A STOCHASTIC MODEL OF FLOW VERSUS CONCENTRATION APPLIED TO TRAFFIC ON HILLS

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In the fundamental relationship between flow and concentration, flow (in vph) increases with concentration (in vehicles/mile) until a critical point is reached. After this critical point, flow decreases to zero as concentration increases to saturation. This is a deterministic model relating flow rate q to concentration k. In this paper this deterministic model is extended by allowing a probabilistic distribution of concentrations for a given mean value of flow. The specific application is to traffic proceeding up a two-lane hill. In this stochastic model, platoons arrive at the bottom of the hill in a Poisson fashion with parameter λ and at the top of the hill in a Poisson fashion with parameter μ. Because the size of platoon at the top of the hill is generally considerably larger than that at the bottom, $\lambda > \mu$. The distributions of platoon sizes at both the bottom and the top of the hill are additional parameters in the formulation. Vehicles on a hill represent a birth and death process where arrival of vehicles at the bottom corresponds to births and arrival of vehicles at the top of the hill corresponds to deaths. Because the lower bound on the number of vehicles is zero and the upper bound is determined from the length of the hill and the length of vehicles, there are a finite number of possible states. These states are incorporated into a finite Markov chain with a transition matrix determined by λ, μ, and the distribution of platoon sizes at the bottom and top of the hill. The transition matrix generates the probability of various concentrations on the hill as a function of the input parameters and time t. Hence, instead of two concentrations corresponding to a mean flow rate, we generate a probability distribution that varies with time for a whole range of concentrations. The Markov process also generates certain dynamic properties of the system such as relative stability. These and other stochastic properties of the Markov process are included to provide an extension of the classical flow-concentration deterministic model.

•ONE of the most complicating features in the analysis of traffic flow is its probabilistic nature. Two identical roadways may have an average flow of 500 vph, but the various parameters that measure the performance of the roadway could be significantly different in any given time interval. Some of these parameters are speed, concentration, number of passes completed and aborted, number of accidents, and percentage of slow-moving vehicles. The actual state of these traffic parameters fluctuates over time, so we must usually be satisfied with measuring an average value and perhaps some extreme values.

Solomon (12) observed that variation in speed from the normal flow of traffic was a leading cause of accidents. Very slow or very fast vehicles are involved in an abnormally high percentage of accidents. This study emphasizes the role and need for a more detailed analysis of traffic flow, in particular the need for a probabilistic model that treats random fluctuations in traffic behavior as a function of time.

It is the purpose of this paper to analyze the stochastic nature of one important traffic parameter, concentration. The classical relationship between flow and concentration is a deterministic one (6, 8): The flow rate increases with concentration until a critical point is reached, after which the flow rate decreases to zero. In this paper the deterministic model embodied in the fundamental diagram of flow and concentration is extended by allowing a probabilistic distribution of concentrations for a given average value of flow, which is itself a random and changing quantity. Furthermore, the relative stability of traffic flow will be measured by observing the rapidity with which a low concentration is transformed into a high concentration and vice versa. Hence, certain dynamic characteristics of traffic behavior will be presented.

The model to be presented is a Markov birth and death process. Along with the requisite mathematical development, data and results of a field study that tested the feasibility and utility of the model are included. It should be noted, though, that the model developed in this paper is but a tool and not an end in itself. The model will produce, with appropriate input, information about probabilities of various concentrations as a function of time and other factors.

The two-lane hill is a frequently encountered configuration that causes disruption and turbulence in the normal flow on level roadways. It is a physical setting in which concentration of vehicles obviously and directly affects flow rate. Every driver has experienced the agony of heavy traffic proceeding uphill; the speed of the platoon is controlled by the slowest moving vehicle when passing is not permitted or is too risky. As the concentration increases, so does the probability of encountering a slow-moving truck. The two-lane hill will be the physical setting for the stochastic extension of the fundamental diagram of flow and concentration.

BIRTH AND DEATH STOCHASTIC MODELS

The birth and death process is one type of stochastic process in which the time parameter is continuous and state space is discrete. Usually a population of individuals (or things) is considered where the size of the population at time t is X(t). During the interval t to t + Δt, the population may increase (birth) or decrease (death). We will consider the number of vehicles on the hill as our population of individuals; when a vehicle arrives at the bottom of the hill it is a birth, and as a vehicle reaches the crest of the hill it is a death. As t varies, vehicles will enter and depart the hill, i.e., births and deaths, and X(t) will then denote the number of vehicles on the hill at time t. If we then can obtain the probabilistic description of X(t) we have a probabilistic description of the number of vehicles on the hill for any time t.

