Revised Queueing Model of Delay at All-Way Stop–Controlled Intersections

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An existing M/G/1 queueing model of delay at all-way stop–controlled intersections is updated to reflect recent empirical evidence of driver behavior. In particular, the queueing model has been expanded to account for turning and coordination between drivers who get priority at the same time. An exact expression for an important variable in the model, the variance of service time, could not be found. However, an approximate expression could be verified using a Monte Carlo simulation. A queueing model can produce values of delay that are similar to strictly empirical models, but that apply to a much greater range of intersection volumes.

In a series of empirical studies, Kyte has developed statistical models of delay at all-way stop–controlled (AWSC) intersections (*1,2*). Kyte's models formed the basis for the "Interim Material on Unsignalized Intersection," recently published by TRB's Committee on Highway Capacity and Quality of Service in *Transportation Research Circular 373* (*3*). The procedure presented in that circular, TRC 373, is quite simple, consisting only of the application of two equations. The first equation calculates subject approach capacity from intersection movement data; the second equation calculates delay from the approach's volume-to-capacity ratio. The most obvious drawback to the TRC 373 procedure is that its relation for capacity applies to a rather narrow range of traffic conditions.

It might be possible to build a queueing model that closely replicates the results of the TRC 373 procedure over its valid range of inputs, but permits extrapolation to all possible traffic conditions. A queueing model incorporates more features of traffic theory and is more intuitive, but it would be somewhat more difficult to calculate.

A particularly interesting queueing model was developed by Richardson (*4*), who was able to build a model consisting of four single-server queues—one for each approach—that could be iteratively solved for the mean and variance of service time at each approach. Richardson assumed exponential arrivals, but was able to avoid making any restrictive assumptions about the probability distribution of service times. With knowledge of the approach's volume, its service time, and the variance of service times, Richardson was able to calculate delay at an approach as if he were looking at a simple M/G/1 queue.

Unfortunately, Richardson's model cannot be made to replicate TRC 373. Richardson did not include any means of adjusting for more delay due to left turns or due to coordination of drivers at multilane approaches, or for less delay due to right turns. It is not obvious that such delay adjustments can be properly incorporated into Richardson's model, because the expression for the variance of the service time (necessary for delay calculation) becomes exceptionally complex.

The purpose of this paper is to create and evaluate a revised M/G/1 queueing model, generally following Richardson's framework, which encompasses the same traffic behavior as the TRC 373 procedure and can give comparable results.

REVIEW OF RICHARDSON'S MODEL

Richardson hypothesized that the service time for any vehicle on the subject approach was the sum of two quantities: (a) the vehicle's minimum headway, defined as the time it takes for the vehicle to execute its maneuver once it gains priority, and (b) the amount of time it must wait for one or more vehicles on the conflicting approaches. Richardson assumed that the minimum headway was constant for all approaches. Richardson further assumed that the time necessary to wait for a conflicting vehicle was also a constant and slightly less than the minimum headway. Consequently, the only source of variation in service time at an approach would be the presence or absence of a conflicting vehicle. Richardson was able to demonstrate that his assumptions were consistent with the limited amount of AWSC intersection data available at that time.

Average service time at any approach, s, is

$$s = t_m + (t_m - t_c)\rho_c \tag{1}$$

where

t_m = minimum headway,
t_c = adjustment to conflicting vehicle minimum headway, and
ρ_c = utilization ratio and also the probability of having a conflicting vehicle (the probability that there is at least one vehicle on either of the two conflicting approaches).

$$\rho_c = 1 - (1 - \rho_{c1})(1 - \rho_{c2}) \tag{2}$$

where ρ_{c1} is the probability of having a vehicle on the first conflicting approach and ρ_{c2} is the probability of having a vehicle on the second conflicting approach. From conventional queueing theory, the probability of having a vehicle on a conflicting approach can be found from the product of the arrival rate and the service time. For the ith conflicting approach,

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$$\rho_{ci} = \lambda_{ci} s_{ci} \tag{3}$$

where λ_{ci} is the arrival rate (or volume) for conflicting approach i and s_{ci} is the service time for conflicting approach i.

Where there is more than one lane at an approach, Richardson split the traffic evenly across lanes and treated each lane as if it were independent.

$$\rho_{cin} = 1 - (1 - \rho_{ci}/n)^n \tag{4}$$

where ρ_{cin} is the probability of having a vehicle at the stop line of at least one lane at an approach with n lanes. Richardson did not attempt to assign right-turning vehicles to the right-most lane or the left-turning traffic to the left-most lane.

It is important to note that a separate set of Equations 1 to 3 (and possibly Equation 4) exist for each approach and that the approaches are interdependent. These equations represent a set of nonlinear simultaneous equations, which are sufficient to solve for the average service time and the utilization ratio of each approach. Richardson found that the method of successive approximations worked very well as a solution technique.

