch planning.

TOPAZ (technique for the optimum placement of activities in zones) seems to be of benefit in the sketch-planning stage. It was first used in the Melbourne, Australia, metropolitan area. The basic idea behind TOPAZ, as envisioned by Brotchie, Sharpe, and Toakley (1, 2, 3), was to use readily available mathematical allocation schemes to organize land use development in an urban area. Initially, the minimization of public service and travel was the main siting objective, although it was recognized from the start that costs certainly were not the only items of concern. Public service costs included those for water, sewerage, local streets, hospitals, and schools, to name a few. A prediction was made of how much would be needed by 1985 for high- and low-density residential land and industrial land. TOPAZ then was employed to determine where to allocate the needed land use areas so as to minimize the public service and

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travel costs. All solutions were constrained so that areas available for development in each zone of the city were not "filled" above capacity. The minimum cost allocations obtained via TOPAZ proved to have some interesting ramifications for development policies in Melbourne.

USE OF TOPAZ IN BLACKSBURG

An endeavor similar to the one in Melbourne was launched in Blacksburg, a small but expanding town of 22,000 people (including students) in southwest Virginia. This endeavor was intended as a prototype but actually may prove to have some worthwhile practical benefits, for Blacksburg is at present involved in a court case related to attempts to annex part of the adjacent county. (Service costs, of course, are important items in annexation considerations, especially for small rural communities.)

The town was divided into 61 zones, which are shown in Figure 1 along with existing land development. (Some of the 61 zones were combined later in the analysis and thus are not shown in Figure 1.) Zonal delineation was done, as in many planning studies, primarily on the basis of slope of the land, depth of bedrock, soil type, availability of existing utilities, existing land use development, natural drainage areas, and man-made boundaries (e.g., US-460 bypass). Figure 2 shows the land slopes, and Figure 3 shows the proposed water system improvements for the area. These are presented to give some idea of the kind of information needed as input to TOPAZ.

The data for the zonal delineation study also were utilized in part in the determination of the per acre establishment capital costs and benefits. Costs were divided into 5 categories: building unit, water, sewerage, local streets, and electricity (supplied by local developers and the town). These costs vary, of course, according to land slope, the need to excavate in bedrock, soil type, nearness to existing services, and so on. For example, in zone 1, which has a slope range of 12 to 20 percent and bedrock very close to the surface, it was determined from conversations with town officials and local land developers that public service capital costs would be about 120 percent higher than those in the lowest cost zones. Costs per acre also varied with the type of activity or land use being considered. In the case of Blacksburg, 16 activities were employed (Table 1). Examples of the costs per acre used in Blacksburg are given in Table 2.

The determination of benefits naturally proved to be rather difficult. Our interest was in indicating the benefits an activity or land use type would receive from being located in places that had certain amenities, such as a good view of the mountains, nearness to other activities, and good landscaping. As a very rough measure of all these, we used land values. We do not feel that this measure is entirely adequate, but at least we have attempted to include some representation of items other than costs. Typical land values are also given in Table 2.

Travel costs are also taken into account in TOPAZ. A gravity model is used to make estimates of zone-to-zone movements based on existing and future amounts of each activity in each zone. From a mathematical standpoint, the inclusion of the gravity model makes the determination of the optimal allocation of activities a very difficult matter. However, the main advantage of TOPAZ is that it incorporates an iterative solution scheme that is very fast and gives solutions that, although not necessarily global optima, seem to be very close. The Appendix contains a mathematical formulation and small numerical example of TOPAZ.

The main elements in Blacksburg's transportation system have been surveyed and coded in a manner similar to that done in most large-scale transportation studies. Interzonal travel costs for each daily trip predicted by means of the gravity model were obtained by summing costs on each link on the minimum time path between zones. These costs then were multiplied by the expected repetitions of that daily trip for each year up to and including the horizon year. Overall travel costs probably could be expected to be relatively low because we assumed a repetition rate of 200 trips per year and a cost per mile of $0.065. (The number of trips is not too low because Blacksburg is a university town, and there are many times during the year when the 13,000-member student body is not in full attendance. Although low, the cost-per-mile figure can be adjusted and tested via sensitivity analysis using TOPAZ.) The horizon year was 1990.

Figure 1. Zones and existing development.

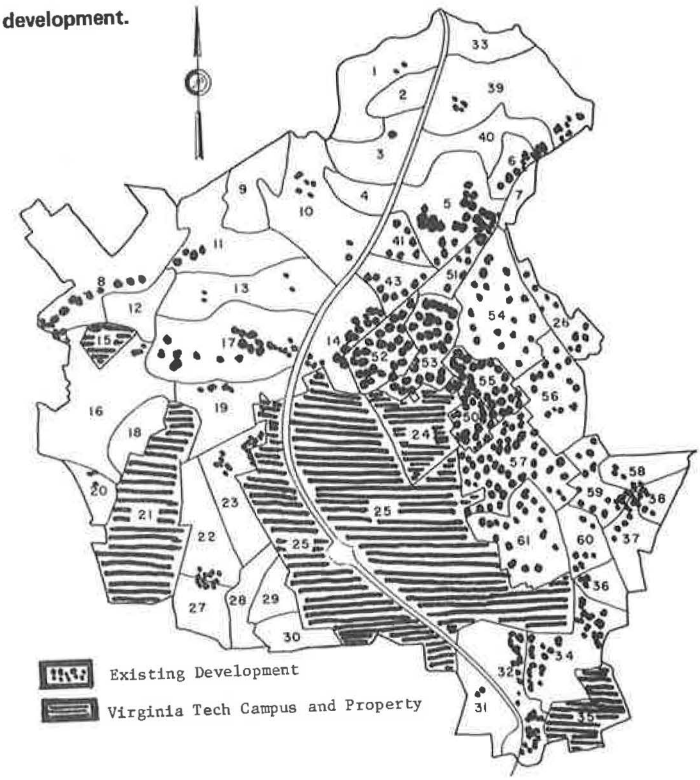


Figure 2. Slope of land.