One of the simplest birth-death processes is the one used to derive the stochastic nature of a single-server queuing system with Poisson arrivals and an exponential service time. In this system the probability of a birth in a small interval Δt is assumed to be proportional to the length of Δt and likewise for a death. Usually λ is the average birth rate, and μ is the average death rate. If we denote the probability of n individuals in the system at time t as $P_n(t)$, then the forward Kolmogorov difference equations are (making standard assumptions)

$$P_n(t + \Delta t) = (1 - \lambda\Delta t - \mu\Delta t)\, P_n(t) + (1 - \lambda\Delta t)\, \mu\Delta t\, P_{n+1}(t)$$
$$+ (1 - \mu\Delta t)\, \lambda\Delta t\, P_{n-1}(t) \qquad n \geq 1$$
$$P_o(t + \Delta t) = (1 - \lambda\Delta t)\, P_o(t) + (1 - \lambda\Delta t)\, \mu\Delta t\, P_1(t)$$

What this system means is that we can be in state n at time t + Δt in three possible ways:

1. Be in state n at time t and have no births and no deaths in the interval Δt,
2. Be in state n + 1 at time t and have no births but one death in the interval Δt, or
3. Be in state n - 1 at time t and have no deaths but one birth in the interval Δt.

The usual procedure is to let Δt become very small and obtain a system of differential-difference equations whose solution determines various properties of the process X(t). A closed-form solution for the time-varying X(t) is very difficult in this example, as in most other models, but the so-called steady-state solution is readily attained and widely known. The steady-state solution is appropriate for large values of t wherein initial transient influences become dampened. In this example the steady-state solution would be

$$P[X = n] = (1 - \lambda/\mu)\, (\lambda/\mu)^n$$

This steady-state distribution is very simple, and as such we can calculate important properties of the birth-death process in the steady state; e.g., the average number of individuals in the system is $\lambda/(\mu - \lambda)$, whereas the expected time in the system for each individual is $1/(\mu - \lambda)$.

Modeling this single-server queuing system mathematically permits both analysis and synthesis of the system. For example, we can predict the change in system characteristics by varying the parameters λ and μ, or we might have an optimization problem in which we want to minimize the expected number of individuals in the system subject to constraints on λ and μ. These same comments of course apply to a valid model of traffic proceeding up a hill. We would like to have a model with which we could perform the following operations (however, we do not deal with such applications specifically in this paper):

1. Predict changes in system characteristics if the passenger or transport arrival rate or both change,
2. Predict changes in system characteristics if physical changes in the roadway are made, and
3. Optimize various objective functions subject to constraints on the parameters.

For example, we might want to predict changes in the probability distribution of concentrations of a hill if heavy transport vehicles increased in density by 25 percent or if the speed capabilities of trucks decreased 15 percent because of heavier loads.

CHARACTERISTICS OF THE TRAFFIC MODEL

In describing a multiple birth and death process that approximates the flow of traffic proceeding uphill on a two-lane highway, we should first see how this model differs from the single-server queuing model. In the single-server queuing model we assume that individuals arrive at a service station, form a waiting line, and wait their turn for service. Only one individual is served at a time. This is not the case in the traffic system, for as soon as a vehicle arrives at the foot of the hill it begins service, i.e., climbing the hill. The traffic model is then a self-service model; we let $X(t)$ be the number of vehicles on the hill at time t. Another difference is that in the single-server queuing model we assume that all arriving customers are homogeneous in the sense that their service distributions are all the same. In the traffic model we have two types of customers or arrivals: transport vehicles and passenger vehicles. The performance of each type of vehicle on the hill is considerably different. A third crucial difference lies in the nature of the arrivals and services: In the queuing model we assumed that individuals arrive in a Poisson fashion and are serviced in an exponential fashion. In the traffic model that is simply not true, for vehicles do not flow freely or randomly on the highway, especially on hills where platoons are formed behind slow-moving vehicles. What happens is vehicles arrive at the foot of the hill in bulk and leave the hill in bulk.

The multiple birth and death process for traffic flow on hills includes features not present in the ordinary queuing model, i.e., self-service, nonhomogeneous vehicles, bulk arrivals, and bulk finishes. $P_{ij}(t)$ is defined as the probability of i cars and j trucks on the hill at time t; the possible states of our system will be pairs of nonnegative integers (i, j). We make two assumptions.

First, the distribution of times between platoon arrivals at both the bottom and the top of the hill is exponential with parameters $1/\lambda$ and $1/\mu$ respectively. Equivalently, platoon arrivals at the bottom and the top of the hill can be shown to be Poisson with parameters λ and μ respectively. The measurement of time between platoon arrivals by our own convention shall be from the front of the lead car of a platoon to the front of the lead car of the next platoon.

Second, given a platoon arrival, the change in the state of the hill can be by more than one vehicle. Thus, a discrete distribution A_{ij} gives the probability of i cars and j trucks arriving in a platoon at the bottom of the hill for all combinations of i and j.

$$\sum_{ij} A_{ij} = 1 \qquad A_{ij} \geq 0$$

Similarly, a discrete distribution f_{ij} gives the probability of i cars and j trucks arriving in a platoon at the top of the hill and departing the system for all combinations of i and j.

