DESIGN OF DENSITY-MEASURING SYSTEMS FOR ROADWAYS

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A Kalman filtering methodology for the estimation of traffic densities on multilane roadways is tested by using aerial photography data. The method gives very satisfactory estimates even when the sensor separation is as great as 3,000 ft. A systematic procedure is given for designing and calibrating a density-measuring system for a roadway.

•A SERIES OF PAPERS (1, 2, 3) demonstrated that it is possible to obtain accurate estimates of the number of vehicles on a section of a roadway by using the technique of Kalman filtering on noisy measurements of flow and velocity taken by sensors at the entrance and exit of the section. The technique was used with sensor data obtained at the Lincoln Tunnel in New York City during the tunnel control experiment (4) carried out from 1966 to 1969. The data corresponded to the particular configuration and conditions of the experiment; namely, the sensors were placed about ½ mile (0.8 km) apart, and lane-changing was illegal and therefore infrequent (but not altogether absent).

Many questions arose as a result of the investigation discussed in the series of papers. Would the method work equally well in a freeway environment where lane-changing is much more frequent? How close must the sensors be placed in order to yield accurate density estimates? What are the trade-offs between cost and accuracy? The purpose of this paper is to provide some answers to these questions. A partial answer to the question of the effectiveness of the Kalman filtering approach in measuring freeway densities has been provided by the successful use of a somewhat different Kalman filtering procedure by Nahi and Trivedi (5).

The data used in this paper were obtained by the System Development Corporation through an analysis of aerial photographs of sections of the Long Island Expressway taken at the rate of one frame every 2 seconds. Using these data, we could assume the placement of imaginary sensors at any location of the observed section of the expressway, which was about ¾ mile (1.2 km) long. The aerial data also provided exact counts of vehicles between the assumed location of sensors, and these counts were used as a benchmark for estimating the effectiveness of the density estimation algorithm. In view of the nature of the aerial data, the assumed sensors gave a very accurate measurement of the flow and a reasonably accurate measurement of the speed of each vehicle past the sensor point. Noisy flow measurements could be simulated by the addition of noise to the aerial data for flow. The density estimation algorithm used (1) in this investigation was chosen because of its advantage of linearity in both the state equations and the observation equations entering in the Kalman filtering procedure and because of the satisfactory quality of what is taken as observation.

PROBLEM STATEMENT

Let us briefly review the essential features of the density estimation algorithm. Suppose there are $N+1$ sensors placed at discrete distances on a roadway for the measurement of count and velocity and that these sensors divide up the roadway into N sections identified by the index of the upstream sensor. Let Y_k^i be the number of vehicles in section i at time k, where $i = 1, 2, \ldots, N$ and $k = 1, 2, 3, \ldots$, and let n_k^i be the number of vehicles that pass over sensor i between times k and $k+1$, where $i = 1, 2, \ldots,$

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N + 1 and k = 1, 2, 3,.... Then the following equation, which merely states the conservation of cars, should be true:

$$y^i_{k+1} = y^i_k + n^i_k - n^{i+1}_k \tag{1}$$

If there are no lane-changing effects or errors on the flow measurements, Eq. 1 is an exact description of the transition from time k to time k+1. However, in general it is necessary to add a "noise" term in Eq. 1 to account for occasional deviations, yielding the state equation

$$y^i_{k+1} = y^i_k + n^i_k - n^{i+1}_k + w_k \tag{2}$$

where w_k is the noise term, which is assumed to have the following statistical properties:

$$\left.\begin{aligned} E[w_k] &= 0 \\ E\{w_j w_k\} &= \begin{cases} 0, & j \neq k \\ Q, & j = k \end{cases} \end{aligned}\right\} \tag{3}$$

From the measurements of velocities and counts at the entrance and exit of section i, one can generate a rough measurement of y^i_k, which we will call z^i_k, using the travel-time algorithm (1). The travel-time algorithm involves smoothness arguments on the distribution of velocities inside the section, and these arguments allow the computation of some average travel time for the cars that enter section i between times k and k + 1, and hence also for the last car to exit during this period. All the cars that entered the section while the last vehicle was traversing it are assumed to be still within the section. As might be expected, the rough count z^i_k deviates from y^i_k, and if we denote the difference between y^i_k and z^i_k by v_k, we have the observation equation,

$$z^i_k = y^i_k + v_k \tag{4}$$

Experience has shown that it is not unreasonable to assume that the noise term v_k has the following statistical properties:

$$\left.\begin{aligned} E\{v_k\} &= 0 \\ E\{v_j v_k\} &= \begin{cases} 0, & j \neq k \\ R, & j = k \end{cases} \end{aligned}\right\} \tag{5}$$

