IMPLEMENTATION OF OPERATIONAL NETWORK EQUILIBRIUM PROCEDURES

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Operational network equilibrium procedures are being developed for the fixed-demand single-mode case. The basis of these procedures is described in the light of the historical development of approaches to the problem of predicting equilibrium flows in transportation networks. The procedures are described as capacity restraint methods that have the following advantages over traditional approaches: On each iteration, improvement of the solution is ensured; and following each iteration, a measure that indicates the maximum amount of error remaining can be calculated. This paper describes network equilibrium procedures being made operational as a part of the UMTA Transportation Planning System. These procedures are described in light of their theoretical and mathematical background. Although significant theoretical work has been done on the variable-demand network equilibrium problem, the first developmental step being taken is to provide an efficient fixed-demand equilibrium procedure. It is expected, however, that expansion to the variable-demand case will be possible within the general algorithmic framework being developed. The paper begins by stating the general (variable-demand) network equilibrium problem. This problem is then formulated mathematically, and the nature of its solution is discussed. Previous work to develop efficient solution techniques is discussed. The results of much of the previous work are summarized as a general equilibrium algorithm for the fixed-demand problem. Finally, based on this general algorithm, current development work is described.

•THE PROBLEM of predicting flow equilibrium in transportation networks is in determining the values of interzonal flows and costs and link flows and costs. (Cost is used in a very general sense to represent in a single variable a combination of things such as travel time, fares, operating expenses, and discomfort.)

These are the output variables; the inputs are the structure of the transportation network, sets of link supply and interzonal demand functions, and flow distribution rules. Because deterministic and static, or steady-state, inputs are used, the output variables are also deterministic and static. They therefore represent constant or average conditions over a period such as a peak hour or an average day.

The components of the inputs to the flow equilibrium problem listed above are described as follows:

1. The network—A network is composed of nodes and ordered pairs of these nodes termed links. Links connect two nodes and allow flow to occur in only one direction between them. Some of the nodes are zones at which trips enter and leave the network.
2. Supply functions—Each link has associated with it not only a flow but also an impedance to flow in the form of a travel time or generalized cost. The relationship between link flow and link cost is expressed by a supply function that indicates how cost increases as flow increases. Typically, the supply function for each link may have an asymptote at a maximum flow level or capacity.

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3. Demand functions—Each zone pair has associated with it a demand function that relates the origin-destination (O-D) travel cost to the volume of travel that will flow from origin to destination. In the variable-demand case, this volume of travel decreases as the cost increases. In the fixed-demand case, the volume of travel remains constant for all levels of cost.

4. Flow distribution rules—A flow distribution rule that describes how travelers route themselves over links to move from an origin zone to a destination zone is assumed to exist. This rule can imply either individual route choice, systemwide control of route choice, or some combination of these. For the representation of highway travel by private vehicles, the common assumption is that individuals choose a minimum cost route. The results of this flow distribution rule are that all routes chosen from any origin to any destination will have equal travel costs and that all other routes will have higher travel costs. These results are termed Wardrop's first principle (26) or a user-optimized flow pattern (4).

For the representation of travel by vehicles belonging to a single authority, such as a railroad providing freight service, a logical assumption is that the single authority wishes to maximize its total consumer's surplus and that its flow distribution rule is to make routing decisions with this objective in mind. The result of this flow distribution rule has been termed Wardrop's second principle or a system-optimized flow pattern.

Our concern is with the prediction of user-optimized flow patterns, although the relationships of these two flow patterns will also be explored. When user-optimized flow patterns are obtained, Wardrop's first principle states that there will be a unique travel cost for each zone pair.

MATHEMATICAL FORMULATION

The mathematical relationships that exist between the components of the equilibrium problem will be detailed here for the user-optimized problem. This has been done in the literature in a number of ways, based on alternative mathematical descriptions of network flows. The approach used here has been borrowed largely from Kulash (11).