$$\sum_{ij} f_{ij} = 1 \qquad f_{ij} \geq 0$$

For example we may have

$$f_{10} = 0.1$$

$$f_{01} = 0.1$$

$$f_{11} = 0.2$$

$$f_{2,1} = 0.3$$

$$f_{3,1} = 0.3$$

Given that a platoon arrives at the crest and that the state of the hill is (1,1), we must use conditional probability to find the correct probabilities of f_{10}, f_{01}, f_{11}; the modified distribution would be

$$f_{10} = \frac{0.1}{0.1 + 0.1 + 0.2} = 0.25$$

$$f_{01} = 0.25$$

$$f_{11} = 0.50$$

The aforementioned process should really not be called a birth and death process, for the standard terminology of a birth and death process requires that, in a small interval of time Δt, only a single birth or death has positive probability. (Rosenshine of Pennsylvania State University suggested the name multiple birth and death.) Finally we must observe that in the queuing model presented earlier it was implicitly assumed that one could have any number of individuals in the system. Certainly this is not the case on the hill where there are physical limitations due to the actual length of the roadway and corresponding lengths of vehicles. Assume then that N and M are upper limits to the number of cars and trucks on the hill; if the state of the system is at some point N and M, then no more vehicles can enter the hill until some vehicles in the system leave. Hence the process under consideration will have two reflecting barriers: the state (0,0) and the state (N,M) where $N + M = Q$. Q is the maximum number of vehicles that can be physically present on the hill at any one time when we consider the average length of cars and trucks and make plausible assumptions about the proportion of each present. However, for the sake of illustration we simplify the problem by calling N and M the respective upper limits for cars and trucks present on the hill.

MODEL FORMULATION

We now consider the Kolmogorov differential-difference equations that describe this multiple birth-death process with reflecting barriers. Because the required notation is a bit abstruse for a general model let us first set $N = 4$ and $M = 2$. $P_{ij}(t)$ is the probability of i cars and j trucks on the hill at time t where of course for each $t \in (0, \infty)$

$$\sum_{(i,j)} P_{ij}(t) = 1$$

We shall also specify the conditional probability distribution of arrivals and finishes given that an arrival or finish has occurred. Let A_{ij} be the conditional probabilities of i cars and j trucks arriving, given that an arrival has occurred, and f_{ij} be the conditional probabilities of i cars and j trucks finishing the hill (i.e., reaching the crest), given that a finish has occurred. In our example set

$$A_{10} = 0.5 \qquad f_{10} = 0.6$$

$$A_{01} = 0.3 \qquad f_{01} = 0.17$$

$$A_{11} = 0.1 \qquad f_{11} = 0.17$$

$$A_{21} = 0.1 \qquad f_{21} = 0.50$$

(The fact that the possible sets of bulk arrivals and bulk finishes are the same is only coincidental in this example.)

The usual procedure in birth-death processes is to write $P_{ij}(t + \Delta t)$ where Δt is some very small interval in terms of $P_{ij}(t)$. For example, we set i = 2 and j = 2 and consider $P_{22}(t + \Delta t)$; i.e., we want to write an expression for the probability of being in state (2,2) at time $t + \Delta t$. There are the three mutually exclusive and exhaustive ways of being in state (2,2) at time $t + \Delta t$:

1. Be in state (2,2) at time t and have no arrivals or no finishes in Δt,
2. Be in state (i,j) at time t and have (2-i, 2-j) arrivals and no finishes in Δt, or
3. Be in state (i,j) at time t and have (i-2, j-2) finishes and no arrivals in Δt.

When t is very small we cannot have both an arrival and a finish in t time so that no other possibilities are available. Hence, when t is sufficiently small $P_{22}(t + \Delta t)$ is approximately equal to the sum of the following three expressions:

1. Prob—no arrivals or finishes in Δt and system in state (2,2) at time t,
2. $\sum_{(i,j)}$ Prob—(2-i, 2-j) vehicles arrive and no finishes in Δt and the system is in state (i,j) at time t, and
3. $\sum_{(i,j)}$ Prob—(i-2, j-2) vehicles finish and no arrivals in Δt and the system is in state (i,j) at time t.

We can now write in more classical terminology the Kolmogorov equations where we have utilized the independence of the probability of being in a given state and the event of arrivals and finishes (except at boundaries) plus the fact that the Prob [2-i, 2-j) arrivals] = Prob [(2-i, 2-j) arrivals | an arrival] × Prob [an arrival]. $P_{22}(t + \Delta t) = (1 - \lambda\Delta t)(1 - \mu\Delta t)P_{22}(t) + (1 - \mu\Delta t)\lambda\Delta t\,[A_{21}P_{01}(t) + A_{11}P_{11}(t) + A_{01}P_{21}(t) + A_{10} \times P_{12}(t)] + (1 - \lambda\Delta t)\mu\Delta t\,[f_{10}P_{32}(t)]$. Rearranging terms, dividing by Δt, and neglecting terms on the order $(\Delta t)^2$ yield

$$\frac{P_{22}(t + \Delta t) - P_{22}(t)}{\Delta t} = (-\lambda - \mu)P_{22}(t) + \lambda\,[A_{21}P_{01}(t) + A_{11}P_{11}(t) + A_{01}P_{21}(t) + A_{10} \times P_{12}(t)] + \mu f_{10}P_{32}(t)$$

Now we let $\Delta t \to 0$ and on the L.H.S. we have by definition $P'_{22}(t)$, i.e., the derivative of $P'_{22}(t)$. In this example we have 15 possible states so that employing the same limiting procedure to each of the 15 possible states would yield a linear system of 15 homogeneous differential equations of the first degree; the system would have the following simple form:

$$P'(t) = A\,P(t)$$

where A is a 15 × 15 matrix and not a function of t.

