ANALYSIS OF THE RESIDUAL STRESSES IN CONCRETE DUE TO TEMPERATURE VARIATION, SHRINKAGE, AND CREEP

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Because the physical properties (thermal expansion and shrinkage) and mechanical properties (strengths and moduli of elasticity) of the components of concrete are different, concrete is considered a multiphase material. For the sake of simplicity, it is assumed in this paper that there are two phases: the cement stone (i.e., the hardened cement paste) as the matrix and the aggregate particles as the discrete phase. Because the phases are prevented from free deformation by their strong bond, hampered strains impose balanced stresses on cement stone and aggregate; thus, the concrete gets in a residual stress condition. Three elementary models (disk, circular cylinder, and sphere) simulating a cement disk of outer shell and an aggregate disk of inner core will be applied to illustrate the evaluation of residual stresses in concrete due to temperature variation, shrinkage, and creep. Diagrams are also presented for easier applicability of the method.

•CONCRETE is a building material consisting of two phases, the viscoelastic cement stone (i.e., hardened cement paste) and the elastic aggregate, where differential phase properties affect the development of strains and stresses in a definite manner.

Such nonhomogeneous, non-isotropic materials cannot freely develop motions upon physical and mechanical influences because the identical deformations on phase interfaces necessarily impose inner constraints and the concrete gets in an equilibrium condition of residual stresses.

In what follows, some ideas will be presented that are related to the development of residual stresses in each phase, due to phase differences in concrete. An exact solution is very difficult; therefore, it has to be approximated by introducing certain elementary assumptions. Phases are assumed to fit together without slip and gaps to meet the principle of compatibility. This leads to the assumption that, in any point of the interface, the strains of cement stone and aggregate in a given direction are equal.

The main difficulty affecting concrete residual stress calculations is that knowledge of the mechanical properties of the concrete and its phases is required, as well as the particular physical characteristics of each phase and the resultant concrete characteristics. Although the strains can be measured directly as deformations, the stresses should at most be deduced from strain values. The effective moduli of elasticity of the cement stone and aggregate (E_s and E_a) required for the determination of the stresses in cement stone and aggregate are by no means the ones obtained by some of the usual methods from uniaxial loading but are complex characteristics obtainable only indirectly from certain measurements in a concrete of given composition and quality. Thus, the effective E_s and E_a cannot be determined by independent tests on the cement stone or aggregate, or, if they are, such results can only be applied with approximate corrections. Such a simplified method, based on the standard modulus of concrete E_c and on the absolute volumes of cement stone and aggregate v_s and v_a, is presented below.

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Concrete stress σ_c is assumed to be composed of stresses σ_s and σ_a in the cement stone and the aggregate respectively, leading to the relationship

$$\sigma_c = c\sigma_s + (1 - c)\sigma_a \tag{1}$$

Strains ϵ being identical, Eq. 1 leads to

$$E_s = E_c/[c + (1 - c)n] \tag{2}$$

where $n = E_a/E_s$ and $c = f(v_s)$.

E_c, v_s, and v_a are known for a given case, whereas n can be determined by the suitable deformation condition as follows. In the case of low stresses, E_c may be replaced by the initial tangential modulus

$$\begin{aligned} E_{co} &= 550{,}000\rho \\ \rho &= f_c/[f_c + 200] \end{aligned} \tag{3}$$

where f_c, in kips/cm^2, is the concrete cube strength. In general, the modulus of deformation E_c for stress at any time t is

$$E_c = \nu E_{co} \text{ kip/cm}^2 \tag{4}$$

where

$$\begin{aligned} E_{so} &= \frac{E_{co}}{c + (1 - c)n} \\ E_s &= \nu E_{so} \\ \nu &= \frac{\nu_o}{1 + \varphi} \\ \nu_o &= \frac{1}{2}\left[1 + \left(1 - \frac{\sigma}{\sigma_\mu}\right)^{1/2}\right] \end{aligned} \tag{5}$$

The rate of creep φ at any time t is given as

$$\varphi = k_o k_r \delta_{\varphi_n} \tag{6}$$

k_o, the factor characterizing the time of beginning of application of the sustained load, can be expressed as

$$k_o = 3.9 \times e^{0.77t^{1/6}} \tag{7}$$

k_r is a factor expressing the effect of the moisture content n_r (in percentage) of the surroundings:

$$k_r = \frac{115 - n_r}{100 - 0.7\, n_r} \tag{8}$$

δ is a process function of the creep, which is

$$\delta = 1 - 3^{-0.1t^{1/2}} \tag{9}$$

φ_n is the final value of creep, and σ is either the prism or cylinder strength σ_p or the tensile strength σ_t of the concrete in kip/cm^2.