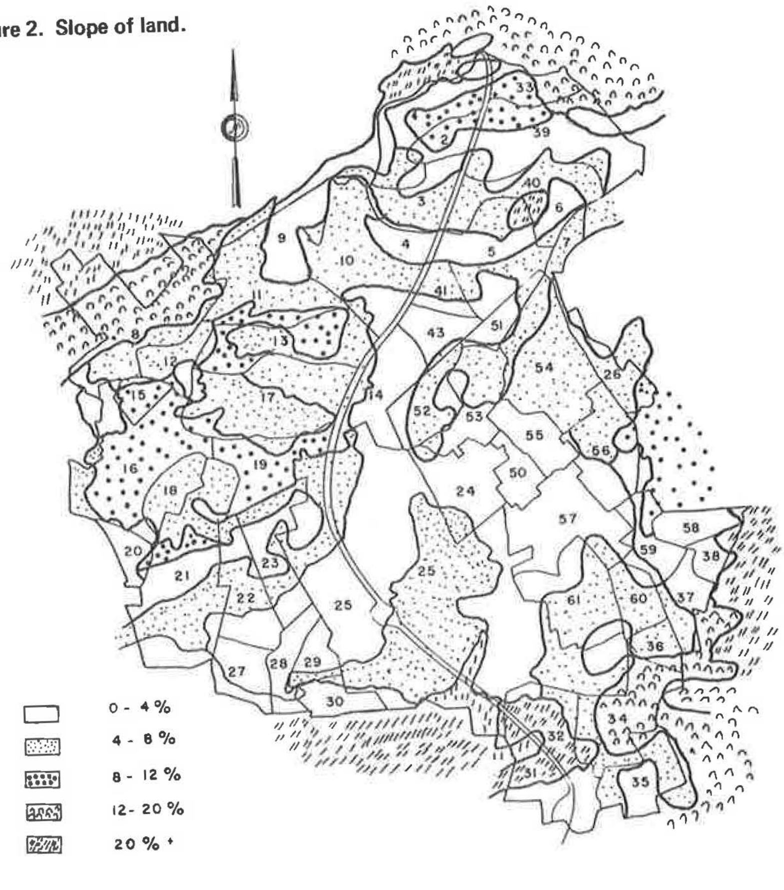


Figure 3. Priorities of water system improvements.

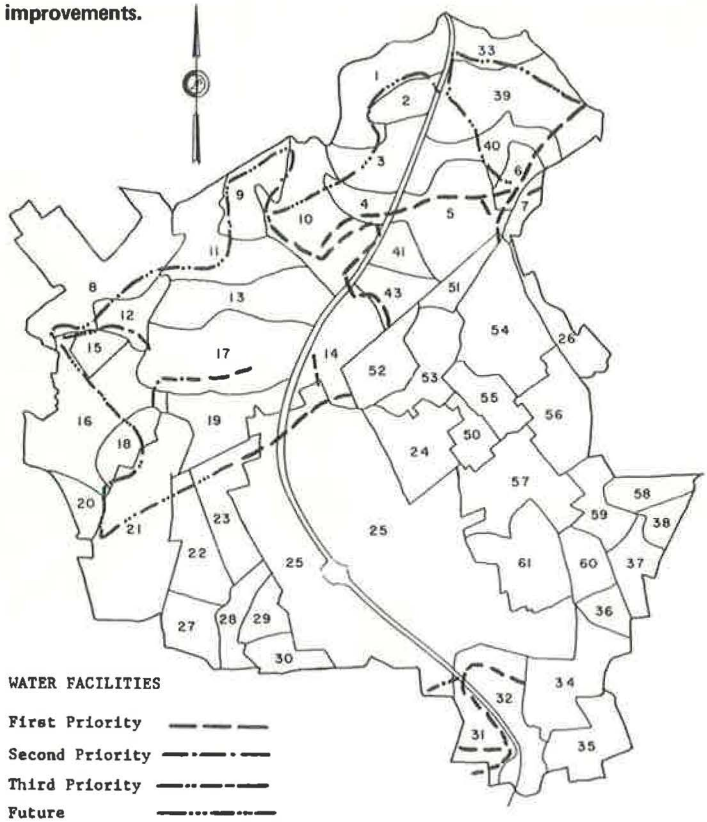


Table 1. Land use activity codes.