The following notation is used:

a = a typical link connecting two nodes,
k = a typical O-D pair,
m = a typical path for a given O-D pair, and
P_{km} = $(a_1, \ldots, a_n)$ = the set of links on path m connecting O-D pair k.

The links included in each P_{km} constitute a single path from the origin to the destination of k. This path must be free of loops, and all links included in the path must be used in proceeding from origin to destination.

This notation can be used to define the following variables:

f_a, c_a = flow and cost on link a,
f^{km}, c^{km} = flow and cost on path m for O-D pair k, and
f^k, c^k = flow and cost for O-D pair k.

The relationships between these variables are the following:

1. The network structure gives rise to flow relationships for interzonal flows:

$$f^k = \sum_{\text{all } m} f^{km} \tag{1}$$

for all k; for link flows:

$$f_a = \sum_{\text{all } k,\, m} f^{km} \tag{2}$$

for all a and for which $a \in P_{km}$; and for path costs:

$$c^{km} = \sum_{\text{all } a} c_a \tag{3}$$

for all k,m and for which $a \in P_{km}$.

2. The supply relationships are

$$c_a = s_a(f_a) \tag{4}$$

for all a where s_a is a function.

3. The demand relationships are

$$f^k = d^k(c^k) \tag{5}$$

for all k where d^k is a function.

4. The flow distribution rule, for a user-optimized flow pattern, gives rise to the following equilibrium relationships:

$$c^{km} \begin{Bmatrix} = c^k & \text{if } f^{km} > 0 \\ \geq c^k & \text{if } f^{km} = 0 \end{Bmatrix} \tag{6}$$

for all k.

The properties of the solution to the user-optimized network equilibrium problem can be obtained by defining an equivalent optimization problem. This can be done as follows:

1. For each demand function d^k (Eq. 5), define the inverse function g^k such that

$$c^k = g^k(f^k) \tag{7}$$

2. Define a new link function, $\underline{S}_a$, as follows:

$$\underline{S}_a(f_a) = \int_0^{f_a} s_a(x)dx \tag{8}$$

3. Define a new interzonal function, $\underline{G}^k$, as follows:

$$\underline{G}^k(f^k) = \int_0^{f^k} g^k(x)dx \tag{9}$$

The equivalent optimization problem is then

$$\text{Maximize } Z = \sum_k \underline{G}^k(f^k) - \sum_a \underline{S}_a(f_a) \tag{10}$$

subject to Eqs. 1 to 3.

This equivalence is proved by a number of mathematicians, including Gibert (9) and Murchland (17) for the general case and by Dafermos (5) for the fixed-demand case. In the fixed-demand case, the function g^k cannot be obtained. There is, however, an analogous optimization problem:

$$\text{Minimize } Z = \sum_a \underline{S}_a(f_a) \tag{11}$$

subject to Eqs. 1 to 3.

After the equivalency of the two problems has been demonstrated, the mathematics of nonlinear convex programming was used to prove that the solution of both problems exists, is unique, and is stable.

REVIEW OF EQUILIBRIUM APPROACHES

Problem Formation

It appears that the first recognition of the difference between user-optimized and system-optimized network flows was described by Pigou in 1920 (20), who demonstrated for a simple two-link, two-node network. Current interest in the problem, however, dates from Wardrop's statements of the two kinds of problems in 1952 (26).

Subsequent work on the formulation of the network equilibrium problem was done by Beckmann, McGuire, and Winsten (1), Prager (21), and Jorgensen (13). Jorgensen showed that if the supply functions (Eq. 4) are used to define a new set of functions $S_a^{*}(f_a)$ by using the relation

$$S_a^{*}(f_a) = 1/f_a \int_0^{f_a} S_a(v)dv \qquad (12)$$

then any flow pattern that is user-optimizing with respect to the set of cost functions $S_a(f_a)$ is at the same time system-optimizing with respect to the set of cost functions $S_a^{*}(f_a)$.