When the service time has been found for each approach, the variance of the service time, σ^2, can be found from

$$\sigma^2 = t_m^2(2t_m - t_c - s)/(t_m - t_c) + (2t_m - t_c)^2(s - t_m)/(t_m - t_c) - s^2 \tag{5}$$

The delay can be found from

$$D = (2\rho - \rho^2 + \lambda^2\sigma^2)/[2\lambda(1 - \rho)] \tag{6}$$

where ρ is the utilization ratio and λ is the volume on the subject approach.

Richardson's model is elegant. It requires few assumptions but provides a good description of driver behavior at AWSC intersections.

IMPLICATIONS OF TRC 373

TRC 373 implies that driver behavior at an AWSC intersection is much more complex than the simple queueing relations of Richardson's original model. According to TRC 373 there are increases and decreases in subject approach delay due to turning, and there are increases in delay associated with the need for coordination among drivers at multilane approaches.

To account for turning and coordination in Richardson's model it is necessary to allow the average minimum headway, t_m, to vary at each approach according to the amount of turning and the presence of additional vehicles on the subject or opposing approaches. Thus, average minimum headway would be a complex expression

$$t_m = t_b + \sum_r t_r p_r \tag{7}$$

where

t_b = base minimum headway,
t_r = adjustment for condition r, and
p_r = probability that condition r exists.

From TRC 373 and Kyte's original data, it appears that the following conditions would be important in estimating capacity:

- One left-turning vehicle among the subject and opposing approaches,
- A left-turning vehicle at both the subject and opposing approaches,
- One right-turning vehicle among the subject and opposing approaches,
- A right-turning vehicle at both the subject and opposing approaches,
- A second vehicle on the subject approach (multilane approaches),
- One vehicle on the opposing approach, and
- Two vehicles on the opposing approach (multilane approaches).

Some of these conditions are mutually exclusive. All of these conditions apply to either the subject or opposing approach, as there is no need to deal separately with turning on conflicting approaches in this queueing model.

In addition, it is plausible that the adjustment to conflicting approach minimum headway, t_c, would be related to intersection geometry. Multilane approaches would require conflicting vehicles to travel greater distances before yielding priority to a vehicle on the subject approach.

A set of preliminary values for t_b, t_r, and t_c in units of seconds is given in Table 1. These parameters were chosen to match the capacity as given by the raw data and regression equations of Kyte and Marek (2). Because of the major conceptual differences between Kyte's regression models and queueing models, these parameters were obtained by trial and error and should be considered accurate to just 0.25 sec.

Some observations about the parameters given in Table 1 would help interpret the queueing model. First, the difference between the two values of t_c (1.0 sec) is roughly the amount of time it would take a vehicle to travel past the additional width of a four-lane road. Note that t_c is subtracted from the value of the average minimum headway of the conflicting approaches, so its value should become smaller as the intersection becomes wider. Second, the parameters of the first

TABLE 1 Preliminary Parameters for Enhanced M/G/1 Queueing Model for Delay at AWSC Intersections

Traffic Condition	Wait, seconds
Base Minimum Headway, t_b	3.6
Adjustments for Traffic Conditions, t_r	
1. One left vehicle	1.0
2. Two left vehicles	1.0
3. One right vehicle	-0.5
4. Two right vehicles	-1.0
5. Second vehicle on subject approach	1.0
6. One vehicle on opposing approach	0.25
7. Two vehicles on opposing approach	1.0
Adjustments to Conflicting Minimum Headway, t_c	
One-lane approach	0.5
Two-lane approach	-0.5

two conditions (left-turning vehicles) are the same. Because these conditions are mutually exclusive, the model is insensitive to having two left-turning vehicles on two approaches instead of one left-turning vehicle. Third, t_r and t_c have been rounded to the nearest 0.25 sec.

The capacity of an intersection under certain artificial conditions can be estimated directly from the parameters given in Table 1. For example, the capacity of a single-lane approach at a fully saturated intersection with no turning would be exactly 500 vehicles per hour [3,600/(3.6 + 0.25 + 3.6 + 0.25 − 0.5)]. The capacity of a two-lane approach under similar conditions would be 615 vehicles per hour [(2 × 3,600)/(3.6 + 1.0 + 1.0 + 3.6 + 1.0 + 1.0 − 0.5)]. The corresponding values from TRC 373 are 525 and 625, respectively. Some other simple comparisons are given in Table 2.

Unfortunately, many of the conditions that permit hand calculation of capacity are outside the range of TRC 373. Nonetheless, it can be seen that the M/G/1 model can produce reasonable estimates of capacity under a wide range of traffic conditions.