From now on, w_k will be referred to as system noise and v_k will be referred to as observation noise. The quantity Q from Eq. 3 is then the variance of the system noise, and R from Eq. 5 is the variance of the observation noise. With the state equation (Eq. 2) and the observation equation (Eq. 4), one can use Kalman filtering techniques to generate optimal estimates, $\hat{y}^i_k$, of the y^i_k. The equations for the Kalman filter are given in the Appendix. Here, it is sufficient to recall that the Kalman filter produces, recursively, best estimates of y^i_k as a weighted average of a value obtained by using the previous best estimate and the flow data, as shown in Eq. 1, and the rough count z^i_k. The weights of the averaging process are the Kalman gain, G_k, defined in the Appendix, and its unit complement.

Before the Kalman filter can be used, four parameters must be set: the initial mean of the state vector, the covariance matrix Σ_0 of the initial state, Q, and R. For any finite run, the accuracy of the filter estimates is dependent on the choice of these parameter values of which Σ_0, Q, and R are particularly important. In practice, it is difficult to guess the best values for Σ_0, Q, and R, especially when one is designing a surveillance system for an unknown environment. A systematic methodology should therefore include a procedure for choosing appropriate values of Σ_0, Q, and R.

In view of these observations and those made in the introduction, we pose the following questions:

1. Is there a systematic way of choosing the initial values for the Kalman filter in this particular application?
2. How sensitive is the algorithm to the distance between sensors?
3. Is there a systematic methodology for designing a surveillance system for roadways?

CHOICE OF INITIAL VALUES OF THE KALMAN FILTER

Because we are working with linear state equations and linear observation equations, the effect of the choice of values for the initial mean of the state vector and the initial covariance matrix Σ_0 on the accuracy of the filter estimates is minimal (6). However, the choice of Σ_0 affects the rate of convergence of the estimates. In practical terms, we are faced with the problem of choosing Σ_0, Q, and R so that the filter will yield good estimates of the state (the number of vehicles between sensors) for a finite run of the algorithm.

At this point we must define how we measure the goodness of the estimates. The Kalman filter is supposed to be an unbiased estimator, but this is never the case in practice. For each section i between sensors, let us define the following measures:

$$m_i = \sum_{k=1}^{T} (y_k^i - \hat{y}_k^i) \tag{6}$$

$$s_i = \sum_{k=1}^{T} (y_k^i - \hat{y}_k^i)^2 - m_i^2 \tag{7}$$

where T is a finite time horizon and y_k^i and $\hat{y}_k^i$ are respectively the number of cars in section i at time k and its corresponding estimate. Over a certain finite time horizon, one always finds a certain amount of bias in the filter estimates. The quantity m_i is this bias and is never zero in practice. It is one measure of the goodness of the estimates. The quantity s_i measures the amount of dispersion of the estimates about the actual values of the state and as such is also a measure of the goodness of the estimates. Experience has shown that by appropriately choosing Σ_0, Q, and R it is possible to bring both $|m_i|$ and s_i down to a certain point, beyond which varying the values for Σ_0, Q, and R will only decrease $|m_i|$ at the expense of increasing s_i, and vice versa. It is therefore necessary to take both m_i and s_i into account when considering the performance of the filter. We therefore chose a measure of error ϵ_i for each section i, defined by

$$\epsilon_i = \tfrac{1}{2} (m_i^2 + s_i)^{1/2} \Big/ \sum_{k=1}^{T} y_k^i \tag{8}$$

The values of the parameters Σ_0, Q, and R are then to be chosen so that ϵ_i is minimized for each section of the roadway.

