A RAPID SIGNAL TRANSITION ALGORITHM

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An increasing number of traffic signal systems are being brought under computer control. Although the designs of these control systems vary in detail, nearly all can be described as a sequence of fixed-time signal patterns. One of the primary advantages of computer control is the ability to change signal patterns in response to variations in traffic conditions. Associated with each change is a transition period wherein the signal settings transform from one fixed-time pattern to the next. Experience has shown that these transition periods can have disruptive effects on traffic operations. Hence, the more frequent these signal-pattern changes are, the greater is the need for careful design of the signal settings during the transition period. This paper presents a new algorithm that is easily implementable in real time by a digital computer and that is designed to minimize the duration of the transition period and yet service traffic demand along all approaches to every intersection. The results of an evaluation effort that compared this logarithm with two others are given. The results indicate that this method compares favorably with respect to the others considered.

•TRAFFIC signal systems controlled by computers are becoming increasingly common. Computer control permits the changing of signal patterns to respond to changing traffic conditions. For each change in pattern there is a transition period during which the signal settings convert from one fixed-time pattern to another. Because these transition periods can have disruptive effects on traffic, careful design of the signal settings during the transition period is necessary.

In general, transition policies may be classified as (a) a smooth, staged transition that restricts the change in offset per cycle or (b) a procedure designed to minimize delay. The first is "smooth" from the viewpoint of the control system; such a policy could, however, produce poor offsets over the duration of the transition period at several network nodes. The second is intuitively appealing and could well produce excellent results; such procedures, however, imply certain assumptions in the delay-control model that may not be realistic.

The algorithm presented here is designed to minimize the time to complete the signal transition, subject to the condition that all traffic demands can be serviced during this period. As such, there are no assumptions embedded in the methodology; the objective is explicitly satisfied. There is some question, of course, as to whether such an objective provides good service both during transition and subsequently, relative to other candidate procedures.

As part of the effort to extend and apply the UTCS-1 simulation model (1), an activity was undertaken to evaluate the performance of three signal transition algorithms:

1. "Immediate" transition;
2. "Second-generation" policy transition; and
3. The subject Rapid Signal Transition (RAST).

The first of these is normally implemented by standard multi-dial controllers. Essentially, the signal dwells in green for the main street until the new offset is attained. The "second-generation" transition (2-GT) algorithm is designed to minimize the sum of offset changes that must be experienced at all signals in the network; the transition parameter that satisfies this objective is computed in the process (2). The RAST algorithm is described below.

RAST ALGORITHM

Objectives

The RAST algorithm is designed to satisfy the following objectives:

1. Transform the signal control system from one offset pattern to the next immediately; i.e., minimize the number and duration of intervening nonoptimal signal intervals.
2. Keep the interval sequence unchanged.
3. Provide intervals of sufficient duration to service demand during transition.
4. Minimize the duration of transition at the "worst" node.

Parameter Definitions

The following terms are used in describing the algorithm:

X = reference time displacement in seconds.
$a_i^{(k)}$ = elapsed time to the offset of interval k, node i, from the reference time R for the existing pattern. Note that

$$-\frac{C_1}{2} < a_i^{(k)} \le \frac{C_1}{2}$$

where R is the start of transition and C_1 is the cycle length of the signal pattern being terminated at time R.
$A_i^{(k)}$ = offset of interval k of signal i with respect to reference time $\hat{R} = R + X$ for new signal pattern of cycle length C_2.
$p_i^{(k)}$ = duration of signal interval k at i.
$\bar{p}_i^{(k)}$ = minimum duration of signal interval k at node i.
$V_{ji}^{(k)}$ = critical lane traffic volume serviced by signal interval k at node i for approach j, expressed in vehicles per hour.
$\hat{C}_i$ = minimum signal cycle time possible during transition period at node i,

$$= \sum_{k=1}^{K_i} \bar{p}_i^{(k)}$$

K_i = total number of intervals for signal at node i.
$\hat{k}$ = "key" interval at beginning of transition (may be modified by algorithm).
k_o = interval used as reference for defining the signal offset for the new pattern.

The parameters a_i, A_i, p_i, X, R, and $\hat{R}$ are shown in Figure 1.

Preliminary Determinations

The necessary preliminary steps are as follows:

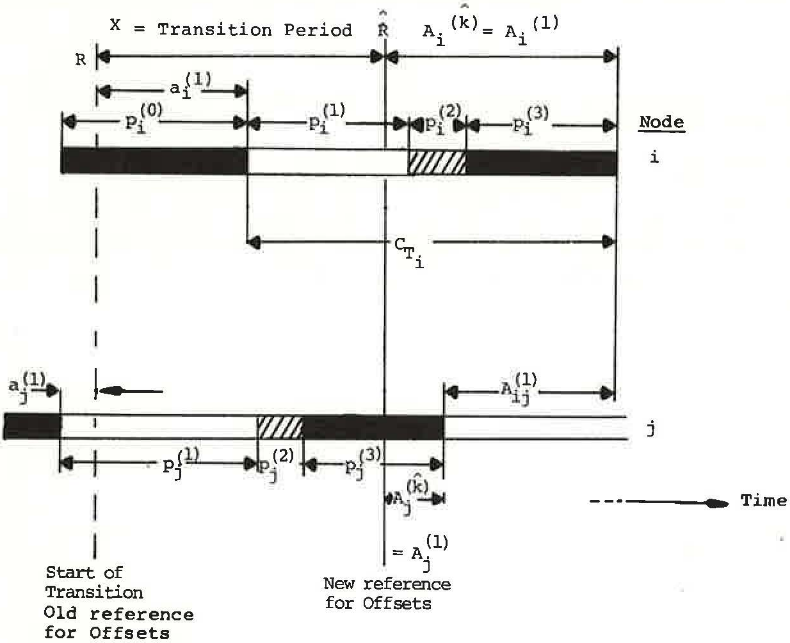
1. For each interval $\bar{k}$, determine

$$\bar{p}_i^{(\bar{k})} = \max_j \left\{ (p_i^{(\bar{k})})_{min},\ S_j + H_j \left(\frac{V_{ji}^{(\bar{k})} \cdot C_1}{3,600} - 1 \right) \right\}$$

where

$(\bar{p}_i^{(\bar{k})})_{min}$ = minimum allowable green interval duration servicing the appropriate component of traffic volume, $V_{ji}^{(\bar{k})}$. This value is specified externally and dictated by practical considerations such as pedestrian crossing time.
S_j = start-up loss for queue on approach j.
H_j = mean queue discharge headway for approach j.
$\bar{k}$ = "major" intervals, i.e., that subset of the K_i intervals that are variable during the transition period. For example, amber, all-red intervals, and possibly intervals servicing turning traffic may be held fixed.