Activity	Code	Activity	Code
Single-family houses	1	Town parks	9
Apartments	2	Primary schools	10
Town houses	3	Secondary schools	11
Planned unit development	4	Public and semipublic land	12
Mobile homes	5	Industry	13
Convenience commercial	6	Streets[a]	14
Regional commercial	7	University	15
Neighborhood parks	8	Undeveloped land	16

[a]This category was not actually used in the analyses because all areas were gross areas, that is, including local streets and alleys.

Table 2. Establishment cost and benefit values.

Cost or Benefit (dollar/acre)	Zone	Single Family	Apartment	Town House
Building unit cost	1	64,000	184,000	144,000
	2	54,000	154,000	120,000
	3	55,000	155,000	121,000
	4	54,000	154,000	120,000
Sewerage system capital cost	1	2,960	3,460	3,460
	2	1,810	2,310	2,310
	3	1,810	2,310	2,310
	4	1,810	2,310	2,310
Amenity benefit[a]	1	-1,500	-2,250	-2,250
	2	-1,500	-2,250	-2,250
	3	-1,500	-2,250	-2,250
	4	-2,000	-3,000	-3,000

[a]Land values. The minus signs indicate negative costs, that is, benefits.

The remaining sets of information required for input to TOPAZ are the estimates of areas available for development in each zone and the estimates of the areas of each activity (land use) required by the horizon year. The first set of estimates is obtained fairly readily through a typical land use survey. However, there is a definitional question as to what constitutes land available for development. Should land with a slope greater than 20 percent be considered available? Is land already dedicated for an industrial park or zoned commercial really available for other uses? These and similar questions become quite perplexing. Our approach has been to assume that almost all vacant land is available. By acting in this manner, we leave ourselves in a flexible position, for we can come back later if we so desire, incorporate restrictions of various sorts (e.g., zoning and general policy), and determine the increased costs brought about by these restrictions. In this way, we are able to set up trade-off situations where we can ask, for example, whether the increased costs occasioned by, say, a certain zoning ordinance are more than offset by the anticipated benefits (excluding land values).

The second set of estimates, the amounts of land use areas needed by the horizon year, is perhaps the least reliable input to TOPAZ. These areas are obtained by taking the forecast population figure for the overall region and applying certain proportions to it. The population of Blacksburg plus the student body is expected to grow from 22,000 people at present to 40,000 people within the next 20 years. Of the increase of 18,000 people, 9,000 are expected to be students and 9,000 permanent residents. In this latter group, it is anticipated that 6,000 will wish to live in single-family houses. Based on 3.2 persons per family and 3 single-family units per gross acre (including streets and other services), about 626 acres of single-family homes will be needed. Similar reasoning is employed to obtain estimates of the other activity areas required. The amounts of these areas could vary somewhat, of course, especially since we are assuming currently accepted development standards, current zoning density restrictions, and a similar pattern of demands for land use as at present. But again, we can do some sensitivity analyses to see how land allocations may change. We could, for instance, determine what happens when the demand for town houses increases while that for single-family houses decreases.

RESULTS OF THE APPLICATION TO BLACKSBURG

A series of runs were made with TOPAZ and the Blacksburg data. It will not be possible to report on all the results here, but we will attempt to highlight the important ones.

TOPAZ requires that a feasible solution be assumed initially. This solution then is upgraded to an optimal one (or close thereto). We started with a solution that the town's planner particularly desired to test because it designated growth in many of the areas for which the town anticipated providing water and sewer extensions. The initial solution is shown in Figure 4. It includes, predominantly, incursions to the northwest side of town in zones 9, 10, 11, 13, and 17. TOPAZ automatically "costs out" all initial solutions; the following costs and benefits were obtained:

Benefit or Cost	Millions of Dollars
Establishment benefits	-3.9
Building unit costs	66.2
Water system costs	2.2
Sewer system costs	2.1
Local street costs	3.1
Electric system costs	0.8
Travel costs	19.5
Total	90.0

The size of the benefit and cost items should be of interest at this point. The 3 items of the largest magnitudes (establishment benefits and building unit and travel costs) are the ones for which the town probably would have the least concern because it does not have to pay for these directly. Travel costs are about 22 percent of the total, a relatively low figure because most travel is for short distances in a small town. (Travel costs could be more substantial in a large city, however; this finding was borne out in Melbourne to some extent.) The costs of direct concern to the town total about $8.2 million.

The optimal land use pattern generated by TOPAZ starting with the initial solution (Fig. 4) is shown in Figure 5. The benefit and cost figures for this pattern are as follows:

Benefit or Cost	Millions of Dollars
Establishment benefits	-5.8
Building unit costs	65.9
Water system costs	2.0
Sewer system costs	1.8
Local street costs	2.9
Electric system costs	0.7
Travel costs	16.6
Total	84.1