Experimentation with the choice of these initial values leads to the observation that the accuracy of the filter estimates does not depend on the individual values of Σ_0, Q, or R; instead, it depends crucially only on the choice of the ratio ρ defined by

$$\rho = Q/R \tag{9}$$

By keeping ρ constant, it is possible to vary Σ_0, Q, and R individually by one or two orders of magnitude without affecting the accuracy of the filter estimates. This is an important observation, because it means that in designing a surveillance system for a roadway the engineer has to worry about the choice of the proper value for only one unknown parameter, instead of three, as the equations of the Kalman filter would seem to indicate. This result is also justifiable theoretically. It can be shown that for our particular application of the Kalman filter the steady-state feedback gain G_∞ is of the form

$$G_\infty = \frac{1}{1 + \frac{R}{\Sigma_\infty + Q}} \tag{10}$$

where Σ_∞ is the steady-state variance of the estimator. But $\Sigma_\infty \to 0$ and $G_\infty \to (1 + 1/\rho)^{-1}$, where ρ is given in Eq. 9. In other words, the steady-state gain approaches a function of ρ alone, and, because the steady-state gain is the most crucial factor in determining the accuracy of the filter estimates in the long run, it is not surprising to find the filter performance dependent on ρ alone.

The next logical step at this point is to investigate how the optimal values for ρ relate to the operating conditions of the corresponding sections. In our experiments with the choice of initial values, it was found that the optimal values for ρ (i.e., those values that yield minimum ϵ_i, where $i = 1, \ldots, N$ for the various sections) do not have any systematic relation to the separation between sensors, the mean speed of vehicles in the section, or the mean density of vehicles in the section. They are, however, related to the average frequency of lane-changing in the section. In general, a high frequency of lane-changing inside the section implies a high optimal value for ρ, and vice versa. The results are shown in Figure 1, where the solid line indicates the mean and the dotted lines indicate the spread of the optimal values for ρ versus the level of lane-changing frequency. This dependence of ρ on the frequency of lane-changing is understood if one recalls that ρ is the ratio of the system noise variance to the observation noise variance. System noise is largely caused by lane-changing, and observation noise accounts for the crudeness of the travel-time estimates (used as noisy observations of the state). If the quality of the travel-time estimates is uniform, the observation noise variance is more or less constant, and ρ is a monotonically increasing function of the system noise variance. Because higher frequencies of lane-changing imply higher values for the system noise variance, they also imply higher values for the optimum ρ.

This discussion pertains to the density estimate of a single lane that is affected by lane-changing. If we apply the Kalman filter combining all lanes of a roadway, then the system noise may reasonably be expected to be close to zero, and hence the optimum ρ is also very close to zero.

OPTIMAL SENSOR PLACEMENT

As mentioned, we set up imaginary sensors along the roadway to measure count and velocity and used these measurements to generate estimates of the numbers of vehicles between sensors. We first created several sections of roadway by placing sensors 500 ft (152 m) apart in each of the three lanes of the expressway, chose optimal values for the initial parameters for each section between sensors, and ran the estimation algorithm for all the sections. We then increased the distance between sensors from 500 ft to 4,000 ft (1219 m) in 500-ft steps, repeating the estimation experiment at each step. In the cases of wide sensor separation, overlapping sections were defined to ensure an adequate number of samples. For any given sensor separation, the minimum errors obtained in all the sections were averaged to remove possible effects of geometric peculiarities of the roadway. Our objective was to find out how the accuracy of the estimation algorithm was affected by the separation between the sensors. The results are shown in Figure 2, where the average minimum error is plotted against the sensor separation. It is significant that the error remains almost constant at a low of approximately 12 to 17 percent as the sensor separation is increased from 500 ft to 3,000 ft (914 m), beyond which it starts to rise slowly.

The same experiments were repeated with all three lanes treated together (2). For any given sensor separation, a section was defined across the three lanes, and an estimate was obtained for the total number of vehicles in the three lanes within the section.

Figure 1. Optimum value of ρ versus frequency of lane-changing.

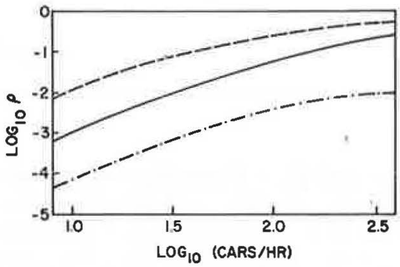

Figure 2. Percent error in density estimation versus sensor separation when all lanes are treated singly.