Figure 1. RAST parameter definition.



$A_{ij}^{(1)}$ new specified offset $= A_i^{(1)} - A_j^{(1)}$, for interval 1.

$\hat{k} = 1$ at both intersections

$\bar{p}_i^{(\bar{k})}$ = minimum duration of green interval $\bar{k}$ during transition period servicing traffic on the appropriate approaches.

2. At each node i, identify as the "key interval" that major interval $\bar{k}$ that is active at time R. If a minor (e.g., amber) interval is active, then the following major interval is identified as such. Denote this interval by the superscript $\hat{k}$. If $a_i^{(\hat{k})} < 0$ and $|a_i^{(\hat{k})}| > \bar{p}_i^{(\hat{k})}$, then select the following major interval as $\hat{k}$ instead, and recompute $a_i^{(\hat{k})}$. In Figure 1, $\hat{k} = 1$ at both intersections: $a_i^{(1)} > 0$, $a_j^{(1)} < 0$.

3. The offset $A_i^{(\hat{k})}$ of interval $\hat{k}$ for the new signal pattern at node i with respect to some common reference time $\hat{R}$ for the entire network is specified externally. This new reference time $\hat{R}$ is displaced from the current reference time R by a period X, which is a solution variable.

4. From Figure 1,

$$A_i^{(\hat{k})} = a_i^{(\hat{k})} + \sum_{\substack{k=\hat{k},\hat{k}+1,\ldots,K_i,\\ 1,2,\ldots,\hat{k}-1}} p_i^{(k)} - X$$

$$= a_i^{(\hat{k})} + C_{T_i} - X$$

where C_{T_i} is the (unknown) transition signal cycle at node i.

Necessary Conditions

The following conditions are to be met:

1. Satisfy all specified $A_i^{(\hat{k})}$ (new offsets).
2. Minimize transition time at the critical node (i.e., the node that takes the longest time to complete transition).
3. Determine all $p_i^{(k)} \geq \bar{p}_i^{(k)}$.

At each node i in the subnetwork the given parameters are $(p_i^{(\bar{k})})_{min}$; S_i; $V_{ji}^{(\bar{k})}$; H_i; $A_i^{(k_o)}$; $(p_i^{(k)})_{new\ pattern}$. For the subnetwork, C_1 = existing cycle length and C_2 = new cycle length.

Procedure

This algorithm, which is essentially an application of elementary game theory, is described in terms of a step-by-step procedure:

1. Determine $\bar{p}_i^{(k)}$ for all major signal intervals at each node i; then $\hat{C}_i$.
2. Identify interval $\hat{k}$ and its reference offset $a_i^{(\hat{k})}$ at each node i.
3. Calculate $\hat{X}_i = a_i^{(\hat{k})} + \hat{C}_i$ at each node i. This is the minimum possible value of X_i.
4. Determine the new reference offsets $A_i^{(\hat{k})}$ at each node with respect to the (unknown) reference time $\hat{R} = R + X$ consistent with the specified new offsets $A_i^{(k_o)}$ as follows: Initially, determine $A_i^{(\hat{k})}$,

$$A_i^{(\hat{k})} = (A_i^{(k_o)} + \emptyset_i) \bmod C_2$$

where

$$\emptyset_i = \begin{cases} 0 \text{ if } \hat{k} = k_o \\ \displaystyle\sum_{k=k_o}^{\hat{k}-1} (p_i^{(k)})_{new} \text{ if } \hat{k} \neq k_o \end{cases}$$

5. Scan all nodes, n = 1, 2, . . . , N, in sequence, where node n denotes an "anchor" node. For each n, compute

$$A_{in}^{(\hat{k})} = (A_i^{(\hat{k})} - A_n^{(\hat{k})}) \bmod C_2;\ i = 1, 2, \ldots, N$$

and then

$$\delta_{in} = A_{in}^{(\hat{k})} + \hat{X}_n \quad \text{for } i = 1, 2, \ldots, N$$

This value of δ_{in} is the time required for the signal at node i to attain the new synchronization if the transition period is set at $\hat{X}_n$, i.e., if the time to complete transition at the anchor node n is minimized. This value is subject to the constraint

$$\delta_{in} \geq \hat{X}_i$$

which, if satisfied, ensures that the constraints on minimum-interval durations for the signal at node i will be satisfied. If

$$\delta_{in} < \hat{X}_i$$

then the value of $\hat{X}_n$ must be revised (increased),

$$\hat{X}_n = (\hat{X}_n)_{old} + \hat{X}_i - \delta_{in},$$

and all previously calculated values of δ_{in} must be suitably revised prior to continuing the sweep over i.

6. Define $\Delta_n = \max_i (\delta_{in})$. This is the worst case (longest transition) in the network, if we minimize the time to complete transition at the anchor node n. The algorithm seeks the minimum value of Δ_n. Each node in sequence takes on the role of the anchor node, generating a new set of δ_{in} and another value of Δ_n. Then the critical anchor node $\hat{n}$ is located:

$$\Delta_{\hat{n}} = \min_n [\Delta_n] = \min_n \left\{ \max_i (\delta_{in}) \right\}$$

The transition period is $X = \hat{X}_{\hat{n}}$. It is seen that the minimum time of transition for the entire network is $\Delta_{\hat{n}}$ and the transition time at node $\hat{n}$ is the minimum possible, subject to constraints on minimum phase durations.