One of the easiest methods of obtaining the 15 forward Kolmogorov equations is to write the approximate probabilities of moving from state (i,j) at time t to state (i′,j′) at time $t + \Delta t$. This is shown in detail in Figure 1, where for brevity we have omitted the Δt associated with each λ and μ and terms on the order $(\Delta t)^2$. The first line in Figure 1 indicates that, if the process is in state (0,0) at time t, it can conceivably be in states [(0,0), (1,0), (0,1), (1,1), (2,1)] at time $t + \Delta t$. In particular the probability that the process will be in one of these states is 1. The forward Kolmogorov equations for state (2,2) were obtained by setting $P_{22}(t + \Delta t)$ equal to the column entries below (2,2) multiplied by their respective states. In fact this is how all the forward Kolmogorov equations could be obtained. Note that in the row associated with state (3,0) at time t are the conditional probabilities with superscripts attached. If the process is in state (3,0) at time t and an arrival occurs, then it is impossible for the arrival to contain two cars and one truck; hence, the probabilities of the other three possibilities must be amended. In our example $A_{10}^{(3)}$, $A_{01}^{(3)}$, and $A_{11}^{(3)}$ are the conditional distributions of arrivals, given that an arrival occurs and that the arrival can be in only one of three states, (1,0), (0,1), or (1,1). In this example

$$A_{10}^{(3)} = \frac{0.5}{0.5 + 0.3 + 0.1}$$

$$A_{01}^{(3)} = \frac{0.3}{0.5 + 0.3 + 0.1}$$

$$A_{11}^{(3)} = \frac{0.1}{0.5 + 0.3 + 0.1}$$

The Kolmogorov equations for this example produce a system of 15 linear differential, first-degree, and homogeneous equations of the form

$$P'(t) = A\ P(t)$$

At this juncture we can proceed in one of two directions:

1. Find the solution of this system of linear differential equations with various initial boundary conditions, or
2. Find the steady-state probabilities by making t very large, i.e., set $P'_{ij}(t) = 0$ and solve the system of 15 linear equations, subject to $\sum_{ij} P_{ij} = 1$.

The form of the solution to the linear system of differential equations in the first direction is

$$P(t) = e^{At}\ P(0)$$

Methods for obtaining this solution and solutions for the second direction are discussed in the next section.

SOLUTION PROCEDURE

For the example in the previous section we had 15 possible concentrations, i.e., N = 4 and M = 2. Suppose that the arrival of platoons at the bottom of the hill is 5/min and at the top of the hill is 3/min so that we set $\lambda = 5$ and $\mu = 3$ in Figure 1. The conditional distributions $\{A_{ij}\}$ and $\{f_{ij}\}$ will be the same as those given earlier. The average flow rate of platoons at the bottom of the hill is 5/min, but of course in some minutes there may be 0 platoons and in other minutes 10 platoons moving through the system in such a way that the limits N and M are not violated.

$B_{(0,0)}(t)$ is a 15-component vector that represents the probability of being in each of the 15 states at time t given that at time 0 the state was (0,0). Of course, for different values of t these probabilities are different, but the sum of the 15 components is 1. To obtain $B_{(0,0)}(t)$ we must solve the following set of 15 linear differential equations:

$$B'_{(0,0)}(t) = A\ B_{(0,0)}(t)$$

where A is the 15 × 15 matrix in Figure 1 with $\lambda = 5$ and $\mu = 3$. The general form of such systems is given by

$$B_{(0,0)}(t) = \ell^{At}\ B_{(0,0)}(0)$$

where $B_{(0,0)}(0)$ is the boundary condition and represents a starting state of (0,0) at time 0. The solution procedure is to obtain the matrix ℓ^{At}; there are two well-known procedures for deriving this matrix (2, 10). The first relies on obtaining 15 distinct eigenvalues of the matrix A so that A can be diagonalized, and the second is simply a series expansion of the matrix ℓ^{At}. We have chosen the second method for computational expedience.

The series expansion for the matrix ℓ^{At} is

$$\ell^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots$$

Figure 1. Transition matrix at t + Δt.