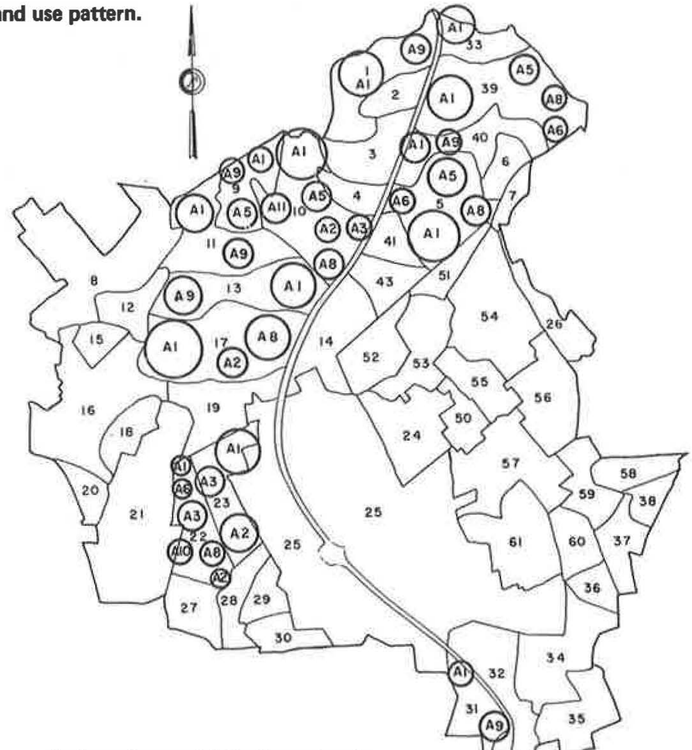
Total overall costs have been reduced $5.9 million for the initial solution, but the makeup of component changes is of interest. Establishment benefits have risen $1.9 million, indicating that land uses have been placed in areas with more amenities. Travel costs have decreased by $2.9 million, while costs of direct concern to the town have decreased only $0.6 million. Thus, it appears that the town's anticipated strategy of locating some major water and sewer mains on the northwest side will increase their direct costs only slightly but will put an added travel burden on the public and perhaps induce people to go where their "benefits" would not be quite so great. These results are borne out by a close survey of the zonal allocations shown in Figures 4 and 5. The proposed expansion shown in Figure 4 to the northwest—zones 9, 10, 11, 13, and 17—is not shown in Figure 5. Instead, much more use is made of the closer in, currently built-up zones to the north and east of town. This TOPAZ-generated alternative obviously presents a quite different land use development scheme from the one currently being considered. (Interestingly enough, this TOPAZ scheme does not allocate much land to areas being considered for annexation by the town.) A warning is in order, however. There may be other benefits not taken into account in TOPAZ that more than make up for the additional costs (and lack of benefits) to be incurred in the initial solution. Yet, the trade-offs are more explicit now: Are the additional benefits not considered in TOPAZ worth the extra $5.9 million in costs and foregone land value benefits, $0.6 million of which is in direct costs to the town? Perhaps only the political process can answer this question. But at least the consequences are clearer, and new and apparently worthwhile alternatives have been generated.

ADDITIONAL RESULTS

To provide added perspective to the results given above and to demonstrate some of the versatility of TOPAZ, we have made an extra set of analyses based on the following objectives: maximize overall costs-benefits, minimize direct town costs, maximize direct town costs, minimize travel costs, and maximize travel costs.

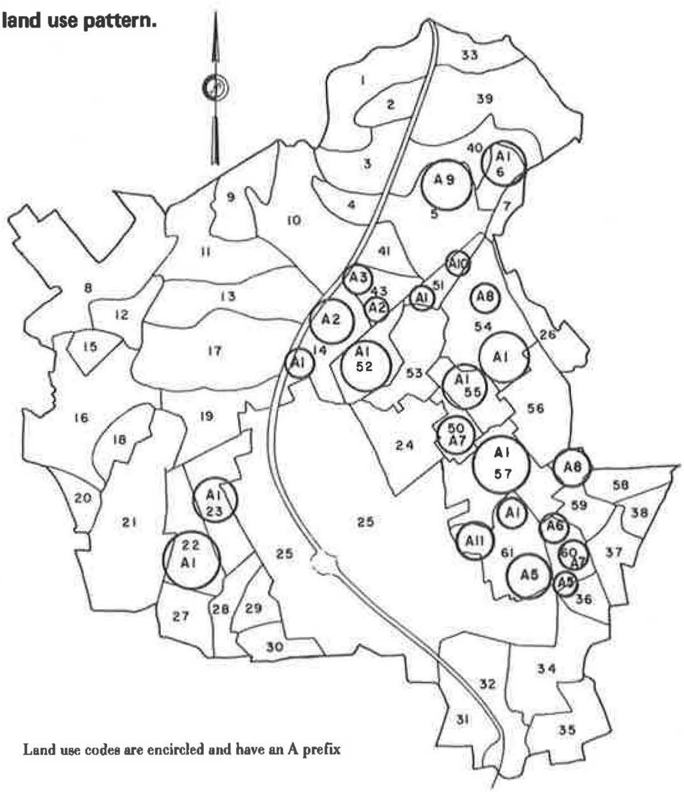
The purpose of the first analysis was to see what the worst land use pattern would be and thereby to provide both a datum by which to judge schemes with intermediate cost consequences and an indication of where growth definitely ought not to go. The resultant maximum value was $107 million, of which $86.6 million was for establishment costs minus benefits and $20.4 for travel. We now can see that the town's anticipated scheme would be about 25 percent of the way toward the worst case on an overall cost-benefit scale. The worst land use pattern itself (not shown) is somewhat as one might expect; activities are allocated to the most expensive peripheral zones.

Figure 4. Initial land use pattern.



Land use codes are encircled and have an A prefix

Figure 5. Optimal land use pattern.