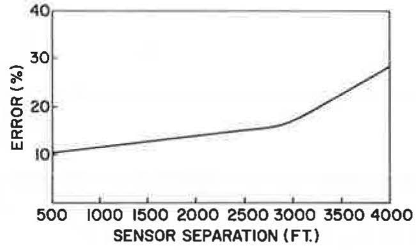

Figure 3. Percent error in density estimation versus sensor separation when all lanes are treated together.

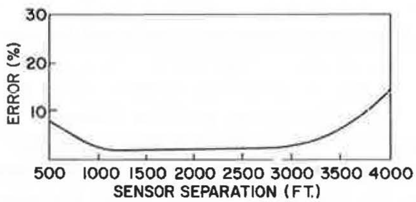

Table 1. Errors in density estimates for different sensor separations.

	Errors (percent)								
Sensor Separation (ft)	Separate Lanes			Combined Lanes			Speed (ft/sec)		
	Max.	Min.	Mean	Max.	Min.	Mean	Max.	Min.	Mean
500	15.0	10.8	11.7	14.3	4.7	8.1	92.2	74.7	84.2
1,800	20.0	7.1	11.8	5.3	0.8	1.8	91.8	75.2	83.1
1,500	21.4	9.7	13.7	7.5	1.9	2.9	91.4	75.2	83.9
2,000	23.8	9.5	14.7	3.5	0.8	1.8	89.2	75.5	82.8
2,500	21.4	10.7	15.4	2.7	1.6	1.8	89.4	76.2	83.2
3,000	22.4	11.1	17.1	—[a]	—	2.1	90.8	76.6	83.6
3,500	32.0	18.0	23.8	—	—	5.7	89.5	76.9	83.4
3,850	45.1	24.8	28.0	—	—	13.1	89.1	77.3	83.0

Note: 1 ft = 0.3048 m; 1 ft/sec = 0.3048 m/s.

[a]The maximum and minimum figures are not given where the sample size was one.

The results, shown in Figure 3, lead to three observations. First, the errors in this case were uniformly smaller than they were when the lanes were treated independently. This is to be expected because by combining the lanes we remove the errors that may result from lane-changing of vehicles. Second, as before, the error curve has a flat region. In this case, it occurs between 1,000 and 3,000 ft (305 to 914 m), where the error stays at a constant low level of about 2 percent. Third, the error curve has a slight dip between 500 and 1,000 ft. This may have been because the true counts have integer values whereas the estimated counts do not, and a count error may be a greater fraction of the true value when the sensor separation, and hence the vehicle count, is small. Such an effect is not observed in the case of lanes treated individually, because lane-changing in that case raises the level of error as the sensor separation is increased.

Clearly the insensitivity of the accuracy of the estimation algorithm to the separation between sensors has significant practical implications for the design of surveillance systems, simply because it costs less to place sensors farther apart.

The numerical results corresponding to Figures 2 and 3 are given in Table 1. Also given in this table are the average speeds corresponding to the various runs of the Kalman filter. It is seen that all the runs correspond to relatively light, free-flowing traffic. Therefore, the tests of the estimation algorithm given here are somewhat incomplete in that they do not show how well the algorithm works during periods of heavy traffic. Undoubtedly, periods of high-density, slow-speed traffic will be handled well if one leads up to them starting with light traffic. It is not clear, however, how well the filter initializes during periods of persistently heavy traffic. A test of the algorithm for such traffic was not possible because of lack of data but would be very desirable.

GENERAL DESIGN METHODOLOGY

We can now suggest a general methodology that uses the estimation algorithm (1) for designing a density measuring system for a roadway:

1. Obtain some aerial data of the roadway—about 1 hour's duration of traffic, with photographs taken at 1 frame per 2 seconds, is more than adequate. Ideally, one should obtain data for different degrees of congestion, because the accuracy of the estimation depends on the degree of congestion (1).
2. Compute the speed and flow past imaginary sensors placed at varying distances on the roadway.
3. Start with reasonable guesses for the initial count of vehicles in a section and the initial variance Σ_0 of this count. Run the estimation algorithm for values of ρ varying from, say, 10^{-4} to 1, keeping the observation noise variance R at some constant level. Find the best value of ρ for each section, corresponding to the minimum error, and compute the average minimum error for each value of the sensor separation, varying the sensor separation in steps of 500 ft between 500 and 4,000 ft.