	0,0	1,0	2,0	3,0	4,0	0,1	1,1	2,1	3,1	4,1	0,2	1,2	2,2	3,2	4,2
0,0	$1-\lambda$	λA_{10}				λA_{01}	λA_{11}	λA_{21}							
1,0	μ	$1-\lambda-\mu$	λA_{10}				λA_{01}	λA_{11}	λA_{21}						
2,0		μ	$1-\lambda-\mu$	λA_{10}				λA_{01}	λA_{11}	λA_{21}					
3,0			μ	$1-\lambda-\mu$	$\lambda A_{10}^{(3)}$				$\lambda A_{01}^{(3)}$	$\lambda A_{11}^{(3)}$					
4,0				μ	$1-\lambda-\mu$					λ					
0,1						$1-\lambda-\mu$	λA_{10}				λA_{01}	λA_{11}	λA_{21}		
1,1	$\mu f_{11}^{(3)}$	$\mu f_{01}^{(3)}$				$\mu f_{10}^{(3)}$	$1-\lambda-\mu$	λA_{10}				λA_{01}	λA_{11}	λA_{21}	
2,1	μf_{21}	μf_{11}	μf_{01}				μf_{10}	$1-\lambda-\mu$	λA_{10}				λA_{01}	λA_{11}	λA_{21}
3,1		μf_{21}	μf_{11}	μf_{01}				μf_{10}	$1-\lambda-\mu$	$\lambda A_{10}^{(3)}$				$\lambda A_{01}^{(3)}$	$\lambda A_{11}^{(3)}$
4,1			μf_{21}	μf_{11}	μf_{01}				μf_{10}	$1-\lambda-\mu$					λ
0,2						μ					$1-\lambda-\mu$	λ			
1,2						$\mu f_{11}^{(3)}$	$\mu f_{01}^{(3)}$				$\mu f_{10}^{(3)}$	$1-\lambda-\mu$	λ		
2,2						μf_{21}	μf_{11}	μf_{01}				μf_{10}	$1-\lambda-\mu$	λ	
3,2							μf_{21}	μf_{11}	μf_{01}				μf_{10}	$1-\lambda-\mu$	λ
4,2								μf_{21}	μf_{11}	μf_{01}				μf_{10}	$1-\mu$

Figure 2. Transition matrix for illustrative example.

(a)

	0,0	1,0	2,0	3,0	4,0	0,1	1,1	2,1	3,1	4,1	0,2	1,2	2,2	3,2	4,2
0,0	.158	.103	.060	.029	.012	.063	.077	.088	.057	.040	.017	.034	.051	.062	.148
1,0	.143	.099	.064	.034	.015	.055	.070	.090	.062	.047	.014	.029	.046	.06;	.171
2,0	.121	.093	.069	.041	.020	.043	.060	.092	.068	.058	.010	.022	.039	.059	.204
3,0	.101	.085	.072	.046	.024	.033	.051	.094	.073	.068	.007	.016	.034	.058	.238
4,0	.091	.078	.068	.043	.023	.029	.047	.097	.075	.070	.006	.014	.033	.060	.265
0,1	.143	.093	.052	.024	.009	.065	.078	.090	.056	.037	.019	.038	.057	.069	.164
1,1	.135	.089	.054	.027	.011	.056	.071	.093	.061	.043	.015	.032	.052	.069	.192
2,1	.123	.087	.059	.031	.014	.047	.063	.096	.066	.051	.012	.023	.044	.065	.217
3,1	.107	.082	.062	.035	.016	.039	.055	.098	.071	.059	.009	.019	.039	.063	.246
4,1	.097	.077	.061	.035	.017	.033	.051	.100	.074	.062	.007	.017	.037	.064	.270
0,2	.134	.082	.044	.020	.008	.066	.078	.091	.055	.035	.021	.092	.064	.077	.184
1,2	.123	.079	.046	.022	.009	.056	.071	.095	.061	.041	.016	.034	.057	.076	.214
2,2	.114	.078	.050	.025	.011	.047	.064	.098	.066	.048	.013	.027	.049	.072	.239
3,2	.104	.076	.053	.027	.012	.039	.056	.101	.071	.054	.009	.021	.042	.069	.264
4,2	.098	.075	.055	.029	.013	.035	.052	.102	.073	.058	.008	.018	.039	.067	.278

(b)

0,0	1,0	2,0	3,0	4,0	0,1	1,1	2,1	3,1	4,1	0,2	1,2	2,2	3,2	4,2
.120	.086	.058	.031	.014	.046	.062	.096	.067	.052	.012	.025	.044	.065	.223

The number of terms required to approximate ℓ^{At} of course depends on the size of t; for large values of t a larger number of terms are required.

When t = 1 min, the probabilities of being in each of the 15 states, given one starts in a particular state, are given in Figure 2. If at t = 0 the state of the hill is (0,0), then the probability that the state of the hill is (0,0) at t = 1 min is 0.1589, whereas the probability that the state of the hill is (4,2) at t = 1 is 0.1481. The probabilities of various concentrations at t = 1 depend quite naturally on the state of the hill at t = 0 (each row in Fig. 2 is different). The probability of being in state (0,0) at t = 1, given that the hill is in state (0,1) at t = 0, is 0.1433. If we want to know the probabilities of various concentrations when t = 2, the matrix ℓ^{At} would be needed where t = 2.