It has been observed that drivers exhibit different stopping behavior (full stop versus partial stop) depending on their intended movement and the presence of a conflicting vehicle. To some degree this behavior is reflected in the parameters given in Table 1. However, randomness in stopping behavior has not been accounted for in this M/G/1 model.

Richardson assigned vehicles to lanes as if they were all identical. At multilane approaches, turning vehicles should be assigned to the most appropriate lane. Thus, Equation 4 must be made more elaborate to handle the few situations in which there are insufficient through vehicles to balance flows across all lanes.

$$\rho_{ncin} = 1 - \prod_{j=1}^{n} (1 - \rho_{cij}) \qquad (8)$$

where j ranges over all lanes. The lane probabilities, ρ_{cij}, at any given approach are still computed with the same mean service time (similar to Equation 3).

CALCULATION OF DELAY

The revised queueing model contains two choices for calculating delay. First, an estimate of subject approach capacity could be obtained from the average service time as found from Equations 1 to 4, then delay could be calculated from the empirical delay relation in TRC 373. Second, it is possible to use Equation 6 as provided in the M/G/1 queueing model. Regrettably, Equation 6 requires knowledge of the variance of service time, which is no longer given by Equation 5.

Finding an exact expression of the variance of service time would be very difficult. However, it is possible to find a good approximation of the variance by discarding minor sources of variation. If the variation associated with turning and coordination in Equation 7 can be ignored, the variance in service time can be approximated by

$$\sigma^2 \approx (t_m - t_o\rho_c)^2(1 - \rho_c) + [t_m + t_z - t_c + t_o(1 - \rho_c)]^2\rho_c - s^2 \qquad (9)$$

where t_o is mean time necessary for coordination between vehicles on subject and opposing approaches when there is a conflicting vehicle, and t_z is the maximum of the average headways on conflicting approaches when there is a conflicting vehicle. Note that t_m now applies only to the subject approach, as each approach can have a distinctly different average minimum headway. Because Equation 9 ignores many sources of variation, its results are assured to be smaller than an exact expression.

A quick inspection of Equation 6 shows that some error in the variance might be tolerable. The question arises: How much error exists in Equation 9 and what is the effect of this error on delay?

ESTIMATING ERROR IN APPROXIMATE VARIANCE

In absence of an exact expression for the variance of service time, a Monte Carlo simulation was used to calculate the correct variance. Separate sets of simulations were performed for one-by-one, one-by-two, and two-by-two intersections. The following procedure was used.

1. Generate randomly a set of movement volumes for an intersection;
2. Determine the average service time and the utilization ratios of each approach with Equations 1 to 3, 7, and 8;

TABLE 2 Comparison of M/G/1 Capacities with TRC 373 Capacities

Traffic Condition	M/G/1	TRC 373
Single lane; no turning; all approaches at capacity	500.	525.
Double-lane; no turning; all approaches at capacity	616.	625.
Single-lane; no turning; no conflicting traffic; subject and opposing approaches at capacity	935.	950.*
Double-lane; no turning; no conflicting traffic; subject and opposing approaches at capacity	1286.	1050.*
Single-lane; no turning; only subject approach has volume	1000.	1100.*
Double-lane; no turning; only subject approach has volume	1565.	1200.*
Single-lane; all approaches at capacity, 25% left turns	446.	375.
Single-lane; all approaches at capacity, 25% right turns	535.	650.

*Outside the stated range of TRC 373 capacity relation.

3. Stop if the capacity of any approach is exceeded running the M/G/1 model;
4. Determine the service time for 2,000 separate vehicle arrivals at the subject approach with a random set of vehicles at an intersection;
5. Calculate the sample mean and the sample variance of the service time;
6. Estimate the delay from the sample mean and variance; and
7. Calculate the delay from the Equation 9 and the computed information from Step 2.

This procedure was applied 10,000 times for each of the three intersection geometries. After eliminating unfeasible volumes in Step 3, there remained 55 cases for the one-by-one intersection, 90 cases for the one-by-two intersection, and 183 cases for the two-by-two intersection. Because the random set of vehicles for each trial in Step 4 were created with the probabilities obtained in Step 2, the results of the Monte Carlo simulation should be quite consistent with the analytical model.

The results for the two-by-two intersection are illustrated in Figures 1 and 2. The results for the other two intersections are similar. The anticipated underestimate of the variance is apparent for all intersections, but the effect on the calculation of delay is small.

When interpreting Figures 1 and 2 it is helpful to note that there is no consistent relation between variance and delay or between the error in variance and delay.

If desired, the small downward bias in delay could be removed by simply adding a correction factor to the Equation 9. The correction factor would be approximately 0.7 sec^2 for a two-by-two intersection, 0.5 sec^2 for a one-by-two intersection and 0.2 sec^2 for a one-by-one intersection. The delay values shown in Figure 2 are uncorrected. Because correction factors are dependent on the chosen set of parameters, they would need to be rederived with a Monte Carlo simulation each time a new set of parameters was selected.