7. The excess time available at each signal i is

$$E_i = \delta_{i\hat{n}} - a_i^{(\hat{k})} - \hat{C}_i$$

This excess is allocated in proportion to requirements of the dominant approaches. Determine the excess of green time provided at each node to the dominant approach (i.e., the approach that services the higher volume in that direction) j serviced by interval $\bar{p}^{(\bar{k}_j)}$:

$$\Delta s_j = \bar{p}^{(\bar{k}_j)} - \left\{ S_j + H_j \left(\frac{V_{ji}^{(\bar{k}_j)} \cdot C_i}{3,600} - 1 \right) \right\}$$

As noted earlier, $\Delta s_j > 0$ only if $\bar{p}_i^{(\bar{k}_j)} = (p_i^{(\bar{k}_j)})_{min}$. Denote as $\bar{k}_1$ and $\bar{k}_2$ the two dominant (variable) intervals that service the demand in the respective directions, j_1 and j_2. (This analysis can be extended to consider additional phases.) The objective is to allocate the excess green time E_i between these intervals as follows:

$$\frac{\Delta s_{j_1} + e_{\bar{k}_1}}{V_{j_1}} = \frac{\Delta s_{j_2} + e_{\bar{k}_2}}{V_{j_2}}$$

$$e_{\bar{k}_1} + e_{\bar{k}_2} = E_i$$

Hence, the total excess green time above that required to satisfy demand is allocated so as to be proportional to the critical per-lane demands $V_{ji}^{(\bar{k}_j)}$ in the respective directions serviced by the signal intervals $\bar{k}_1$ and $\bar{k}_2$. Then,

$$e_{\bar{k}_1} = \frac{V_{j_1}}{V_{j_1} + V_{j_2}} \left\{ \Delta s_{j_2} + E_i - \frac{V_{j_2}}{V_{j_1}} \Delta s_{j_1} \right\}$$

(Note that $0 \le e_{\bar{k}_1} \le E_i$ must be asserted.) Then, $e_{\bar{k}_2} = E_i - e_{\bar{k}_1}$. Hence, $p^{(\bar{k}_1)} = \bar{p}^{(\bar{k}_1)} + e_{\bar{k}_1}$ and $p^{(\bar{k}_2)} = \bar{p}^{(\bar{k}_2)} + e_{\bar{k}_2}$.

8. Step 6 has yielded $X = \hat{X}_{\hat{n}}$; step 7, the interval durations during this transition period. Note that the determination of $\bar{p}_i^{(k)}$ and $\hat{C}_i$ in step 1 utilized C_i; $C_{T_i} = A_i^{(\hat{k})} + X - a_i^{(\hat{k})}$ should have been utilized, but X was unknown at that point. It may, therefore, be necessary to iterate for X. Note also that the node-specific cycle length during the transition period C_{T_i} varies from one node to another.

The offsets for the active intervals prior to the beginning of transition are "optimal" for the old pattern. Starting with interval $\hat{k}$, signal offsets depart from these optimal values and do not attain their new optimal values until the completion of transition at each node. Hence, minimizing the time to complete transition at the critical (worst) node should serve to restrict the duration of turbulence arising from these nonoptimal offsets. The interval durations during transition, however, do satisfy demand on a volume basis. An illustrative case showing all calculations is given in the Appendix.

EVALUATION OF TRANSITION ALGORITHMS

The logic representing all the transition algorithms was introduced into the UTCS-1 model and exercised to perform tests on two networks—the Washington, D.C., UTCS grid (Figure 2) and the Wisconsin Avenue arterial (Figure 3)—for two traffic conditions—morning peak to off-peak and off-peak to morning peak. The final results are shown in Figures 4 through 7. A summary of all results is given in Tables 1 and 2. Full details are provided elsewhere (2).

CONCLUSIONS

The impact of a signal transition policy on the effectiveness of a responsive control

Figure 2. Washington UTCS grid network.