A plot of the minimum error versus sensor separation may then be used in conjunction with economic considerations of installation and operation costs in order to determine the optimum placement of sensors. A high density of sensors entails a high installation and maintenance cost for the sensors themselves and for communication and possibly a high cost of computing power required for processing the sensor data. Reduction of this cost may be traded-off against some degradation of density estimates.

Frequently, the maximum separation of sensors is dictated by the need to detect incidents such as vehicle stoppages with a high reliability. The problem of reliable incident detection is still not completely understood, and it is not clear how one can best combine incident detection and density estimation.

CONCLUSIONS

We have provided, in this paper, some answers to questions concerning Kalman filter application, algorithm sensitivity, and roadway surveillance system design. We have found a systematic way of choosing proper initial values for the Kalman filter as used in the density estimation algorithm (1), and we have shown that the algorithm gives satisfactory results for freeways, at least where the densities are relatively low. Contrary to intuition, the minimum achievable error of the density estimation algorithm is not necessarily a strictly increasing function of the distance between sensors, but the variation of this error versus sensor separation may have a flat region, offering an opportunity for substantial savings in sensor cost.

There are a number of possible extensions of the work presented here. Analytical modeling of the flow of traffic from section to section may improve the overall accuracy of a surveillance system or even provide an analytical solution to the problem of optimal sensor placement and a feedback solution for optimizing traffic flow. A simple correlation of the density estimation in two adjacent sections should improve the overall accuracy by use of the argument of conservation of cars over many contiguous sections of a roadway.

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APPENDIX

THE KALMAN FILTER EQUATIONS

Let $k = 0, 1, 2, \ldots, k$ be an index of time where $k = 0$ corresponds to the initial time. Consider a discrete-time, possibly nonlinear, and time-varying system whose state vector $\underline{x}_k$ is an n-dimensional vector, $\underline{x}_k \epsilon R_n$. Assume that the state propagates according to the stochastic difference equation

$$\underline{x}_{k+1} = \underline{f}_k(\underline{x}_k) + \underline{w}_k \tag{11}$$

Suppose that the measurement vector $\underline{z}_k$, $(\underline{z}_k \epsilon R_r)$, is related, possibly nonlinearly, to the state vector by the observation equation

$$\underline{z}_k = \underline{h}_k(\underline{x}_k) + \underline{v}_k \tag{12}$$

Assume that $\underline{f}_k(\cdot) : R_n \rightarrow R_n$ and $\underline{h}_k(\cdot) : R_n \rightarrow R_r$ are known, continuous, and sufficiently differentiable. Assume zero-mean, uncorrelated "white" noise in both the state and observation equations, i.e.,

$$\left.\begin{aligned} E\{\underline{w}_k\} &= \underline{0} \\ E\{\underline{v}_k\} &= \underline{0} \\ E\{\underline{w}_k\underline{w}_j'\} &= \underline{Q}_k\,\delta k_j,\ \underline{Q}_k > \underline{0} \\ E\{\underline{v}_k\underline{v}_j\} &= \underline{R}_k\,\delta_{kj},\ R_k > \underline{0} \end{aligned}\quad \text{for all } k \right\} \tag{13}$$

Define

$$\left.\begin{aligned} \underline{x}_0 &= E\{\underline{x}_0\} \\ \Sigma_0 &= E\{(\underline{x}_0 - \overline{\underline{x}}_0)(\underline{x}_0 - \overline{\underline{x}}_0)'\} \end{aligned}\right\} \tag{14}$$

Assume that $\underline{x}_0$, $\underline{w}_k$, and $\underline{v}_j$ are mutually independent for all k, j.

The following notation will be used:

$\hat{\underline{x}}_{k|k}$ = estimate of state vector $\underline{x}_k$ based on the observation $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_k$

$\hat{\underline{x}}_{k+1|k}$ = predicted estimate of the vector $\underline{x}_{k+1}$ based only on the measurements $\underline{z}_1$, $\underline{z}_2$, $\ldots$, $\underline{z}_k$ (i.e., before measurement $\underline{z}_{k+1}$ is made)

$\Sigma_{k|k} = E\{(\underline{x}_k - \hat{\underline{x}}_{k|k})(\underline{x}_k - \hat{\underline{x}}_{k|k})'\}$