To provide uniformity of treatment, the average concrete strain ϵ_c is assumed to be of the form

$$\epsilon_c = c \times \epsilon_s + (1 - c)\epsilon_a \tag{10}$$

which is obtainable from the free deformations of cement stone and aggregate. Here c is a function of the absolute cement volume in the concrete, expressed approximately by

$$c = v_s^{3/2} \tag{11}$$

This last equation is supported by experimental values on thermal deformations (α_c, α_s, and α_a), on shrinkage (ϵ_{cs}, ϵ_{ss}, and ϵ_{as}), and on creep of concretes.

RESIDUAL STRESSES DUE TO TEMPERATURE VARIATION

It is assumed that the thermal expansion coefficients of concrete and its components (α_s, α_a, and α_c) and the Poisson's ratios (μ_s and μ_a) are known. Similarly, the concrete composition (absolute volumes v_s and v_a) and the concrete modulus of elasticity (E_c) should be given. α_c is simply calculated by means of Eq. 10 (1). Approximate calculations of the residual stresses will be attempted by the following three model types:

1. Plane disk or linear model consisting of intercrossing cement stone and aggregate disk elements of absolute volumes v_s and v_a respectively (Fig. 1);
2. Circular cylinder model (Fig. 2); and
3. Spherical model (Fig. 3) with the cement stone as outer shell and the aggregate as core.

Starting from certain accepted equations of statics, kinematics, and strength of materials, some important relationships between the $n = E_a/E_s$ ratio and the stresses σ_s and σ_a in cement stone and aggregate due to 1 deg C of temperature variation have been compiled and are shown in Figures 1, 2, and 3. Because temperature variation may be considered as an instantaneous effect, zero rate of creep φ has been assumed.

The important relationships on the basis of the elasticity equations are given below:

1. For disks (Fig. 1),

$$p = p_s = \sigma_{sd}\, v_s = -p_a = -\sigma_{ad}\, v_a \tag{12}$$

$$p = \frac{(\alpha_c - \alpha_s)}{1 - \mu_s} E_{sd}\, v_s = \frac{(\alpha_a - \alpha_c)}{1 - \mu_a} E_{ad}\, v_a \tag{13}$$

$$\sigma_{sd} = \frac{p}{v_s}\Delta t = -\frac{(\alpha_s - \alpha_a)}{1 - \mu_s}(1 - c)E_{sd}\,\Delta t$$

$$\sigma_{ad} = -\frac{p}{v_a}\Delta t = \frac{(\alpha_s - \alpha_a)}{1 - \mu_s} c\, E_{ad}\,\Delta t \tag{14}$$

$$E_{sd} = \frac{E_c}{c + (1 - c)n_d} \qquad E_{ad} = n_d E_{sd} \tag{15}$$

$$n_d = \frac{1 - c}{c}\,\frac{v_s}{v_a} \times \frac{1 - \mu_a}{1 - \mu_s}\, n_o \beta_d \tag{16}$$

$$n_o = \frac{1 - c}{c}\,\frac{v_s}{v_a} \qquad \beta_d = \frac{1 - \mu_a}{1 - \mu_s} \tag{17}$$

2. For cylinders (Fig. 2),

$$p = \frac{(\alpha_c - \alpha_s)}{1 + \mu_s + (1 - \mu_s)/v_a} \times E_{sc}\, v_s = \frac{(\alpha_a - \alpha_c)}{1 - \mu_a} E_{ac} \tag{18}$$

$$\left.\begin{aligned}
\sigma_{ts,i} &= p\,\frac{1+v_a}{v_s}\,\Delta t & \sigma_{rs,i} &= -\,p\Delta t \\
\sigma_{ts,o} &= -\,p\,\frac{2v_a}{v_s}\,\Delta t & \sigma_{rs,o} &= 0 \\
\sigma_{ta,i} &= -\,p\Delta t & \sigma_{ra,i} &= -\,p\Delta t \\
\sigma_{ta,o} &= 0 & \sigma_{ra,o} &= 0
\end{aligned}\right\} \tag{19}$$