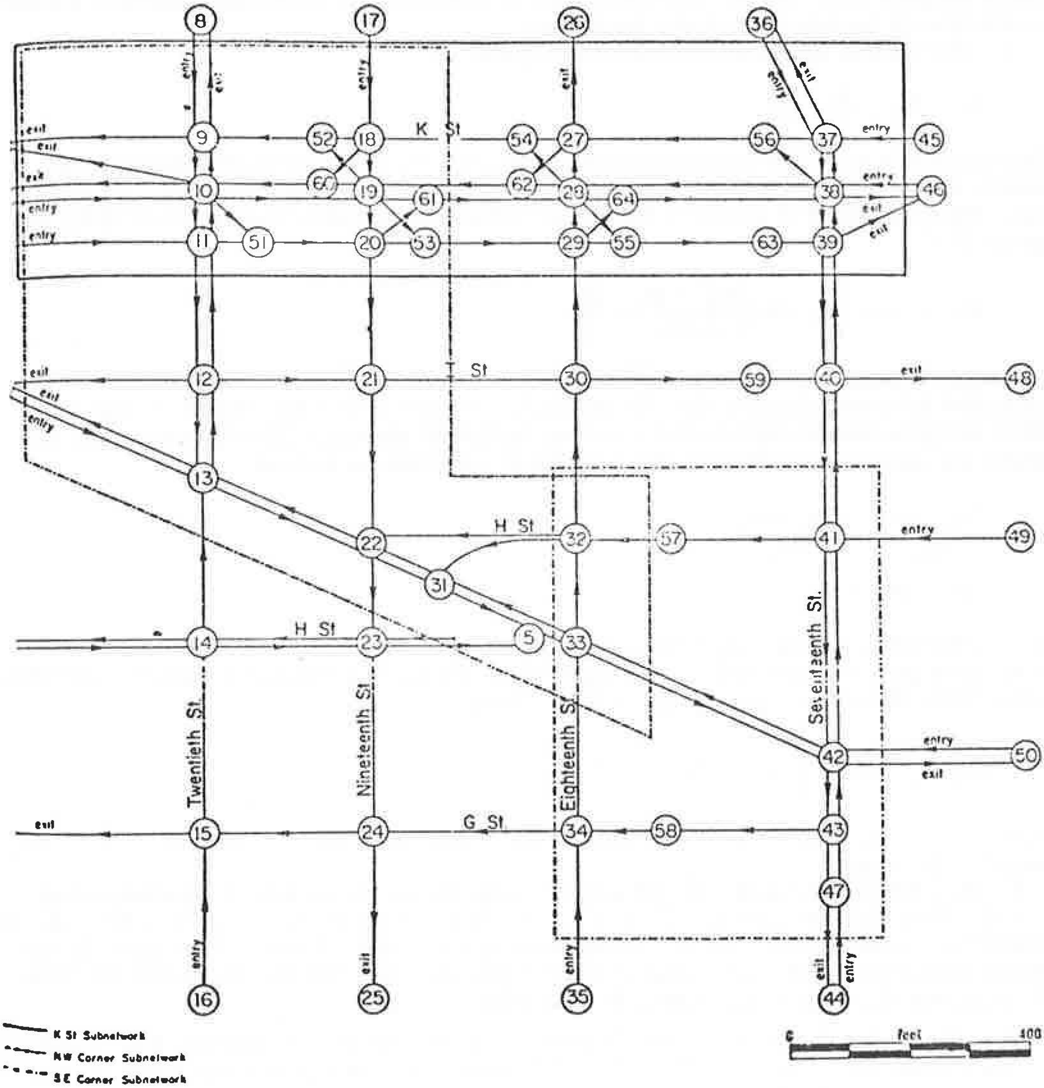


Table 1. Total delay, vehicle-minutes per hour.

Test	Network	Transition Algorithm		
		Immediate	2-GT	RAST
Peak to off-peak	Grid	12,733	11,427	11,127
Off-peak to peak	Grid	8,566	7,801	7,845
Peak to off-peak	Arterial	4,279	4,185	4,272
Off-peak to peak	Arterial	3,361	4,211	3,483
Total		28,939	27,624	26,727

Table 2. Percent reduction in delay relative to immediate transition.

Network	Transition Algorithm	
	2-GT	RAST
Grid	+9.0	+10.9
Arterial	-9.9[a]	-1.5[a]
Overall	+4.5	+7.6

[a]Increase in delay.

Figure 3. Wisconsin Avenue transition test network.

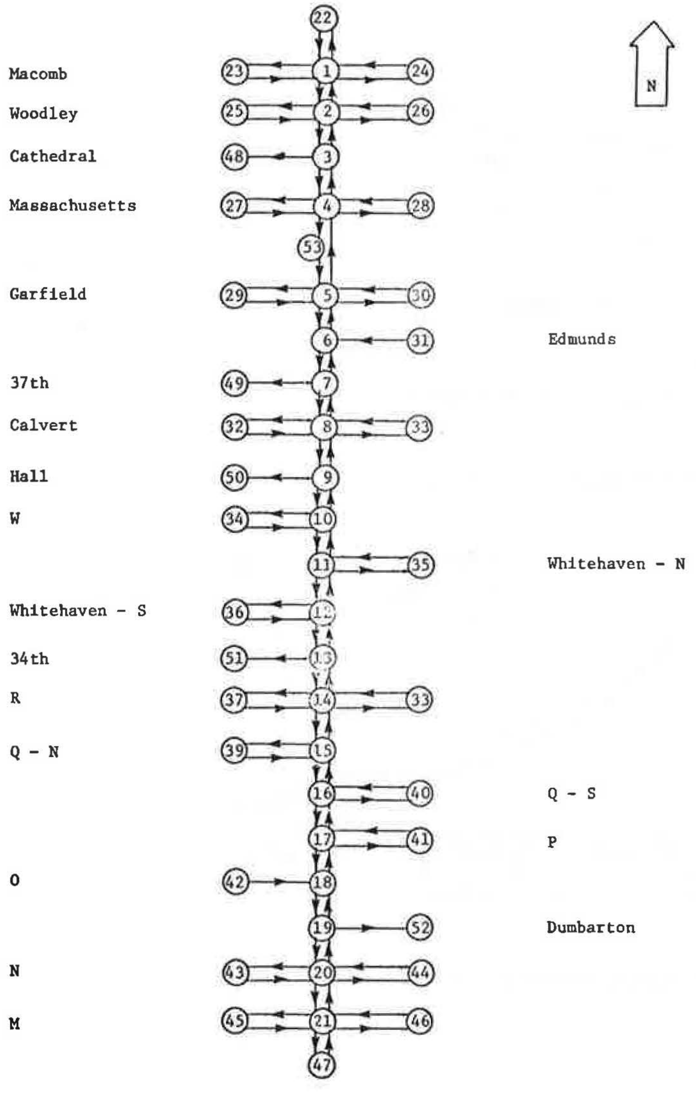


Figure 4. Delay per subinterval, UTCS-1 network, peak to off-peak.

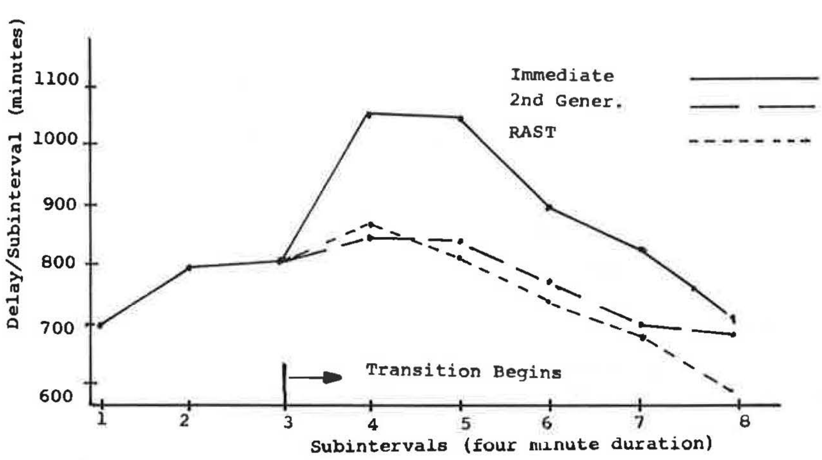


Figure 5. Delay per subinterval, UTCS-1 network, off-peak to peak.