Land use codes are encircled and have an A prefix

The scheme with the lowest direct costs to the town would represent an expenditure of $7.5 million on the 4 direct-cost items. The corresponding maximum scheme would entail an expenditure of $11.3 million. The town scheme, as could be expected, is fairly close to the low end of this range. The minimum and maximum travel cost schemes give cost figures of $15.4 million and $20.7 million respectively. These figures tend to affirm the earlier argument that the town scheme ($19.5 million in travel) encourages longer trip-making and, thus, is more expensive along that line, especially because the areas to be developed under that plan do not have particularly good access. Thus, the town scheme falls toward the top of the range in this respect. On the other hand, the TOPAZ-generated optimal scheme falls near the minimum. These results all are of some consequence but should be judged only in connection with the assumptions implied in TOPAZ.

CRITIQUE OF TOPAZ

There are many assumptions employed in TOPAZ that have not been tested to any great degree (if at all). There are also many areas where the technique can be improved simply with additional time and effort. We will briefly list and discuss items in each of these 2 classes.

We must first recognize the fact that the land use controls needed to implement various results from TOPAZ are, at least in most American cities, almost nonexistent. Nonetheless, urban plans in general have almost always been advisory in nature so that schemes generated by TOPAZ are not likely to be more disadvantageous in this respect. However, we anticipate that, as new towns become more prevalent and as more thought is given to national and statewide land use policies, new and stronger land use controls may be forthcoming that could aid in implementing results from TOPAZ.

Another major drawback with TOPAZ is that it is focused almost entirely on physical planning. Yet, we know that often there is a strong connection between the physical, the economic, the social, and the political. Only by considering land values have we even started to make a rough approximation of private sphere economic, social, and political gains and losses. Still, there are many problems involved with our present approach. First, we have made the mistake of double counting benefits and costs because part of the benefits inherent in land values are those attributable to accessibility. This factor is already considered somewhat through reduced travel costs, but the extent of double counting generated here is unknown. A second aspect of the benefits component is that locational benefits, at least in the economic sense, refer to what a person is willing to pay (demand) for land, not necessarily what he actually pays. We equated actual payments to benefits because we had no clear picture of the demand curve for land in Blacksburg.

Another problem involves capital improvements for water purification plants, sewage-treatment plants, and secondary and primary roads. It would seem desirable to have TOPAZ determine where these major facilities should go to minimize even further the difference between costs and benefits, but such a process is not as yet possible. It is thus necessary to assume some levels of improvements in these facilities (and their locations) and determine the costs that would result at these levels. We have assumed no major facility changes in this regard, mainly because we would have gone to considerable effort to estimate new costs per acre in many zones if facilities of these types were added.

A final major assumption in TOPAZ has some interesting political ramifications. What happens if, say, a water line is constructed through vacant land to an outlying property? Should the cost per acre for water in that outlying zone include the full cost of the line when, eventually, some people will settle in between and possibly reimburse those on the periphery? We have assumed that the reimbursement would occur so that costs per acre would not depend on expenditures for facilities outside the zone itself. (The town currently is evaluating its present policy that requires those on the periphery to pay the full cost and makes no stipulation for eventual reimbursement.)

There are also short-range deficiencies in TOPAZ that can be readily adjusted with more effort. We would make the following changes in this endeavor:

1. Include public service operating costs;
2. Consider economies of scale for public services, that is, lower the service costs per acre when a greater number of acres are developed;
3. Specify costs on an annual basis (involving potential problems in determination of interest rates and service lives);
4. Include, where reasonable, upper and lower limits on the size of any activity allocated to a zone (theoretically it is possible for TOPAZ to allocate, say, 1 acre of regional commercial land use to a zone, and, although this certainly is unrealistic, the chance of its happening is slight);
5. Consider certain parts of the urban area as being available for redevelopment;
6. Include trips with one or both ends external to the study area;
7. Increase sophistication of the gravity and trip generation models, perhaps with a 3-purpose breakdown;
8. Include modal choice and trip assignment; and
9. Include a process for staging improvements.

These changes, along with additional sensitivity analyses of assumed parameter values (e.g., future population levels or number of building units per acre) should help refine TOPAZ into a more workable tool for sketch-plan generation and analysis.

FINAL REMARKS

The application of TOPAZ was intended to be mostly a prototype endeavor. It soon became clear, however, that the results generated by TOPAZ could be of significance to Blacksburg's development policy, especially in regard to annexation and future extensions of water and sewer lines. It is to these kinds of general policy questions that TOPAZ is directed, and, as such, we feel that it is providing some worthwhile alternatives for political consideration. Nonetheless, TOPAZ needs much in the way of refinement.