It might be expected that for large values of t the probabilities of various concentrations are independent of the starting states. This indeed is true and is a well-known fact about Markov processes. In this small illustrative example, this steady state was reached when t = 6 min. Figure 2b shows the steady-state probabilities; the probability of being in state (0,0) in the long run is 0.120 regardless of the starting state. The time to reach steady state is determined largely by the number of states possible, and in this example the number is only 15. In a later section a field study is described where the time to reach steady state is nearly 2 hours.

The steady-state probabilities of the various concentrations shown in Figure 2b represent a significant departure from the deterministic information given by an ordinary flow-concentration diagram. Instead of assigning a fixed value of concentration for a fixed flow, the stochastic model assigns a certain probability of various concentrations corresponding to a certain fixed mean flow of vehicles. Of course an average concentration could be computed, but the knowledge of probabilities of certain extreme conditions is available with this model, along with the time-varying behavior of these concentrations.

MEAN FIRST PASSAGE TIMES

The dynamic properties of the multiple birth and death model are in some ways illustrated in the previous section wherein it was shown that the probability distribution of concentrations changed with time until a steady state was reached. In this section a more natural and useful dynamic property is described: How long does it take for a road jam to dissipate to ordinary conditions, or, put another way, how long does it take to move from a high concentration state to a low state or vice versa? How quickly a hill can become clogged with vehicles is a measure of its relative stability.

The stability of traffic concentration on a hill can be analyzed by finding what is called the mean first passage time. The mean first passage time is simply the average number of minutes required to pass from a particular state to some other state. If the mean first passage time from state (0,0) to state (20,10) is relatively short, then the hill can become clogged in a very short span of time.

The mean first passage time from state j to state k m_{jk} is calculated from the following system of linear equations (4):

$$m_{jk} = 1 + \sum_{i \neq k} P_{ji}\, m_{ik} \qquad j \neq k$$

P_{ji} is the probability of a single transition from state j to some intermediate state i. But in our continuous-time Markov process, $P_{ji}(t)$ involves an unknown number of intermediate transitions before state i is reached. However, if we consider a sufficiently small t, the number of transitions between j and i approaches 1, and we can use the above system of equations in a valid fashion to get a good approximation of the transitions from state i to state k. If we multiply the number of transitions, m_{jk} by the time one transition on the average occurs, we get a valid approximation of the mean first passage times. In the interest of brevity, the mathematical development supporting these comments is omitted (Fig. 3).

Figure 3 shows the mean first passage times for the illustrative example when the time interval used in discretizing the Markov process was Δt = 0.0001. (It should be noted that we found negligible differences for Δt as large as 0.05.) The mean first passage time from state (0,0) to state (4,2) was 2.09 min, whereas the mean first passage time from (2,1) to (0,0) was 2.00 min. Of course, with a larger number of states, the mean first passage times between the extreme pairs of state would be larger.

FIELD STUDY

In this section we shall describe some results of field studies where the feasibility and the validity of the assumptions for the stochastic model were tested.

The crucial mathematical assumption is that platoons of vehicles arrive at both the bottom and top of the hill in a Poisson fashion (8, 9); nonetheless data were collected from four hills in Centre and Blair Counties in central Pennsylvania. In all cases the hypothesis that platoons of vehicles follow a Poisson flow cannot be rejected at the 0.05 significance level when the classical chi-square goodness-of-fit test is used. It should be noted though that these tests were carried out during daylight hours wherein intercity traffic was only moderate; certain peak periods, such as the 5:00 p.m. rush hour, were not tested.

One particular two-lane hill in Centre County was chosen to be studied extensively. Here we collected data to estimate λ, μ, $\{A_{ij}\}$, and $\{f_{ij}\}$. The particular hill was Penn-144 between the towns of Centre Hall and Pleasant Gap; this hill is about ½ mile long, and the bottom of the hill is in Centre Hall. The data were always collected during clear dry weather and between the hours of 9:00 and 11:30 a.m. Our estimates of the necessary parameters are given in Table 1 and are based on five observations during the months of July and August 1972. The tactical procedures used to collect, transcribe, and analyze the data can be found elsewhere (13). There were some difficult problems such as determining whether vehicles were platooned, and in many instances it was not entirely clear whether a vehicle should be classified as passenger or transport.

The estimates for λ and μ were 1.80 platoons/min and 1.28 platoons/min for this hill. The conditional distribution of platoon sizes is given in Table 1. The upper limits N and M were eventually set at 18 and 5 respectively; this permitted a total of 114 states (19 × 6). These limits do not represent upper bounds on the capacity of the hill; with a ½-mile hill, these limits should be four or five times as large. The difficulty encountered was that, with matrices of the size 500 × 500, certain computational procedures become prohibitively costly; hence, practical limits of 13 cars and five trucks were used initially. Later a simple procedure for avoiding the dimensionality difficulty is discussed.

The probability transition matrix for the 114-state model was computed for increasing values of t. The steady-state distribution was reached for t = 2 hours (Table 2). This compares with the illustrative example where after 6 min the steady state was reached. Note that Table 2 shows that, as the number of trucks increases, correspondingly the number of cars increases. For example, the probability that there are no trucks and 18 cars on the hill at any one time is 0.001; however, the probability of five trucks and 18 cars is much larger at 0.070. The conditional probability of 18 cars given that there are five trucks would be 0.070/0.288 = 0.240.