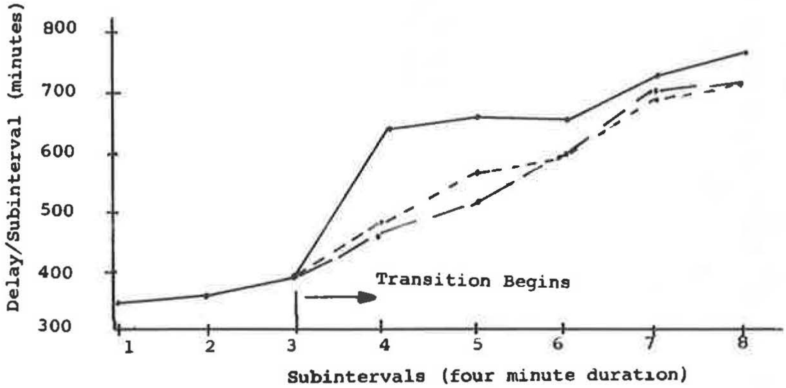


Figure 6. Delay per subinterval, Wisconsin Avenue, peak to off-peak.

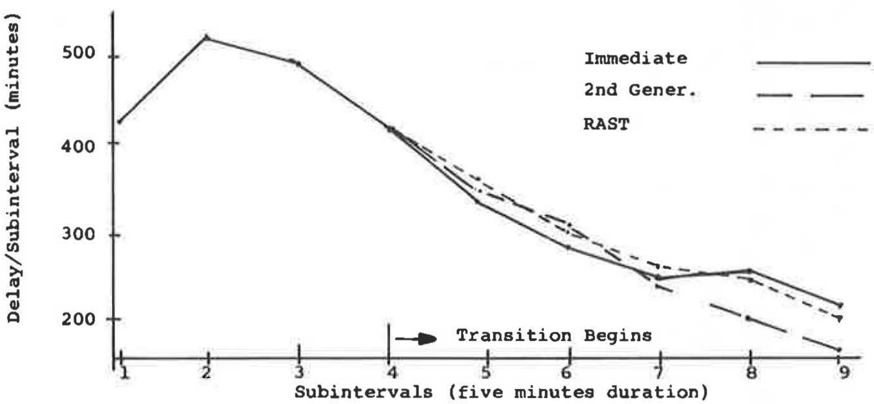
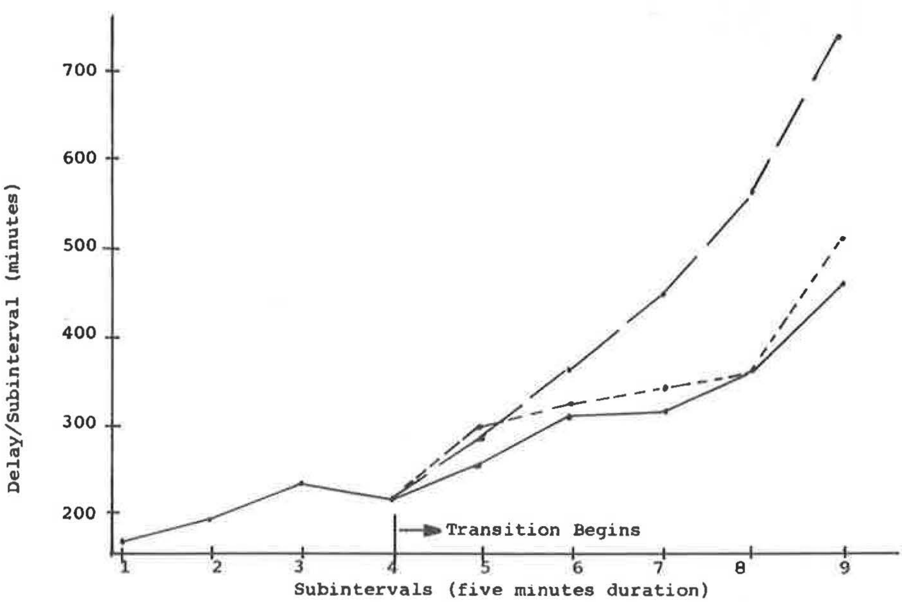


Figure 7. Delay per subinterval, Wisconsin Avenue, off-peak to peak.



system is comparable in importance to the signal optimization procedure for fixed-cycle control policies. In this study, both the 2-GT and RAST algorithms offer significant improvements in traffic operations relative to the "immediate" transition for grid networks (Tables 1 and 2). As should be expected for the Wisconsin Avenue study, the immediate transition policy, which dwells in green facing the arterial until the new offsets are realized, performed relatively well, while the RAST algorithm was strongly competitive. The sensitivity of traffic operations on grid networks to signal transition methodology, however, produced markedly different results, which emphasizes the need for careful treatment of this aspect of control, particularly for a system that changes signal patterns frequently. The RAST algorithm compares favorably with respect to the others considered. This study also demonstrates the utility of traffic simulation as a medium for conducting such evaluations within the framework of a controlled experiment.

ACKNOWLEDGMENTS

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APPENDIX

ILLUSTRATIVE CASE

Given: $S_i = 4$; $\bar{H}_i = 2.4$ seconds/vehicle on all approaches at all nodes i; $(\bar{p}_i)^{(\bar{k})}_{\min} = 15$ seconds for all i, $\bar{k}$; V_{ji} as follows:

	Approach No. j			
i	Main 1	Minor 2	Main 3	Minor 4
1	333	300	467	250
2	367	300	433	200
3	400	300	433	325
4	367	300	400	250

Old interval lengths (seconds), with 4-second amber period prior to each red, are as follows:

$p_i^{(k)}$	Interval No. k				
Node No. i	M.S.G. 1	2	3	4	Main Street Green Offsets ($k_o = 1$)
1	25	4	27	4	10
2	40	4	12	4	55
3	30	4	22	4	40
4	21	4	31	4	36

Figure 8. Old signal pattern facing main street.