$$n_c = n_o\,\beta_c \tag{20}$$

$$\beta_c = \frac{(1-\mu_a)v_a}{1+\mu_s+(1-\mu_s)/v_a} \tag{21}$$

3. For spheres (Fig. 3),

$$p = \frac{(\alpha_c - \alpha_s)}{1+\mu_s+2(1-2\mu_s)/v_a}\,2E_{ss}\,v_s = \frac{(\alpha_a - \alpha_c)}{1-\mu_a}\,E_{as} \tag{22}$$

$$\left.\begin{aligned}
\sigma_{ts,i} &= p\,\frac{1+2v_a}{2v_s}\,\Delta t & \sigma_{rs,i} &= -\,p\Delta t \\
\sigma_{ts,o} &= p\,\frac{3v_a}{2v_s}\,\Delta t & \sigma_{rs,o} &= 0 \\
\sigma_{ta,i} &= -\,p\Delta t & \sigma_{rs,i} &= -\,p\Delta t \\
\sigma_{ta,o} &= 0 & \sigma_{rs,o} &= 0
\end{aligned}\right\} \tag{23}$$

$$n_s = n_o\,\beta_s \tag{24}$$

$$\beta_s = \frac{(1-2\mu_a)/2v_a}{1+\mu_s+2(1-2\mu_s)/v_a} \tag{25}$$

Figure 4 shows the n and E_s values versus v_s, whereas Figure 5 shows the variation of mean σ_s values based on the three models for $\alpha_s = 20.10^{-6}$ deg C, $\alpha_a = 10.10^{-6}$ deg C, $\nu = \nu_o = 1$, $\mu_s = \mu_a = 0$, $\mu_s = 0.2$, and $\mu_a = 0.1$. Because interfaces have been assumed to contact each other slipping-free, approximate values based on the linear model can also be obtained by assuming $\mu_s = \mu_a = 0$.

RESIDUAL STRESSES DUE TO SHRINKAGE

Residual stresses due to shrinkage can be determined by applying the relationships obtained for residual thermal stresses. Evidently, because shrinkage stresses are of a permanent character, the calculation of stresses has to take into account the creep rate φ for the given concrete grade and time. In general, aggregate shrinkage is practically negligible as compared to that of the cement stone; thereby, the second term in Eq. 10 can be omitted. Thus the concrete shrinkage at time t is expressed by

$$\epsilon_{cs} = c \times \epsilon_{ss}$$

At an arbitrary time, shrinkage of concrete or that of the cement stone is given by the process function corresponding to the given conditions

$$\epsilon_{cs} = \delta(t)\epsilon_{cs,\infty}$$

or

$$\epsilon_{ss} = \delta(t)\epsilon_{ss,\infty}$$

The process function can be written as

$$\delta(t) = 1 - e^{-a}$$

where a depends on the given circumstances (cement grade, water-cement ratio, degree of hydration, ambient humidity, and mean drying thickness) and on the time. For relative humidities of about 90, 70, and 40 percent, as well as 10 cm of mean drying thickness, a may be assumed as $0.07t^{1/3}$, $0.10t^{1/3}$, and $0.13t^{1/3}$ respectively (2).

Final shrinkage values of concrete for high early strength portland cement, 10 cm of mean drying value, relative humidities of 90, 70, and 40 percent, and water-cement ratios of 0.4, 0.6, and 0.8 are shown in Figure 6. Figure 7 shows characteristic values (E_c, f_c, φ, ν, ν_o, E_{so}) for 28-day portland cement of $f_o = 245$ kip/cm^2 (3,500 psi) based on the plane disk (linear) model for $\mu_s = \mu_a = 0$. Hence, among characteristics needed to calculate stresses, the ratio n and the modulus of elasticity E_s have been calculated by means of Eqs. 16 and 2 respectively. Tensile and compressive stresses in cement stone and aggregate were obtained with the following formulas:

For tensile stresses,

$$\sigma_{s,s} = (\epsilon_{s,s} - \epsilon_{c,s})E_s = \epsilon_{s,s}(1 - c)\nu E_{so} \tag{26}$$

and for compressive stresses

$$\sigma_{as} = -\epsilon_{cs} nE_s = -\epsilon_{ss}(1 - c)(v_s/v_a)E_s = -\sigma_{ss}v_s/v_a \tag{27}$$

Because there is no uniform temperature or shrinkage along the concrete cross section, in addition to the stresses obtained by assuming uniform distribution, residual stresses develop in elementary fibers due to temperature gradients and variable shrinkage (moisture). It is evident from the informative stress values presented that, under the combined effect of cooling and shrinkage, the tensile strength of cement stone may be exceeded and inner microcracks as well as surface cracks may develop.

STRESS REARRANGEMENT IN CONCRETE DUE TO CREEP

For simplifying the calculation of the stress rearrangement, the effect of diversity of the different kinds of rocks will be neglected. It will also be assumed that their creep may be ignored. However, it should be stressed that the results of the latest experiments clearly show the effect of the different kinds of rocks on the development of shrinkage and creep (17, 18, 19).

The sustained load, in the moment of application, induces initial stresses in the cement stone and aggregate in the concrete. In the course of time, the values of these stresses change as the deformations increase, but still the new stress state developed is a state of equilibrium of stresses.

The eigenstresses developed are determined—by assuming the cross section to be planar—by satisfying the equilibrium of deformations with the help of a simple model wherein the cement stone and skeleton of the aggregate are of plane disk lamination. A more accurate model has been applied by Baker (20) wherein he tried to describe the stress pattern developed within the concrete by a lattice consisting of vertical, horizontal, and diagonal bars.

The experiments conducted on creep showed that the aggregate, possessing no significant anelastic properties, behaves as a large system of elastic rigidity. As a matter of course, the stresses originally developed in the cement stone will decrease owing to its creep, and their major part will be transferred to the skeleton of aggregate.

In order to clarify this question, let us start from the simple model already mentioned above (Fig. 1), where a plane disk lamination of each of the elements (cement stone and aggregate) was assumed.

Figure 1. Plane disk model.

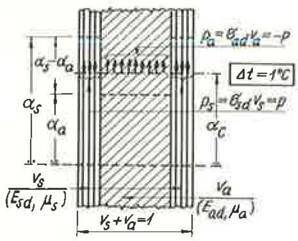


Figure 2. Circular cylinder model.

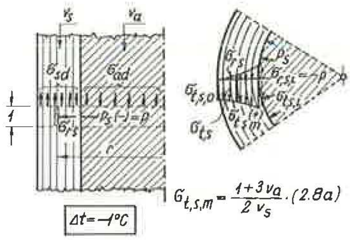


Figure 3. Spherical model.

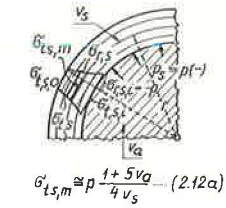


Figure 4. Values of n and E_s versus v_s.

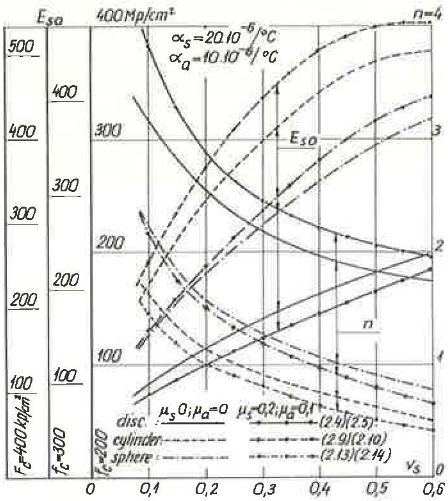


Figure 5. Mean σ_s values versus v_s.

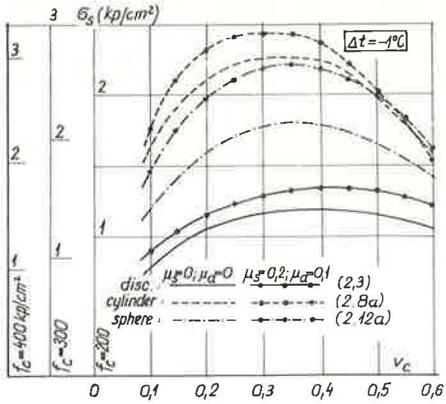


Figure 6. Final concrete shrinkage versus v_s.