$\Sigma_{k+1|k} = E\{(\underline{x}_{k+1} - \hat{\underline{x}}_{k+1|k})(\underline{x}_{k+1} - \hat{\underline{x}}_{k+1|k})'\}$

The discrete-time Kalman filter algorithm is best described by decomposing it into three distinct steps: initializing, predicting, and updating. Start the algorithm by setting

$$\left.\begin{aligned} \hat{\underline{x}}_{0|0} &= \overline{\underline{x}}_0 \\ \Sigma_{0|0} &= \Sigma_0 \end{aligned}\right\} \tag{15}$$

For predicting, generate $\hat{\underline{x}}_{k+1|k}$, $\Sigma_{k+1|k}$ by

$$\left.\begin{aligned} \hat{\underline{x}}_{k+1|k} &= \underline{f}_k\,(\hat{\underline{x}}_{k|k}) \\ \Sigma_{k+1|k} &= \hat{F}_k\,\Sigma_{k|k}\,\hat{F}_k' + Q_k \end{aligned}\right\} \tag{16}$$

where

$$\hat{F}_k \triangleq \frac{\partial f_k(\underline{x}_k)}{\partial \underline{x}_k}\Bigg|\ \underline{x}_k = \hat{\underline{x}}_{k|k} \tag{17}$$

For updating, generate $\underline{x}_{k+1|k+1} = \underline{x}_{k+1|k}, \Sigma_{k+1|k+1}$ by

$$\left.\begin{aligned} \underline{x}_{k+1|\,k+1|\,k} &= \underline{x}_{k+1} + G_{k+1}\,[\underline{z}_{k+1} - \underline{h}_{k+1}\,(\hat{\underline{x}}_{k+1})] \\ \Sigma_{k+1|k+1} &= \Sigma_{k+1|k} - G_{k+1}\,\hat{H}_{k+1}\,\Sigma_{k+1|k} \end{aligned}\right\} \tag{18}$$

where

$$G_{k+1} \triangleq \Sigma_{k+1|k}\,\hat{H}_{k+1}'\,[\hat{H}_{k+1}\,\Sigma_{k+1|k}\,\hat{H}_{k+1}' + R_{k+1}]^{-1} \tag{19}$$

$$\hat{H}_{k+1} = \frac{\partial \underline{h}_{k+1}(\underline{x}_{k+1})}{\partial \underline{x}_{k+1}}\Bigg|\ \underline{x}_{k+1} \hat{=} \underline{x}_{k+1|k} \tag{20}$$

The algorithm is iterative in nature. Starting with an initial guess, it generates a new state estimate each time an observation vector becomes available.

In a particular case (1), the state of the system is a scalar quantity, x_n, the count of vehicles on a roadway section during the nth time interval. The state propagates according to the linear equation

$$x_{n+1} = x_n + \Delta N_n + w_n \tag{21}$$

where ΔN_n is the net input into the section (input at the entrance minus output at the exit) and w_n a random error. The measurement is a rough estimate of the count of vehicles z_n obtained directly from the speed and flow measurements. This estimate was obtained by first estimating the travel time of individual vehicles through the section (1). Other methods for obtaining z_n—for example, a phenomenological relationship between speed and density—can also be used satisfactorily (3). The z_n is related to x_n according to

$$z_n = x_n + v_n \tag{22}$$

Thus, both f(·) and h (·) are simple linear functions, and the preceding formulas are reduced to the simpler form

$$\left.\begin{aligned}
\hat{x}_{k+1|k} &= \hat{x}_{k|k} + \Delta N_k \\
\Sigma_{k+1|k} &= \Sigma_{k|k} + Q_k \\
\Sigma_{k+1|k+1} &= \Sigma_{k+1|k} (1 - G_{k+1}) \\
G_{k+1} &= \Sigma_{k+1|k} (\Sigma_{k+1|k} + R_{k+1})^{-1} \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + G_{k+1} (Z_{k+1} - \hat{x}_{k+1|k})
\end{aligned}\right\} \tag{23}$$

where

$$\left.\begin{aligned}
Q_k &= E\{w_k^2\} \\
R_k &= E\{v_k^2\}
\end{aligned}\right\} \tag{24}$$

and the process is initialized by selecting some initial values $\hat{x}_0$ and $\Sigma_{0|0}$.