Solution Procedures

Based on the foundations laid in the 1950s and early 1960s, solution procedures have been developed that can be divided into four general classes: traffic assignment approaches, mathematical programming approaches, algorithmic approaches with fixed demands, and algorithmic approaches with varying demands.

Traffic Assignment Approaches—This class of solution procedures has by far predominated the other classes in actual application and in number of variants. [For an early survey, see Martin, Memmott, and Bone (15). The most common methods are described in the FHWA Traffic Assignment Manual (8).] Here, it is only necessary to note the major deficiencies of these approaches as methods for solving the network equilibrium problem:

1. Link travel times have often been kept constant, thereby ignoring the existence of link supply functions;
2. Origin-destination trips have often been kept constant, thereby ignoring the existence of travel demand functions;
3. The number of paths traveled between each origin and destination has often been limited to one, making it impossible, normally, to satisfy Wardrop's first principle;
4. The accuracy of the approaches as approximations of equilibrium has not been determined (this includes both their convergence properties, if they involve iterations, and their expected errors upon completion).

These deficiencies are not inherent in the traffic assignment process, and all of them are not true for each assignment procedure. Indeed, the procedure developed by Martin and Manheim (14), and implemented in transportation analysis systems at M.I.T. (14, 22), has only the last deficiency mentioned. Similarly, the package of assignment programs developed by Wigan (27, 28) includes procedures that have all features listed above except proven convergence properties.

Mathematical Programming Approaches—Charnes and Cooper (3) have developed linear programming solutions to network equilibrium problems with fixed demands. Their contribution is the multicopy assignment algorithm, which takes advantage of the specific structure of the linear program they formulate.

Yang and Snell (30) formulated a nonlinear equilibrium problem with fixed demands and developed a solution algorithm based on the maximum principle of Pontryagin. Tomlin (24) formulated a quadratic programming problem involving both the assignment of traffic and the distribution of trip ends over all destinations by using a gravity model. The major problem with all mathematical programming approaches is the prohibitive solution cost for real-sized problems.

Algorithmic Approaches With Fixed Demands—Three major efforts are known that have led to the development of network equilibrium algorithms for the fixed-demand case. These algorithms are significantly more efficient than the mathematical programming approaches. In each case, the improvement over programming approaches is obtained by using each of the following features of the network equilibrium problem:

1. The relationship between the system-optimizing and user-optimizing problems;
2. The theorems of mathematical programming, which are applicable because of the first feature; and
3. The process actually used by travelers to progress to equilibrium.

Expanding on Jorgensen's work, Mosher (16) was the first to formulate the user-optimizing equilibrium problem explicitly and to develop a solution algorithm that can be shown to converge. Dafermos and Sparrow (6) and Dafermos (4, 5) have developed more general algorithms. These algorithms are not limited to linear functions and have been extended explicitly to cases where the supply functions are of the following form:

$$c_a = S_a(f_1, \ldots, f_J) \tag{13}$$

where $f_1, \ldots, f_J$ are a subset of all network links. This extension is useful for representing delays due to two-way traffic on facilities and to intersection flows. A second extension involves the definition of multiple user groups, which can represent different vehicle types or users of different modes. A third set of algorithms for the fixed-demand case has been developed by Bruynooghe, Gibert, and Sakarovitch (2, 10). Their major advance is the elimination of the need to specify paths prior to the start of the procedure. New paths are found as the algorithms progress by using a minimum path procedure.

Algorithmic Approaches With Varying Demands—A number of algorithms have been developed to obtain solutions to the general problem of user-optimized network equilibrium when both demands and supplies vary with travel cost. These are very recent developments developed since 1967.