ACKNOWLEDGMENTS

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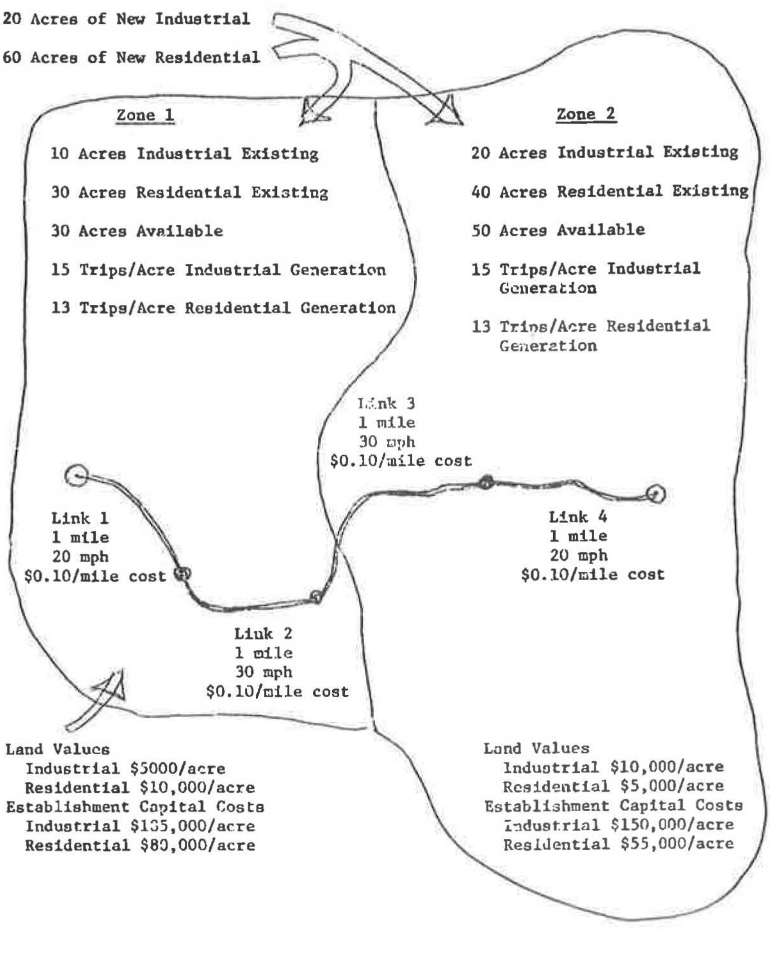
APPENDIX

MATHEMATICAL DESCRIPTION AND EXAMPLE OF TOPAZ

This example is a 2-zone case shown in Figure 6. The centroids of the 2 zones are connected by a highway having 4 distinct links. The lengths of these links are all 1 mile, but the speeds are higher on the middle 2 links (30 mph) than on those connected directly to the centroids (20 mph). Travel costs are the same on all links (10 cents/mile).

Zone 1 is smaller than zone 2, having only 70 acres of land with 30 available for future development. Zone 2 has a total of 110 acres with 50 available for development. It is desired to put 20 acres of industrial and 60 acres of residential land somewhere within the 2 zones. Land values are somewhat equally distributed among activities and zones, whereas service capital costs are $35,000/acre higher for industrial sites and

Figure 6. Representation of 2-zone example.



$25,000/acre higher for residential sites in zone 1 than in zone 2. It is desired to find the allocations of new activities to zones so as to minimize total travel plus establishment costs minus benefits. All new activities must be allocated, and the amount of area available for development in each zone cannot be exceeded.

The notation used is as follows:

X_{ij} = amount of activity i allocated to zone j, acres;
E_{ij} = existing amount of activity i in zone j, acres;
A_i = future amount of activity i to be allocated, acres;
B_j = area available for development in zone j, acres;
c_{sij} = unit establishment benefits or capital costs for service s for activity i in zone j, dollars/acre;
C_{ij} = total establishment costs-benefits for locating activity i in zone j, dollars/acre;
PR_i = daily vehicular trip production rate for activity i, vehicles/day/acre;
AT_i = daily vehicular trip attraction rate for activity i, vehicles/day/acre;
S_ℓ = speed over link ℓ, mph;
L_ℓ = length of link ℓ, miles;
P_{jk} = set of links on the minimum time path from zone j to k;
T_{jk} = minimum highway travel time from zone j to k, min;
M_{jk} = distance over minimum highway travel time path from zone j to k, miles;
d = number of repetitions of daily trips in a year;
y = length of planning horizon, years;
pm_ℓ = vehicular cost to travel over link ℓ, dollars/mile;
z = sum total of all travel costs and establishment costs-benefits, dollars;
z′ = value of the objective function of the linear "transportation problem," dollars; and
K_{jk} = cost over the planning period for a repetitive trip from zone j to k, dollars/daily trip.

The input information for the example is listed below.

1. Activity descriptions: Activity 1 is industrial, and activity 2 is residential.
2. Existing activities, acres: E_{11} = 10, E_{12} = 20, E_{21} = 30, and E_{22} = 40.
3. Areas available for development, acres: B_1 = 30 and B_2 = 50.
4. Areas of activities needed to be developed, acres: A_1 = 20 and A_2 = 60.
5. Trip production and attraction rates, vehicles/acre/day: PR_1 = 15, PR_2 = 13, AT_1 = 15, and AT_2 = 13.
6. Benefits (land values), dollars/acre: c_{111} = -5,000, c_{112} = -10,000, c_{121} = -10,000, and c_{122} = -5,000.
7. Public service capital costs, dollars/acre: c_{211} = 185,000, c_{212} = 150,000, c_{221} = 80,000, and c_{222} = 55,000.
8. Link speeds, mph: S_1 = 20, S_2 = 30, S_3 = 30, and S_4 = 20.
9. Link lengths, miles: 1.
10. Links on minimum time paths: P_{11} = {1}, P_{12} = {1, 2, 3, 4}, P_{21} = {1, 2, 3, 4}, and P_{22} = {4}.
11. Vehicular travel costs, dollars/mile: 0.10.
12. Other information: d = 200 trip repetitions/year and y = 20 years.