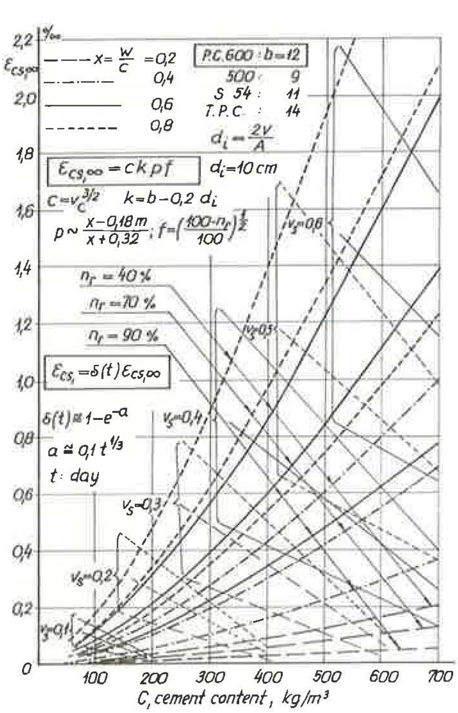
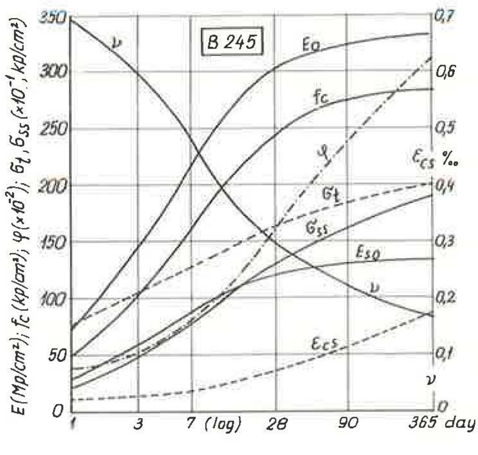


Figure 7. Effects on shrinkage of a concrete of cube strength f_c = 245 kip/cm² (3,500 psi).



Naturally, a model in which both vertical and horizontal and even diagonal laminations are assumed would strongly entangle the problem. In general, the Poisson's ratio has been supposed in the calculations to be different from zero.

Compressive Force

At an instant $t = 0$, i.e., at the moment of application of the load, the compressive force induces the stresses σ_{co} and σ_{ao} in the cement stone and aggregate respectively. The compressive force may be calculated from the formula

$$F = F_{so} + F_{ao} \tag{28}$$

where the following relationships may be assumed:

$$F_{so} = v_s A_c \sigma_{so} = A_s \sigma_{so}$$

$$F_{ao} = v_a A_c \sigma_{ao} = A_a \sigma_{ao}$$

$$v_s A_c + v_a A_c = A_c$$

The common deformation is

$$\epsilon_o = \epsilon_{so} = \frac{F_{so}}{A_s E_{co}} = \frac{F_{ao}}{A_a E_{ao}} = \frac{F}{A_c E_c} \epsilon_{ao}$$

from which

$$F_{ao} = F_{co} \frac{E_{ao}}{E_{co}} \times \frac{v_a}{v_s} = F_{so} n_o \gamma_a = \frac{1 - c}{c} F_{so} \tag{29}$$

where

$n_o \gamma_a = \dfrac{1 - c}{c}$,

$\gamma_a = \dfrac{v_a}{v_c}$, and

$E_{ao} = n_o E_{co}$.

From Eq. 28,

$$F_{so} = cF$$

$$F_{ao} = (1 - c)F$$

From these

$$\left.\begin{aligned} \sigma_{ao} &= \epsilon_o E_{ao} = n_o \sigma_{so} = \frac{(1 - c)F}{A_a} \\ \sigma_{so} &= \epsilon_o E_{so} = \frac{cF}{A_s} = c\sigma_{co} \\ \sigma_{co} &= \frac{F}{A_c} \end{aligned}\right\} \tag{30}$$

Effect of Creep

As a result of creep, the initial compressions increase with the time. The increment of deformation, i.e., the creep, is

$$\epsilon_\ell = \epsilon_o \varphi$$

where φ is as given.