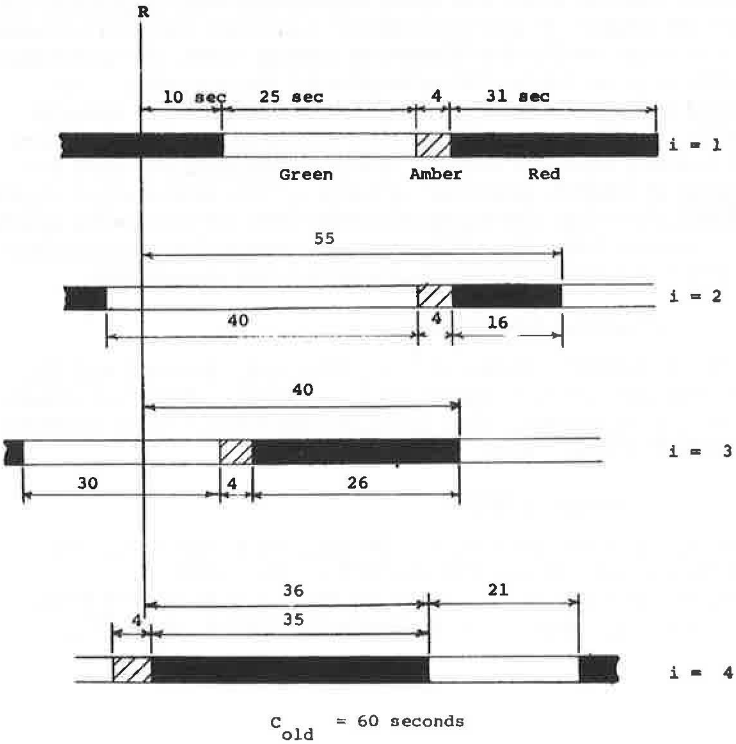
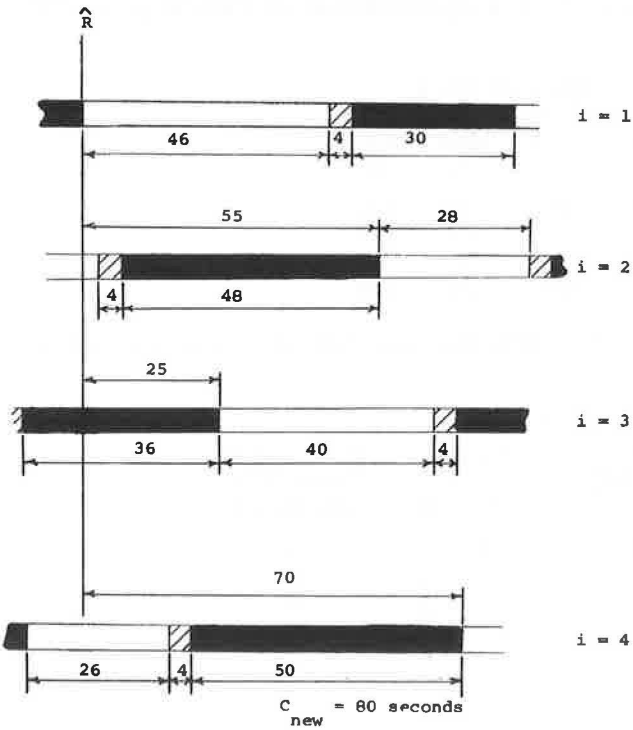


Figure 9. New signal pattern facing main street.



New interval lengths (seconds), with 4-second amber period prior to each red, are as follows:

$p_i^{(k)}$ Node No. i	Interval No. K: M.S.G. 1	2	3	4	Main Street Green Offsets (k_o = 1)
1	46	4	26	4	0
2	28	4	44	4	55
3	40	4	32	4	25
4	26	4	46	4	70

Figure 8 is a schematic of the old signal pattern, and Figure 9 is a schematic of the new signal pattern. Note the disparate cycle lengths between the two patterns. Figure 10 shows an example transition diagram.

Procedure

1. $$\bar{p}_1^{(1)} = \max\left[15,\ 4 + 2.4\left(\frac{\overbrace{333}^{j=1} \cdot 60}{3{,}600} - 1\right),\ 4 + 2.4\left(\frac{\overbrace{467}^{j=3} \cdot 60}{3{,}600} - 1\right)\right]$$

$$= \max\,[15,\ 14.9,\ 20.3] = 20.3 \text{ seconds}$$

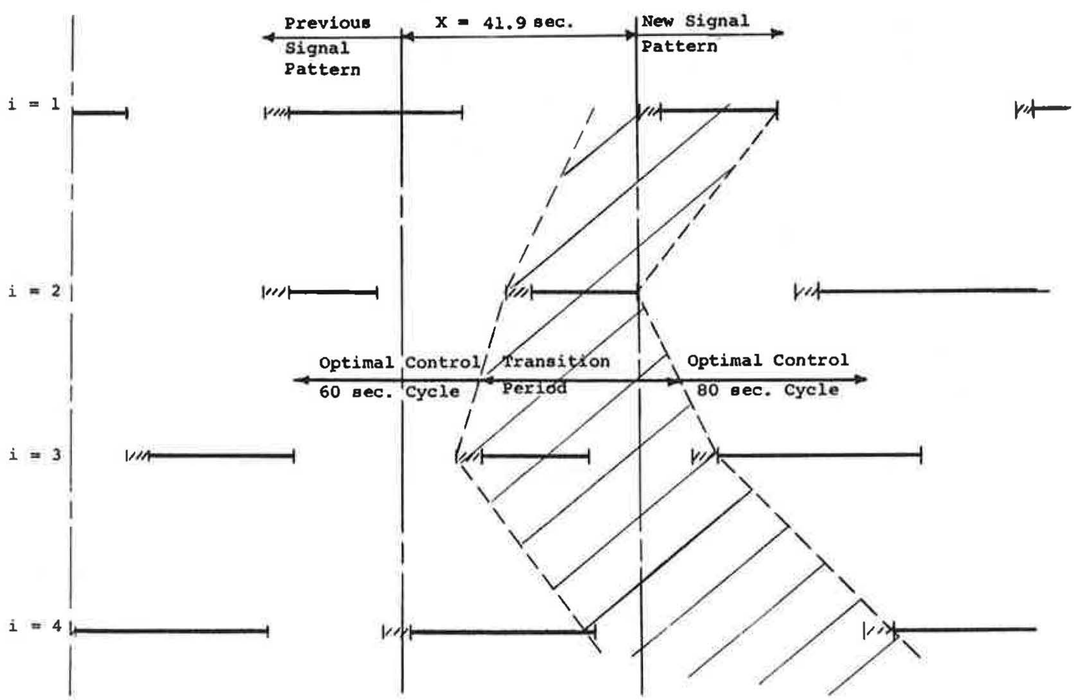
$\bar{p}_1^{(3)} = \max\,[15,\ 13.6] = 15.0$