Table 3 gives a sample of the mean first passage times. Note that the mean first passage time from state (0,0) to (18,5) is 63.84 min but from (18.5) to (0,0) the time is 109.47 min. Hence on the average the time to reach saturation from zero concentration is over twice as long as to go from saturation to zero concentration.

This type of information is, of course, not available from the classical flow-concentration diagram, which specifies the flow rate for a given concentration. Conversely a given flow rate would correspond to two distinct concentrations. If the flow rate were x, then concentrations y and z could generate the flow shown in Figure 4. The stochastic flow-concentration model generates a probability distribution for the various concentrations rather than just two points y and z corresponding to a particular flow x. One might expect that this probability distribution should be some type of bimodal distribution with modes at y and z. To some degree this was in fact found to be true.

Figure 3. Mean first passage times for illustrative example.

	0,0	1,0	2,0	3,0	4,0	0,1	1,1	2,1	3,1	4,1	0,2	1,2	2,2	3,2	4,2
0,0	--	1.08	2.71	6.01	12.7	2.64	1.77	1.35	2.27	3.41	14.1	6.07	3.05	2.31	2.09
1,0	1.43	--	2.02	5.45	12.2	3.50	1.87	1.27	2.01	3.09	15.0	6.48	3.22	2.29	1.93
2,0	2.18	1.32	--	3.91	10.8	4.14	2.49	1.18	1.75	2.48	15.7	7.12	3.49	2.32	1.69
3,0	2.64	1.94	1.62	--	7.54	4.58	2.88	1.39	1.52	1.74	16.3	7.57	3.80	2.38	1.44
4,0	2.78	2.20	2.24	3.34	--	4.67	2.98	1.37	1.76	0.77	16.3	7.62	3.82	2.37	1.08
0,1	1.36	1.75	3.10	6.31	13.0	--	1.36	1.38	2.36	3.50	11.6	4.78	2.39	2.01	1.99
1,1	1.82	1.73	2.97	6.14	12.8	3.09	--	1.01	2.09	3.27	14.2	5.22	2.45	1.32	1.76
2,1	2.00	1.79	2.71	5.81	12.4	3.82	2.25	--	1.52	2.84	15.3	6.77	2.79	1.81	1.49
3,1	2.48	1.84	2.51	5.31	11.8	4.37	2.64	1.22	--	1.94	16.0	7.31	3.56	1.86	1.16
4,1	2.66	2.16	2.41	5.15	11.2	4.52	2.85	1.16	1.70	--	16.1	7.45	3.63	2.17	0.67
0,2	2.03	2.16	3.35	6.50	13.1	1.90	1.76	1.48	2.42	3.54	--	1.92	1.60	1.71	1.87
1,2	2.24	2.21	3.31	6.42	13.0	2.85	1.80	1.34	2.27	3.38	12.5	--	.093	1.32	1.60
2,2	2.34	2.20	3.20	6.28	12.9	3.36	2.06	1.17	2.05	3.14	14.7	6.11	--	0.81	1.24
3,2	2.52	2.23	3.10	6.11	12.7	4.19	2.28	1.02	1.79	2.83	15.6	6.95	3.13	--	0.70
4,2	2.61	2.26	3.02	6.00	12.5	4.43	2.75	0.89	1.67	2.55	16.0	7.34	3.45	1.91	--

Table 1. Field study parameters.

Bottom of Hill		Top of Hill	
Arrivals	Value	Arrivals	Value
A_{10}	0.734	f_{10}	0.635
A_{20}	0.098	f_{20}	0.146
A_{30}	0.016	f_{30}	0.018
A_{40}	0.009	f_{40}	0.018
A_{50}	0.003	f_{70}	0.004
A_{01}	0.095	f_{01}	0.041
A_{11}	0.032	f_{11}	0.055
A_{21}	0.006	f_{21}	0.037
A_{02}	0.006	f_{31}	0.009
		f_{61}	0.009
		f_{71}	0.014
		$f_{2,2}$	0.005
		$f_{6,2}$	0.005
		$f_{12,2}$	0.004

Note: $\lambda = 1.80$, and $\mu = 1.28$.

Table 2. Steady-state probabilities for field study.

Cars	0	1	2	3	4	5
0	0.021	0.009	0.006	0.005	0.003	0.001
1	0.016	0.010	0.007	0.006	0.004	0.002
2	0.013	0.010	0.008	0.007	0.005	0.003
3	0.012	0.010	0.009	0.008	0.005	0.004
4	0.010	0.010	0.009	0.008	0.007	0.005
5	0.009	0.009	0.009	0.009	0.008	0.006
6	0.008	0.009	0.009	0.009	0.008	0.007
7	0.007	0.008	0.009	0.009	0.009	0.008
8	0.006	0.008	0.008	0.009	0.010	0.009
9	0.006	0.007	0.008	0.009	0.010	0.010
10	0.005	0.006	0.008	0.009	0.011	0.011
11	0.005	0.006	0.007	0.009	0.012	0.012
12	0.004	0.005	0.007	0.009	0.012	0.013
13	0.003	0.005	0.006	0.009	0.012	0.015
14	0.003	0.004	0.005	0.008	0.013	0.018
15	0.002	0.003	0.005	0.008	0.013	0.021
16	0.002	0.003	0.004	0.007	0.013	0.027
17	0.001	0.002	0.003	0.006	0.013	0.036
18	0.001	0.002	0.003	0.005	0.011	0.070
Total	0.128	0.126	0.130	0.149	0.179	0.288

Table 3. Mean first passage times for field study, in min.