As an extension of the final fixed-demand algorithm described previously, Gibert (9) developed what appears to be the first variable-demand algorithm with proven convergence properties. Expanding on the work of Gibert, Murchland (17) has described the network equilibrium problem with varying demands in a way that explicitly brings out the relationships between the system- and user-optimized problems. Rather than specify exactly the steps of an algorithm, Murchland gives four principles for their development and states that a number of algorithms should be developed based on these principles and then tested to determine the most efficient one. The principles stated are the following:

1. The algorithm should have as its goal the minimization of either the equivalent system-optimizing problem or its dual. Murchland suggests the use of an error indicator, δ, which is the difference between the objective functions for these two problems.
2. Because these two objective functions are equal at equilibrium, the algorithm can be stopped when δ is sufficiently small.
3. As the algorithm continues, δ can be minimized by forming linear combinations of old and new flow patterns.
4. Because the final solution will typically have flows on a number of paths between all origins and destinations, any single iteration method that will assign flows to a number of paths should improve the speed of convergence.

Murchland has used these principles to develop a research-oriented network equilibrium computer program.

Two approaches to network equilibrium with varying demands have been developed in the United States. The first, by Wilkie and Stefanek (29), applies control theory to the user-optimized equilibrium problem. The second approach, by Kulash (11), is an initial effort involving only linear equations.

Finally, an algorithm has been developed by Netter and Sender (18, 19) that addresses explicitly the multiple user group, multiple dimensioned supply function (as in Eq. 13), and multiple dimensioned variable demand function problem. Netter and Sender show that multiple solutions exist unless the supply functions have a very restricted form. The algorithm is shown to converge to one of the multiple solutions; which one depends on the starting point chosen.

A GENERAL EQUILIBRIUM ALGORITHM FOR FIXED DEMANDS

The previous work done in developing network equilibrium solution procedures can be summarized by stating the features of these procedures that are essential to ensure convergence to the equilibrium solution, as agreed on by a number of authors, and that represent a minimum departure from existing traffic assignment procedures. (The restriction to minimum departures from existing procedures eliminates further consideration of approaches involving significantly more flow variables than used in traffic assignments.) These will be presented within the general algorithmic framework developed by Murchland (17), inasmuch as it can be applied to the fixed-demand problem (all f^k's fixed). After the elements of this framework are listed, some of the options available for each element will be described, emphasizing the suitability of existing production-oriented procedures as parts of equilibrium algorithms.

1. Step 1—Develop an initial network solution, S.
2. Step 2—Determine the best direction in which to proceed to obtain a new trial solution.
3. Step 3—Develop a trial solution, S_t.
4. Step 4—Use an optimization procedure to obtain the best next solution, as a combination of S and S_t. Symbolically, $S = C(S, S_t)$ where C is some combination.
5. Step 5—Determine whether S is a satisfactory final solution. If it is not, return to step 2.

Step 1—Initialization

Because any solution for which the flow conservation relationships hold is appropriate, this step can be accomplished very efficiently by assigning total demands in an all-or-nothing manner to the minimum cost paths corresponding to zero flow. This step concludes with an updating of all link and O-D cost variables. Normally, O-D cost variables will be set equal to the cost on the new minimum path for the O-D pair.

Step 2—Direction for Trial Solution

A new demand level for each O-D pair can best be obtained by adopting a value that equals the old value plus a fraction of the difference between the old value and the value predicted by the demand function at the current minimum path cost.

A number of authors show that the path over which new travel should occur for each O-D pair is the minimum cost path; its choice is assumed in the proofs of convergence. As an alternate, a multiple-path approach, using the link travel costs on the previous solution, can be used. A multiple-path solution for which the average travel cost is less on these new paths than on the old paths, using the old set of link costs, will also be satisfactory.

An important option for the whole algorithm is whether new solutions are developed separately for each O-D pair or at one time for the entire system. The choice of this option will determine whether steps 2, 3, and 4 are done in sequence separately for each O-D pair, or just one time, with an O-D pair loop within each step.

Step 3—Develop Trial Solution

With the directions developed in step 2, the trial solution can be developed by using standard loading and link cost updating procedures to determine all flow variables (new f_a and updated c_a and c^k) associated with this trial.

