A REVIEW OF THE TRAFFIC FLOW PROCESS

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This paper summarizes the results of Wardrop, Edie, Haight, Breiman, and others in the study of space and time distributions of speed and other traffic characteristics. In order to help the traffic engineer properly assess mean speeds and other traffic characteristics, the underlying methodology is clarified and its application illustrated by examples of real-life situations. Such applications involve data from manual counts, several types of detectors, and aerial photography.

•THIS expository paper reviews the work of various investigators on the definition and measurement of traffic parameters. An attempt is made to unify these results to make them more understandable to traffic engineers. The use of the theory is illustrated by examples taken from realistic situations. These examples help the traffic engineer to apply the theory properly in the measurement of mean speeds and other characteristics when various detecting methods are used to record traffic data.

Traffic flow is a rather complex process when one considers some of the characteristic variables that can be associated with an individual vehicle: speed at an instant of time, speed at a particular location, location at an instant of time, number of passengers, distance and time separation of vehicles, and quite a few more. To study this field it is necessary to focus attention on several important variables and consider how they behave under uniform conditions of roadway and environment and under conditions of light to medium flow. As shown by Wardrop's results (17), it is convenient initially to make a simplifying assumption about one of the variables, speed: namely, that any vehicle is considered to have one speed associated with it in order to reflect uniform conditions. Another assumption by Wardrop has to do with the arrival process of a vehicle having a given speed. This paper will dwell mainly on the theory relating to stationary flow.

In the basic works of Wardrop (17) and Lighthill and Whitham (11), the important quantities such as flow, concentration, time-mean speed, and space-mean speed were defined and the relations between them explored. The results of these examinations are remarkable in view of the limited data base available to these and other researchers at that time. Wardrop looked at uniform traffic that was fairly homogeneous in space (over a stretch of roadway) at any instant of time and in time (period of observation) at any location on the road. He then developed several important relationships by means of an ingenious intuitive argument. Lighthill and Whitham required only time homogeneity and developed a local relationship for the uniform condition between flow and concentration by considering road traffic by analogy as a stream and by building a fluid continuum model involving three characteristics of streams: flow (quantity per unit time), concentration (quantity per unit space), and speed (space per unit time). Mathematical relations were studied as to how they varied over space and time so that the situation of traffic on long, crowded roads could be formally modeled under nonuniform conditions. For these time-inhomogeneous conditions, Lighthill and Whitham used their results to study congested flow and bottlenecks.

The variety of definitions applied to measurements in the traffic stream was reviewed by Edie (7). Relations between apparently different definitions of the same characteristics were clarified. First it was brought out that relations between flow, concentration, and speed are meaningful only when their averages are considered. Next it was advocated that the correct type of average be employed, space-mean or time-mean, in forming such relations. This would depend on the type of measurement that was employed: one type made at a point (or short distance) in space taken over a long interval of time and the other type taken at an instant of time (or short interval) taken over a long roadway. Arithmetic means computed from the first type are referred to as time means (averages over time) while those computed from the latter type are referred to as space means (average over distance). In another paper, Edie et al. (8) examine a large sample of speed, concentration, and flow data gathered through the use of electronic instrumentation in the Holland Tunnel in order to study time- and space-inhomogeneous situations.

Recently, three fundamental studies on traffic data and models were made by Breiman (2, 3, 4). The first paper reviews the data base, models, and statistical results for one-way homogeneous multilane traffic flow. The second paper, employing the methodology of stochastic processes, first derives the following relation developed by Lighthill and Whitham under locally homogeneous flow:

$$q = k\,\bar{v}_s \tag{1}$$

where q and k are average flow and concentration and $\bar{v}_s$ is the space mean speed defined by Wardrop. The paper then establishes the relation between the space and time distribution of speeds. In the third paper Breiman provides a further clarification as to interpretation of reduced aerial data and derives the fundamental theorem that relates the space distribution of speeds and headways to obtainable synchronous data involving these variables.

In the following sections a heuristic development is made that reflects the results contained in the papers of Wardrop, Edie, and Breiman. It is important, however, that full recognition be given to the many contributions and studies by other researchers that preceded or were contemporaneous to these. Some, such as those by Weiss and Herman (19), Brieman (1), Thedéen (16), and Renyi (15), consider the statistical properties of traffic under low density, while others, such as Miller (12), Buckley (5), Gafarian et al. (6), and Munjal and Hsu (13), explore the behavior of traffic by empirical investigations and application of the theory.