$\bar{p}_2^{(1)} = 18.9$; $\bar{p}_2^{(3)} = 15$

$\bar{p}_3^{(1)} = 18.9$; $\bar{p}_3^{(3)} = 15$

$\bar{p}_4^{(1)} = 17.6$; $\bar{p}_4^{(3)} = 15$

Figure 10. Illustrative example transition diagram.



Then

$\hat{C}_1 = 20.3 + 4 + 15 + 4 = 43.3$ seconds
$\hat{C}_2 = 41.9$
$\hat{C}_3 = 41.9$
$\hat{C}_4 = 40.6$

2. At $i = 1$, $\hat{k} = 3$; $i = 2$, $\hat{k} = 1$; $i = 3$, $\hat{k} = 1$; $i = 4$, $\hat{k} = 3$ ($k = 1$: green; $k = 3$: red, as shown facing main street). Then $a_1^{(3)} = -21$; $a_2^{(1)} = -5$; $a_3^{(1)} = -20$; $a_4^{(3)} = 1$. Since $a_1^{(3)}$, $a_2^{(1)}$, and $a_3^{(1)}$ are all <0, we must test whether there is sufficient time remaining in these active intervals (3, 1, and 1 respectively) to permit shortening if necessary. Since $|a_1^{(3)}| > \bar{p}_1^{(3)}$, we must identify the other major interval as $\hat{k}$, i.e., define $\hat{k} = 1$ for node 1. Similarly, with $|a_3^{(1)}| > \bar{p}_3^{(1)}$, $\hat{k} = 3$ for node 3, while $\hat{k}$ for node 2 remains 1, since $|a_2^{(1)}| < \bar{p}_2^{(1)}$. Hence we have

i	$\hat{k}$	$A_i^{(1)}$	j_1	j_2	$a_i^{(\hat{k})}$
1	1	0	3	2	10
2	1	55	3	2	-5
3	3	25	3	4	14
4	3	70	3	2	1

where j_1 is the dominant main street approach and j_2 is the dominant minor street approach (values of j_1, j_2 assigned for this example). Note that

$$a_1^{(1)} = a_1^{(3)} + (p_1^{(1)} + p_1^{(2)})_{old} = -21 + 31 = 10;$$
$$a_3^{(3)} = -20 + 30 + 4 = 14$$

3. $\hat{X}_1 = a_1^{(\hat{k})} + \hat{C}_1 = 10 + 43.3 = 53.3$; $\hat{X}_2 = -5 + 41.9 = 36.9$
$\hat{X}_3 = 14 + 41.9 = 55.9$; $\hat{X}_4 = 1 + 40.6 = 41.6$.

4. $A_1^{(\hat{k})} = A_1^{(k_o)} = A_1^{(1)} = 0$
$A_2^{(\hat{k})} = A_2^{(k_o)} = A_2^{(1)} = 55$
$A_3^{(\hat{k})} = A_3^{(3)} = A_3^{(1)} + (p_3^{(1)} + p_3^{(2)})_{new} = 25 + 40 + 4 = 69$
$A_4^{(\hat{k})} = A_4^{(3)} = A_4^{(1)} + (p_4^{(1)} + p_4^{(2)})_{new} = 70 + 26 + 4$
$\mod (80) = 20$

5 and 6. At $n = 1$:

$A_{11}^{(1)} = A_1^{(1)} - A_1^{(1)} = 0$; $\delta_{11} = A_{11}^{(1)} + \hat{X}_1 = 53.3$
$A_{21}^{(1)} = A_2^{(1)} - A_1^{(1)} = 55 - 0 = 55$
$A_{21}^{(1)} = 55$; $\delta_{21} = 55 + 53.3 = 108.3 > 36.9$ ok
$A_{31}^{(3)} = 69$; $\delta_{31} = 69 + 53.3 = 122.3 > 55.9$ ok
$A_{41}^{(3)} = 20$; $\delta_{41} = 20 + 53.3 = 73.3 > 41.6$ ok

$\Delta_1 = \max [53.3, 108.3, 122,3, 73.3] = 122.3$

At $n = 2$:

$A_{12}^{(1)} = A_1^{(1)} - A_2^{(1)} = 0 - 55 = -55 \mod (80) = 25$
$\delta_{12} = A_{12}^{(1)} + X_2 = 25 + 36.9 = 61.9 > 53.3$ ok
$A_{22}^{(1)} = A_2^{(1)} - A_2^{(1)} = 0$; $\delta_{22} = 0 + 36.9 = 36.9 \geq 36.9$ ok
$A_{32}^{(3)} = A_3^{(3)} - A_2^{(1)} = 69 - 55 = 14$; $\delta_{32} = 14 + 36.9 =$
$50.9 < 55.9$ NG

Revised $\hat{X}_2 = \hat{X}_{2\,old} + \hat{X}_3 - \delta_{32\,old} = 36.9 + 55.9 - 50.9 = 41.9$. (This revision applied only to the analysis for n = 2.)