Step 4—Combine S and S_t to Obtain a New Solution

This is the critical step, because it is here that all existing capacity restraint methods fall short of being network equilibrium procedures with proven convergence properties. For convergence, it is necessary that the proportions of old and trial solutions be determined by the procedure itself rather than by the analyst.

A number of options exist with respect to the nature of the combination method, including the characteristics of the function itself, and the procedure for choosing the parameter of this function:

The Combination Function—If O-D pairs are considered separately, which requires saving the route of each path through the network and the corresponding path volume, then two combination functions are suggested:

1. A transfer of volume from the longest path for an O-D pair to the shortest as suggested by Dafermos and Gibert.
2. An increase in volume on the shortest path (a fraction of the trial solution) plus a proportional decrease on all previous paths as suggested by Murchland.

If only systemwide flow changes are made, no path volumes and routes need be saved. Then the only feasible combination function appears to be one corresponding to 2 above, a linear combination of the trial solution and the former solution.

The Combination Function Parameter—If O-D pairs are considered separately and the combination method of 1 above is used, the amount of volume shifted can be calculated based on maximizing the improvement to the objective function, Z (Eq. 11).

If combination method 2 is used or if systemwide flow changes are made, the fraction of the new flow to add to the remaining portion of the old flow can be obtained either by maximizing the change in the objective function, Z, or by minimizing an error measure for the new solution. The details of the former approach are described in the next section. After a new solution is obtained, all link and O-D cost variables should be updated to represent the new flows.

Step 5—Apply Stopping Rule

A number of stopping rules can be envisioned. These will take different forms depending on the method used to determine the combination parameter in step 4:

1. Stop when the change in the objective function Z is small compared to Z itself: $(\Delta Z/Z) \leq \epsilon$.
2. Similarly, if an error function is used, stop when the change in the function is small compared to the function value itself.
3. Stop when a specified number of iterations have been performed.
4. Stop when a specified computing cost, measured in dollars or CPU minutes, has been spent.

Whichever stopping rule is used, the final printout should include measures of the remaining error.

It is useful to summarize the various components that can be used to provide the options discussed and to state their availability.

1. Efficient minimum path, link loading, and link updating capabilities are available in a number of traffic assignment packages. One of these, Dial's STOCH procedure (7), provides an efficient multiple-path assignment capability.
2. A variant of the ability to form linear combinations of two sets of link loadings is included in Wigan's system.
3. The ability to obtain an error measure for any flow pattern that indicates its maximum variation from an equilibrium solution and the nature of such measures are discussed by Murchland and Wigan.
4. The ability to determine the fraction that should be used in forming a linear combination of two flow patterns so as to minimize the error measure is discussed by Murchland, Gibert, and Dafermos.

AN OPERATIONAL ALGORITHM

A number of the options described previously are being investigated, in preparation for specifying additions to the UTP system to incorporate network equilibrium. The basic algorithm serves as the standard of comparison for the efficiency and accuracy of all options developed. This algorithm is basic in that it makes maximum use of available assignment procedures and data structures. Alternatives to this basic algorithm will be judged by comparing their benefits—in terms of increased efficiency and accuracy—to their costs in terms of extra development time and, in some cases, computer storage requirements.

The basic algorithm is described in this section, and the following notation is used:

P^i = set of minimum paths between all zone pairs k, for iteration i;
F_i = set of link flows for all links a, for iteration i;
C_i = set of link costs for all links a, for iteration i;
M_i = set of link supply function slopes for all links a, at the flow levels F_i;
ΔZ_i = change in value of the objective function (Eq. 11); and
$S_a(x)$ = supply function for link a, evaluated at flow level x and

L_1, L_2, L_3, L_4 are analyst-supplied parameters.