Some preliminary calculations are needed before the actual TOPAZ equations are given. First, we must sum the component establishment costs and benefits to get a total for each activity and zone. Thus,

$$c_{ij} = \Sigma\, c_{sij}, \text{ all } i, j \qquad (1)$$

The travel distances and times between zones are found by adding the link distances and times respectively over the minimum time path between the zones (specified beforehand):

$$M_{jk} = \sum_{\ell \in P_{jk}} L_\ell, \text{ all } j, k \qquad (2)$$

and

$$T_{jk} = 60 \sum_{\ell \in P_{jk}} L_\ell / S_\ell, \text{ all } j, k \quad (3)$$

Travel costs over each minimum time path are computed by taking into account the number of times each daily trip (found from the gravity model incorporated in TOPAZ) is repeated within each year in the time span up to the planning horizon date. Therefore,

$$K_{jh} = yd \sum_{\ell \in P_{jk}} pm_\ell L_\ell, \text{ all } j, k \quad (4)$$

Using the 4 equations given above, we can calculate the values for C_{ij}, M_{jk}, T_{jk}, and K_{jk}. The results are listed below.

1. Total establishment costs-benefits, dollars/acre: $C_{11} = 180{,}000$, $C_{12} = 140{,}000$, $C_{21} = 70{,}000$, and $C_{22} = 50{,}000$.
2. Distances between zones, miles: $M_{11} = 1$, $M_{12} = 4$, $M_{21} = 4$, and $M_{22} = 1$.
3. Travel times between zones, min: $T_{11} = 3$, $T_{12} = 10$, $T_{21} = 10$, and $T_{22} = 3$.
4. Travel costs between zones, dollars/daily trip: $K_{11} = 400$, $K_{12} = 1{,}600$, $K_{21} = 1{,}600$, and $K_{22} = 400$.

TOPAZ has as its objective the minimization of the combination of overall travel costs and the establishment costs minus benefits. Travel between zones (and travel costs) is determined with the aid of a gravity model. Establishment costs and benefits are input values. The formulation for TOPAZ can be presented as follows:

$$\min z = \sum_i \sum_j C_{ij} X_{ij} + \sum_j \sum_k K_{jk} \sum_i PR_i (X_{ij} + E_{ij}) \frac{\sum AT_i (X_{ik} + E_{ik}) \frac{1}{T_{jk}^2}}{\sum_n \sum_i AT_i (X_{in} + E_{in}) \frac{1}{T_{jn}^2}} \quad (5)$$

$$\sum_j X_{ij} = A_i, \text{ all } i$$

$$\sum_i X_{ij} = B_j, \text{ all } j$$

$$\sum_i A_i = \sum_j B_j$$

$$X_{ij} \geq 0, \text{ all } i, j$$

The first term in the objective function is the total establishment costs minus benefits. The second term is the gravity model equation; the daily trips between zones j and k multiplied by the travel cost K_{jk}. The term $\Sigma PR_i (X_{ij} + E_{ij})$ takes the existing activity of type i in zone j, E_{ij}, and adds it to the allocated amount, X_{ij}.

This total for activity i then is multiplied by the daily vehicular trip production rate for that activity PR_i, which gives the number of trips produced by that activity in the zone. Summing the trips produced by all activities in the zone then gives the total trips produced by the zone (or, stated in terms of the gravity model, the trip productions of zone j). Similar reasoning applies in the formation of the trip attractions terms, $\sum_i AT_i (X_{ij} + E_{ij})$. After productions and attractions have been determined, the gravity model is used to predict the trips between each pair of zones. This is done by dividing the trip production for zone j according to the trip attractions and squared travel times of a zone k relative to all other zones (all n).

The first set of constraints in Eq. 5 ensures that the future amounts of each activity are allocated. The second set ensures that the exact acreage of land available in each zone is used up. It should be noted here that vacant land, total developable land still remaining after future acreages of all activities are taken into account, is also considered to be an "activity" to be allocated. (This will not be shown in the upcoming example because of the great increase in computation required.) By viewing the left-over land in this manner, we thus can fulfill the third type of constraint in Eq. 5 (which is needed because of the particular computer code employed).

Equation 5 cannot be easily solved for the X_{ij}'s because they occur both in the numerator and denominator of the objective function (and in a nonlinear fashion in the numerator). As a consequence, TOPAZ involves an iterative solution procedure in which enough feasible values of the X_{ij}'s are assumed initially to make the objective function in Eq. 5 linear throughout. This linear version is the standard "transportation problem," which can be solved rapidly with available algorithms and computer codes. [We have utilized a code based on the algorithm by Ford and Fulkerson (9).] The X_{ij}'s that are the solution to the transportation problem are substituted for the initially assumed values in Eq. 5, and this process creates another transportation problem. This procedure continues until the lowest value for z is noted.