Due to the creep, the σ_{co} initial compressive stresses decrease in the cement stone, first (probably due to a self-compaction) with a lower, later with a higher, and then again with a lower rate of change in stress. This latter slowdown may be due to the development of $\sigma_{s\varphi}$ tensile stresses. Meanwhile, only a very small (commonly negligible) $\sigma_{a\varphi}$ compressive stress increment is added to the σ_{ao} initial compressive stresses in the aggregate. During this transfer of compressive stresses, the cross section is in a state of equilibrium. At this state the deformation ϵ_{ϱ} is the value of the slow deformation of the cement stone $\epsilon_{\varrho c}$ reduced by mechanical resistance of the aggregate.

At an instant t, due to the state of eigenstress, $\sigma_{c\varphi} = -\sigma_{a\varphi}\gamma_a$. In this equation, $\sigma_{a\varphi}$ is compressive stress (a negative value) and $\sigma_{c\varphi}$ is tensile stress.

For the simplified estimate of the increment of the compressive stress induced in the aggregate, it seems convenient to apply the reduction factor φ_r that is based on Dischinger's theorem. This factor is as follows:

$$\varphi_r = \frac{c}{1 - c}\left[1 - e^{-(1-c)\varphi}\right] \tag{31}$$

For the determination of the stresses due to shrinkage, the deformation equilibrium equality can be used. According to that, the shrinkage of the concrete may be assumed as being composed of the shrinkages of the cement stone and aggregate.

Because the shrinkage of the aggregate, just like its creep, is very small in comparison to the similar deformation of the cement stone, it is negligible. Thus, the shrinkage of the concrete is governed, first of all, by the cement stone.

The free deformation of the cement stone, as one of its basic characteristics, may be obtained from Eq. 10 as

$$\epsilon_s = \frac{\epsilon_c}{c} = \frac{\epsilon_o}{c}$$

The reduced creep of the concrete is given by the relationship

$$\epsilon_{c,s} = \epsilon_s \frac{\varphi_r}{\varphi_n}$$

Accordingly, the stresses in the aggregate and cement stone due to the creep are

$$\sigma_{a\varphi} = \epsilon_{cs} E_a \tag{32}$$

and

$$\sigma_{s\varphi} = -\sigma_{a\varphi}\gamma_a = -\epsilon_{sc}\gamma_a n_o E_s = -\epsilon_{cs}\frac{1 - c}{c} E_c \tag{33}$$

respectively where, at an instant t, E_c and E_a are as given in Eq. 2, $E_s = \nu E_{so}$, $E_a = n_o E_s$, and $\nu = \frac{\nu_o}{1 + \varphi}$.

Effect of Moment

It is assumed that the moment M applied together with a compressive force F is so low that it causes only a low tensile stress at both edges of the specimen, so that no crack occurs in the cross section. Then the deformation equilibrium constraint equation states that the rotation of the cross section calculated on the basis of either the aggregate or cement stone is the same. The amount of traverse of the stresses may be obtained by the same deduction by the respective application of the formulas.

All that has been said is to be applied also to other kinds of materials.

CONCLUSIONS

Concrete can be considered as a two-phase composite material for the description of both its physical (thermal expansion, shrinkage) and its mechanical (compressive, tensile, and flexural strains, creep) properties. One of the phases is the viscoelastic cement stone, i.e., the porous hardened cement paste, as the matrix; the other is the elastic aggregate particles as the discrete phase. The influence of these two phases on the concrete properties differs significantly.

The two phases in the concrete are forced to deform together causing a mutual restriction, i.e., a state of residual stresses in both the paste and the aggregate. Approximate descriptions of these residual stresses can be obtained by applying formulas of the theory of elasticity (Eqs. 1 through 11) on any of the three elementary models: disk, circular cylinder, and sphere. In these models the cement paste is an outer shell and the aggregate is an inner core. It appears that the results obtained by the disk model (Eqs. 14, 26, 27, 32, and 33) provide the best estimates for practical purposes.

Stresses capable of causing cracking in the concrete can be produced by shrinkage or temperature change without any external loading.

Finally, it is demonstrated that the compression creep causes a rearrangement of the stresses in concrete by reducing the compressive stresses in the paste and by increasing those in the aggregate particles.

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