Step 1—Initialization

Perform an all-or-nothing assignment to the minimum paths corresponding to zero flows on all links [P^0 based on $C^0 = S_a(0)$]. The result will be F_1. Then, update all link costs to correspond to F_1, yielding C_1. At the same time,

1. Estimate supply function slopes at the current flow levels by performing the following calculation for each link a:

$$m_{1a} = \frac{S_a(1.01\, f_{1a}) - c_{1a}}{0.01\, f_{1a}} \tag{14}$$

2. Estimate the initial value of the objective function, Z_1.

$$Z_1 = \tfrac{1}{2} \sum_a f_{1a}(c_{1a} + c_{0a}) \tag{15}$$

Set i = 1 and α, the initial combination size, equal to L_α. Finally, print i and Z_1.

Step 2—Determine Trial Solution Direction

Find new minimum paths, P^{i+1}, based on the link costs C_i.

Step 3—Develop Trial Solution

Assign all travel to the paths P^{i+1}, yielding flows F_t for the trial solution.

Step 4—Obtain New Solution

The parameter λ is determined to (approximately) minimize the (positive) change in the objective function. As derived in the Appendix, the expression for λ (Eq. 25) is

$$\lambda = - \frac{\sum_a c_{ia} \Delta f_a}{\sum_a m_{ia} (\Delta f_a)^2} \tag{16}$$

where

m_{ia} = slope of the supply function for link a at flow level f_{ia} and
$\Delta f_a = f_{ta} - f_{ia}$.

As discussed in the Appendix, λ must be limited to the range $0 < \lambda < 1$. It is shown in the Appendix that Eq. 16 cannot result in a value of λ less than zero. If $\lambda > 1$, λ should be set equal to 1.

Form a new solution, F_{i+1}, by combining F_i and F_t. For each link, this involves

$$f_{i+1,a} = (1 - \lambda) f_{ia} + \lambda f_{ta} \tag{17}$$

Update all link costs to correspond to F_{i+1}, yielding C_{i+1}. At the same time, reestimate supply function slopes between solutions i and i+1 by performing the following calculation for each link, a:

$$m_{i+1,a} = \frac{c_{i+1,a} - c_{ia}}{f_{i+1,a} - f_{ia}} \tag{18}$$

Also, calculate the final estimate of the change in the objective function, ΔZ, and the new value of the function Z_i.

$$\Delta Z = \tfrac{1}{2} \sum_a (f_{i+1,a} - f_{ia})(c_{i+1,a} + c_{ia})$$
$$Z_i = Z_{i-1} + \Delta Z \tag{19}$$

Print i, ΔZ, Z_i, and λ; and set i = i+1.

Step 5—Apply Stopping Rules

The procedure is stopped and the desired assignment outputs are generated if any of the following are true:

1. $-(\Delta Z/Z) \leq L_1$,
2. $-\Delta Z \leq L_2$,
3. $i = L_3$, or
4. CPU minutes $\geq L_4$.

If none of these is true, return to step 2.

CONCLUSIONS

A review of the literature on the network flow equilibrium problem indicates that the problem has a number of interesting properties that are useful in developing solution algorithms. Included are the existence, uniqueness, and stability of a solution and the equivalency of the user-optimized problem and a system-optimized problem. A number of solution algorithms have been developed, and their convergence to a true equilibrium solution can be proved. A number of these algorithms can be made operational by putting together standard components of transportation network analysis systems and simple new evaluation tools. The kinds of computations to be performed by these tools are described in operational terms. The computation costs of these algorithms are expected to be comparable to those of existing restrained capacity assignment procedures.

A basic algorithm is described that involves minimal departures from existing capacity restraint procedures. Efficiency and accuracy results obtained for this algorithm are being used as a base point against which to compare more innovative algorithms.

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APPENDIX

DERIVATION OF THE COMBINATION PARAMETER λ

The combination parameter λ is to be determined to approximately minimize the positive change in the objective function, Z (Eq. 11). This change, ΔZ, is made up of components for each link a, such as the shaded area shown in Figure 1. When Δf_a is positive, as shown in the figure, the contribution to $\Delta Z (\Delta Z_a)$ is positive. Similarly, when Δf_a is negative, ΔZ_a is negative.