As an example of this iterative process involved in TOPAZ, let us substitute values from the input information given earlier into Eq. 5. As indicated in the previous paragraph, it is also necessary to assume or guess a feasible solution to Eq. 5. This is relatively easy, even for large problems, and often the solution the planner thinks is best can be used here. For instance, we can assume that $X_{11} = 20$, $X_{12} = 0$, $X_{21} = 10$, and $X_{22} = 50$ acres. These values meet the constraints since $X_{11} + X_{12} = A_1 = 20 + 0 = 20$, and so on. At this point, if desired, we can actually "price out" this assumed solution. Thus,

$$z = 180{,}000(20) + 140{,}000(0) + 70{,}000(10) + 50{,}000(50)$$

$$+ \frac{400[15(20 + 10) + 13(10 + 30)][15(20 + 10) + 13(10 + 30)]1/(3)^2}{[15(20 + 10) + 13(10 + 30)]1/(3)^2 + [15(0 + 20) + 13(50 + 40)]1/(10)^2}$$

$$+ \text{ costs for travel from zones 1 to 2, 2 to 1, and 2 to 2}$$

Out of these calculations we find that, for the proposed solution,

Establishment costs-benefits	\$6,800,000
Travel costs	1,146,540
Total	\$7,946,540

This total cost figure will provide a good datum to judge the gains registered through TOPAZ.

Now to continue with TOPAZ itself, we substitute the X_{ij}'s for the proposed solution given above into the terms for the trip attractions (but not for the productions) in Eq. 5. Thus, the objective function for that equation becomes

$$z' = 180{,}000\, X_{11} + 140{,}000\, X_{12} + 70{,}000\, X_{21} + 50{,}000\, X_{22}$$

$$+ \frac{400[15(X_{11} + 10) + 13(X_{21} + 20)][15(20 + 10) + 13(10 + 20)]\,1/(3)^2}{[15(20 + 10) + 13(10 + 20)]1/(3)^2 + [15(0 + 20) + 13(50 + 40)]1/(10)^2} \qquad (6)$$

$$+ \text{ the other 3 travel cost terms}$$

This objective function now is linear; for example, the expanded gravity model term in Eq. 6 can be reduced to $5{,}185\, X_{11} + 4{,}493\, X_{21}$. If all the calculations needed for the example were carried out, we thus would find that Eq. 5 becomes

$$\min z' = 188{,}451\, X_{11} + 146{,}869\, X_{12} + 77{,}324\, X_{21} + 55{,}952\, X_{22} \qquad (7)$$

$$X_{11} + X_{21} = 30$$
$$X_{12} + X_{22} = 50$$
$$X_{11} + X_{12} = 20$$
$$X_{21} + X_{22} = 60$$
$$X_{ij} \geq 0, \text{ all } i, j$$

Equation 7 is in the form of the standard transportation problem and can be solved rather easily. The solution in this case would have $X_{11} = 0$, $X_{12} = 20$, $X_{21} = 30$, and $X_{22} = 30$ acres.

To see whether this solution brings any improvement in the objective function of Eq. 5, we could substitute the X_{ij}'s in all places in the objective function. Thus,

$$z = 180{,}000(0) + 140{,}000(20) + 70{,}000(30) + 50{,}000(30) + \frac{400[15(0+10)+13(30+20)][15(0+10)+13(30+20)]1/(3)^2}{[15(0+10)+13(30+20)]1/(3)^2 + [15(20+20)+13(30+40)]1/(10)^2}$$
$$+ \text{ terms for the other 3 travel costs}$$

We then would find that

Establishment costs-benefits	\$1,144,950
Travel costs	6,400,000
Total	\$7,544,950

Here we can see a decrease in costs of about \$400,000 from the solution assumed at the beginning.

Future iterations with TOPAZ may prove to be even more useful in reducing z. To test this, and to show how the next iteration is set up, we will present one more round. To start, we substitute the solution variables from the previous iteration into the trip attraction terms of the objective function of Eq. 5. This becomes

$$z' = 180{,}000\, X_{11} + 140{,}000\, X_{12} + 70{,}000\, X_{21} + 50{,}000\, X_{22} + \frac{400[15(X_{11}+10)+13(X_{21}+20)][15(0+10)+13(30+20)]1/(3)^2}{[15(0+10)+13(30+20)]1/(3)^2 + [15(20+20)+13(30+40)]1/(10)^2} \quad (8)$$
$$+ \text{ the other 3 travel cost terms}$$

This equation in toto reduces to the linear relation

$$z' = 188{,}612\, X_{11} + 146{,}809\, X_{12} + 77{,}463\, X_{21} + 55{,}900\, X_{22} \quad (9)$$

Equation 9 is not significantly different from the one in the previous iteration so that, as it turns out, the solution variables are identical. TOPAZ has reached its stopping point (although in large-scale applications it usually proceeds 4 or 5 iterations before stability at a lower limit is noted). The ultimate solution variables thus have been obtained; $X_{11} = 0$, $X_{12} = 20$, $X_{21} = 30$, and $X_{22} = 30$ acres. The z value of \$1,544,950 may not be a global optimum but, from all indications, it is fairly close.

Insofar as computation times are concerned, we have found TOPAZ to be an extremely fast technique. For the Blacksburg case, there were 61 zones and 16 activities (land uses)—or 976 variables. Computing times for this case on the IBM 360/65 computer in no instance exceeded 3 min and were mostly about 1 min. These times are quoted for 4 iterations with the transportation problem subroutine, the usual number required with TOPAZ in the Blacksburg case.