From	To (0, 0)	(6, 2)	(18, 5)
(0, 0)	0.48	59.71	63.84
(8, 0)	84.27	67.61	53.80
(18, 0)	107.5	82.34	26.28
(0, 1)	25.30	57.55	63.51
(8, 1)	84.57	61.85	53.04
(18, 1)	107.53	82.00	22.86
(8, 2)	87.80	49.85	51.46
(18, 2)	107.87	81.90	18.37
(0, 3)	67.78	61.44	60.76
(8, 3)	94.86	65.61	46.77
(18, 3)	108.52	82.25	13.00
(0, 4)	70.60	65.07	60.57
(8, 4)	96.70	72.93	46.78
(18, 4)	109.1	82.88	07.05
(0, 5)	79.06	69.56	59.24
(8, 5)	99.84	77.87	44.26
(18, 5)	109.47	83.51	00.14

Figure 4. Flow-density diagram.

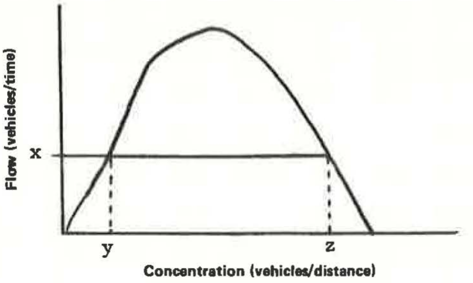
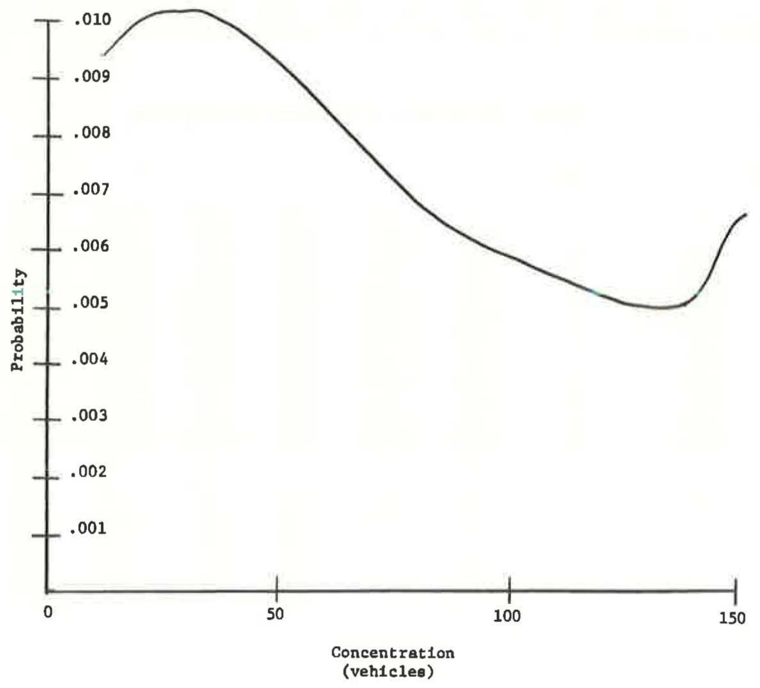


Figure 5. Field study distribution of concentration for total vehicles.



As a demonstration of this phenomenon, the stochastic model used for the Centre Hall Mountain field study was adapted so that cars and trucks were treated identically. Previously, the upper bounds on cars and trucks were 18 and five respectively, which allowed only 23 vehicles on the hill, but this set of bounds requires 114 states. If, instead, all vehicles are treated identically then the model could use 114 as an upper bound. The actual upper bound on the number of vehicles allowed was set at 150. Figure 5 shows the steady-state probabilities in graphical form; the two modes are at state 7 and state 150.

FURTHER CONSIDERATIONS

The stochastic model described in this paper appears to be a valuable tool in the analysis of traffic flow and concentration, although no attempt has been made in this study to apply the methodology to the design and control of roadways. But consideration is now under way for applications in passing safety and speeds and certain other theoretical extensions.

One shortcoming of the model is that the parameters λ and μ are time-homogeneous. Obviously over long enough time spans we should treat λ and μ as functions of time, i.e., $\lambda(t)$ and $\mu(t)$. This extension is certainly feasible, for the only additional difficulty is that the linear differential equations now become time-dependent. The transition matrices should then become even more useful when time intervals over rush hours are included. Furthermore mean first passage times become very important.

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