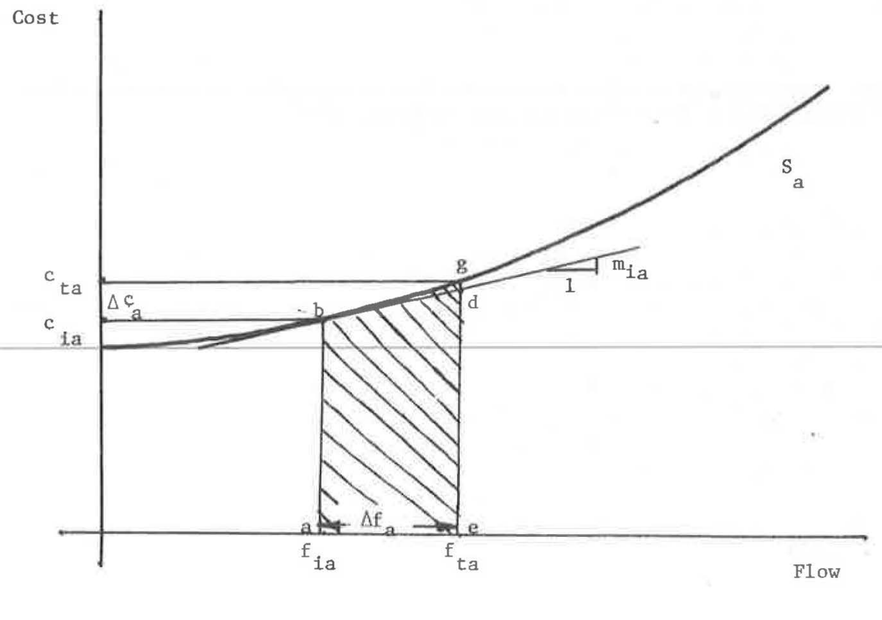
To avoid the necessity of determining c_{ta}, we approximate point g by point d, which can be determined from the following relationship:

$$\Delta c_a = m_{ia} \Delta f_a \tag{20}$$

Then, using the area abde as an approximation for ΔZ_a, we obtain

$$\Delta Z_a = \frac{1}{2} \Delta f_a (2 c_{ia} + m_{ia} \Delta f_a) \tag{21}$$

Figure 1. Relationships used to calculate ΔZ_a for a typical link.



This quantity can be summed over all links to obtain the total change.

$$\Delta Z = \sum_a \tfrac{1}{2} \Delta f_a (2\, c_{1a} + m_{1a} \Delta f_a) \tag{22}$$

To find λ requires that ΔZ be defined as a function of λ. This can be done by replacing Δf_a in Eq. 22 with $\lambda \Delta f_a$, resulting in

$$\Delta Z(\lambda) = \sum_a \tfrac{1}{2} \lambda \Delta f_a (2\, c_{1a} + \lambda m_{1a} \Delta f_a) \tag{23}$$

The valid range for λ is $0 \leq \lambda \leq 1$.

The optimum value for λ can be found by differentiating $\Delta Z(\lambda)$ with respect to λ and setting the derivative equal to zero.

$$0 = \frac{\delta \Delta Z(\lambda)}{\delta \lambda} = \sum_a c_{1a} \Delta f_a + \lambda \sum_a m_{1a} (\Delta f_a)^2 \tag{24}$$

Solving for λ gives the following expression:

$$\lambda = - \frac{\sum_a c_{1a} \Delta f_a}{\sum_a m_{1a} (\Delta f_a)^2} \tag{25}$$

Note that the denominator is always positive. If the numerator is not negative, λ will be negative. This will only occur if the trial solution, evaluated at the former costs C_{1a}, is not so good as solution i. This cannot occur because flows are being shifted from higher to lower cost paths—at the current costs—in steps 2 and 3.

If the value of λ from Eq. 25 is greater than 1, then λ should be set equal to 1. This implies that all of the former solution is being replaced by all of the trial solution t.