Return to beginning of calculation (n = 2) with this new value of $\hat{X}_2$ and repeat procedure:

$A_{12}^{(1)} = 25;\ \delta_{12} = 25 + 41.9 = 66.9 > 53.3$ ok

$A_{22}^{(1)} = 0;\ \delta_{22} = 0 + 41.9 = 41.9 \geq 41.9$ ok

$A_{32}^{(3)} = 14;\ \delta_{32} = 14 + 41.9 = 55.9 \geq 55.9$ ok

$A_{42}^{(3)} = 25 - 55 \bmod (80) = 45;$

$\delta_{42} = 45 + 41.9 = 86.9 > 41.6$ ok

$\Delta_2 = 86.9$

At n = 3:

$A_{13}^{(1)} = 11;\ \delta_{13} = 66.9 > 53.3$ ok

$A_{23}^{(1)} = 66;\ \delta_{23} = 121.9 > 36.9$ ok

$A_{33}^{(3)} = 0;\ \delta_{33} = 55.9 \geq 55.9$ ok

$A_{43}^{(3)} = 31;\ \delta_{43} = 86.9 > 41.6$ ok

$\Delta_3 = 121.9$

At n = 4:

$A_{14}^{(1)} = A_1^{(1)} - A_4^{(3)} = 0 - 20 \bmod (80) = 60;$

$\delta_{14} = 60 + 41.6 = 101.6 > 53.3$ ok

$A_{24}^{(1)} = 35;\ \delta_{24} = 76.6 > 36.9$ ok

$A_{34}^{(3)} = 49;\ \delta_{34} = 90.6 > 55.9$ ok

$A_{44}^{(3)} = 0;\ \delta_{44} = 41.6 \geq 41.6$ ok

$\Delta_4 = 101.6$

$\Delta_{\hat{n}} = \min\,[122.3,\ 86.9,\ 121.9,\ 101.6] = 86.9;\quad \hat{n} = 2$

Then, with node 2 as the critical one, $X = \hat{X}_{\hat{n}} = 41.9$. Also, $A_1^{(1)} = A_{12}^{(1)} = 25$, $A_2^{(1)} = 0$, $A_3^{(3)} = 14$, $A_4^{(3)} = 45$.

7. $E_1 = 66.9 - 10 - 43.3 = 13.6$

$E_2 = 41.9 - (-5) - 41.9 = 5.0$

$E_3 = 55.9 - 14 - 41.9 = 0$

$E_4 = 86.9 - 1 - 40.6 = 45.3$

Node 1

$$\Delta s_3 = 20.3 - \left[4 + 2.4\left(\frac{467 \cdot 60}{3,600} - 1\right)\right] = 0$$

$$\Delta s_2 = 15 - \left[4 + 2.4\left(\frac{300 \cdot 60}{3,600} - 1\right)\right] = 1.4$$

Node 2

$\Delta s_3 = 0$

$$\Delta s_2 = 15 - \left[4 + 2.4\left(\frac{300 \cdot 60}{3,600} - 1\right)\right] = 1.4$$

Node 3

$$\Delta s_3 = 0$$

$$\Delta s_4 = 15 - \left[4 + 2.4\left(\frac{325 \cdot 60}{3,600} - 1\right)\right] = 0.4$$

Node 4

$$\Delta s_3 = 0$$

$$\Delta s_2 = 15 - \left[4 + 2.4\left(\frac{300 \cdot 60}{3,600} - 1\right)\right] = 1.4$$

The allocation of excess time to each of the major intervals is

Node 1

$$e_1 = \frac{467}{467 + 300}[1.4 - 0 + 13.6] = 9.1$$

$$e_3 = 13.6 - 9.1 = 4.5$$

$$p_1^{(1)} = \bar{p}_1^{(1)} + e_1 = 20.3 + 9.1 = 29.4$$

$$p_1^{(3)} = 15 + 4.5 = 19.5$$

Node 2

$$e_1 = \frac{433}{433 + 300}[1.4 - 0 + 5] = 3.8$$

Then $e_3 = 5 - 3.8 = 1.2$

and

$$p_2^{(1)} = \bar{p}_2^{(1)} + e_1 = 18.9 + 3.8 = 22.7 \text{ seconds}$$

$$p_2^{(3)} = \bar{p}_2^{(3)} + e_3 = 15 + 1.2 = 16.2 \text{ seconds}$$

Check: $X + A_{22}^{(1)} \stackrel{?}{=} a_2^{(1)} + \sum_{k=1}^{4} p_2^{(k)}$

$$41.9 + 0 \stackrel{?}{=} -5 + 22.7 + 4 + 16.2 + 4$$

$41.9 = 41.9$ ok

Node 3

Since $E_3 = 0$, $p_3^{(1)} = \bar{p}_3^{(1)} = 18.9$

$$p_3^{(3)} = \bar{p}_3^{(3)} = 15$$

Node 4

$$e_1 = \frac{400}{400 + 300} [1.4 - 0 + 45.3] = 26.7$$

$$e_3 = 45.3 - 26.7 = 18.6$$

$$p_4^{(1)} = \bar{p}_4^{(1)} + e_1 = 17.6 + 26.7 = 44.3$$

$$p_4^{(3)} = \bar{p}_4^{(3)} + e_3 = 15 + 18.6 = 33.6$$

Transition cycle lengths

$$C_{T_1} = \sum_{k=1}^{4} p_1^{(k)} = 29.4 + 4 + 19.5 + 4 = 56.9 \text{ seconds}$$

$$C_{T_2} = \sum_{k=1}^{4} p_2^{(k)} = 22.7 + 4 + 16.2 + 4 = 46.9 \text{ seconds}$$

$$C_{T_3} = \sum_{k=1}^{4} p_3^{(k)} = 18.9 + 4 + 15 + 4 = 41.9 \text{ seconds}$$

$$C_{T_4} = \sum_{k=1}^{4} p_4^{(k)} = 44.3 + 4 + 33.6 + 4 = 85.9 \text{ seconds}$$

As indicated in step 8, we now obtain improved values of $\bar{p}_i^{(k)}$ based on C_{T_i} rather than C_{old}. Only node 4 need be considered, where $C_{T_4} > C_1$:

$$\bar{p}_4 = \max \left[15,\ 4 + 2.4\left(\frac{400 \cdot 85.9}{3{,}600} - 1\right)\right]$$

$$= 24.4 < 44.3 \qquad \text{ok} \qquad \text{No iteration is necessary.}$$