DELAY COMPARISON

Figure 3 illustrates the difference between the TRC 373 delay relation and the M/G/1 queueing model. The random data are for the one-by-two intersection, but the other intersections look similar. The volume-to-capacity ratios were calculated from the queueing model. The two relations produce similar results for delays between 5 and 40 sec. The M/G/1 delay relation does not give any delays less than 4 sec, which is unreasonable. The TRC 373 delay relation did not estimate delays greater than 45 sec, even when the volume-to-capacity ratio approached 1.0.

QUICK COMPARISON WITH TRC 373

A comparison between the TRC 373 and the revised M/G/1 queueing models is given in Table 3. Clearly, the choice of model depends on the trade-off between ease of use versus range of applicability. TRC 373 would be preferred for cases in which hand calculation is required. The revised M/G/1 queueing model would be preferred for computerized assessment of capacity and delay.

CONCLUSIONS

Richardson's M/G/1 queueing model ignored certain aspects of driver behavior at AWSC intersections, which now are

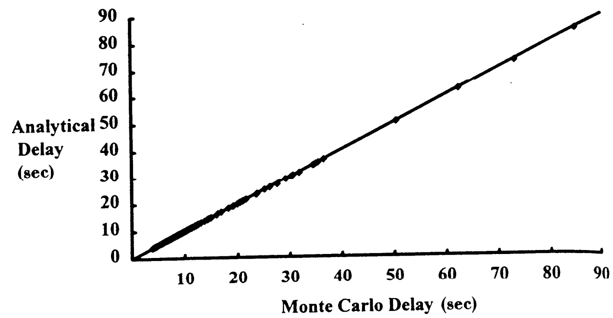

FIGURE 2 Comparison of analytical with Monte Carlo delay for two-by-two intersection.

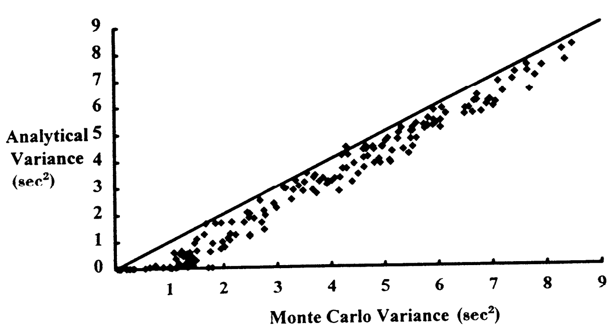

FIGURE 1 Comparison of analytical with Monte Carlo variance for two-by-two intersection.

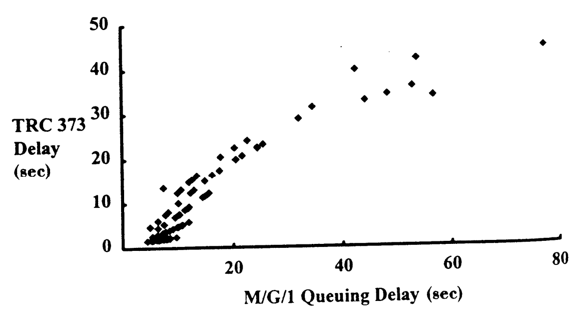

FIGURE 3 Comparison of TRC 373 delay relation with revised M/G/1 queueing model for one-by-two intersection.

TABLE 3 Summary of Model Characteristics

Attribute	TRC 373	Queuing Model
Type of model	Empirical	Theoretical
Ability to provide accurate estimates of capacity and delay over its range of applicability	Good	Good
Range of applicability	Many common situations	Most situations
Number of parameters	9	9
Physical interpretation of parameters	Some	All
Amount of required data	12	12
Hand calculation possible	Yes	No
Ease of understanding	Easy	Difficult

*Outside the stated range of TRC 373 capacity relation.

considered important. These aspects relate to turning and to coordination among drivers who get priority at the same time. A revised model can include a more comprehensive representation of driver behavior; however, it is not possible to obtain an exact formula for the variance of the service time, which is a key variable for the calculation of delay.

Fortunately, a good approximation for the variance of service time is available. The accuracy of the approximation can be verified by Monte Carlo simulation. Furthermore, it is possible to use the results of the Monte Carlo simulation to correct a bias in the approximation that tends to cause a slight underestimate of delay.

The revised queueing model can be made reasonably consistent with the relationships in TRC 373, and it applies to a wider variety of traffic conditions. The queueing model has the same number of parameters as the relations in TRC 373 and the queueing model's parameters are judged to have a stronger physical interpretation. Data requirements are identical.

The queueing model is recommended for computerized evaluation of AWSC intersections having a wide range of traffic conditions.

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