HOMOGENEOUS DISCRETE TRAFFIC STREAM MODELS OF WARDROP

To develop Wardrop's relations, it is necessary to make some formal assumptions as to the possible behavior underlying traffic characteristics in order to study its measurement. We will consider three basic quantities that need to be measured. These are flow, concentration, and speed. As a start, consider a simple model in which the overall process of vehicle speeds $\{V\}$ may be considered as a stream that consists of C (finite) superimposed substreams $\{S\}$. In this model the following assumptions are made to describe it:

1. Any vehicle has associated with it only one speed, v_i.
2. Any vehicle belongs to the ith substream, S_i, only if its speed is exactly equal to v_i.
3. Vehicles are considered as moving points determined from the corresponding locations on the vehicle (i.e., front bumper) and as traveling without interfering with one another.
4. Vehicles proceed on the right lane of a 2-lane section of a 4-lane divided highway, and whenever a point overtakes and passes any point it does so by using the left lane prior to overtaking and immediately merging to the right lane upon passing.
5. For each substream, the vehicles enter one end of a very long roadway at completely random instants of time, constituting a Poisson process of events.

Some of these assumptions could be modified, but in any case there is eventually achieved a homogeneity of traffic after some amount of time has elapsed from the initial entering if one looks at a large section of the road downstream. After such time has elapsed, traffic is called time-homogeneous, which means that any and all time averages converge to a limiting average for long time intervals.

Similarly, traffic is called space-homogeneous if space averages converge to a limiting average for a long enough distance. It has been shown (3) that the limiting averages for both space- and time-homogeneous traffic are equivalent.

It is interesting to note that Wardrop defines a random series of events in time as a series of events in which (a) each event is completely independent of any other event and (b) equal intervals of time are equally likely to contain a given number of events. But this implies that assumption 5, involving Poisson events, would hold. However, Breiman (1) has shown that one can start with an arbitrary homogeneous speed distribution in space and obtain a limiting Poisson spatial distribution under the assumption that cars can pass freely. Similarly, Thedéen (16) concludes that both time and space counts eventually tend to form a Poisson process.

Relationship Involving Space Mean Speed

With these preliminaries we can now present Wardrop's relations on a statistical basis or in a frequency interpretation setting. First look at the process of vehicles in an individual substream, S_i. Since the vehicles in S_i are identified with their own arrival process, which is Poisson or completely random, the quantity q_i (cars per hour) is associated with the arrival rate or traffic intensity parameter, while the time interval T_i between the instants of arrivals of such vehicles obeys the exponential distribution whose density function is given by

$$f_{T_i}(t) = q_i \exp(-q_i t) \qquad 0 \le t < \infty$$

From the expectation of T_i, the average time interval between vehicles passing an observer stationed at a fixed point adjacent to the road is then $1/q_i$. During this time interval, the vehicle is going at fixed speed v_i so that the average distance traveled in this average time is v_i/q_i. This means that, on the average, each vehicle in the ith substream is separately located somewhere along a distance of road that is v_i/q_i units long at any instant of time. It then follows that the average number of substream vehicles per unit length of road (concentration) is given by the reciprocal of this distance or

$$k_i = (v_i/q_i)^{-1} = q_i/v_i \qquad (i = 1, 2, \ldots, C) \tag{2}$$

If $k = \sum_1^C k_i$ denotes the concentration of the entire traffic stream, the discrete frequency distribution in space of vehicles whose speed is v_i is then defined by the multinomial probability

$$p_s(i) = \text{Prob}\,(V_s = v_i) = k_i/k \qquad (i = 1, 2, \ldots, C) \tag{3}$$

Thus $p_s(i)$ is the assigned probability space measure to the body of vehicles in the ith substream. Taking expectations, the space mean speed is obtained as follows:

$$\bar{v}_s = E(V_s) = \sum_1^C v_i p_s(i) = \sum_1^C v_i k_i/k$$

$$\bar{v}_s = \sum_1^C q_i/k \qquad \text{(applying Eq. 2)}$$

$$\bar{v}_s = q/k \tag{4}$$

where $q = \sum_1^C q_i$ is the composite flow of all the substreams or simply the sum of the arrival rates. Equation 4 was first developed by Wardrop and is identical to Eq. 1 here. This is the only valid relation that connects average flow, average concentration, and average speed.

Equation 4 can be used directly in obtaining the space mean speed if one has an unbiased estimate of each k_i/k, which necessitates the observation of vehicle separations on a long roadway at an instant of time. Even reduced aerial data do not provide a long enough distance, as pointed out by Breiman (4). Therefore, we will next consider the alternative method of estimating $\overline{v}_s$ by examining time measurements at a point on the road.

Relationship Involving Harmonic Mean of Time Speeds

Let us consider measurements of speeds as vehicles in the composite stream pass a given point on the road over a long interval of time. We shall designate this time-speed process by $\{V_T\}$ to distinguish it from the process of speeds over space $\{V_s\}$ previously examined. By applying the frequency interpretation for the probability that any vehicle passing the point will have speed $V_t = v_i$, we obtain

$$\text{Prob}\ \{V_t = v_i\} = p_t(i) \tag{5}$$

where $p_t(i)$ is approximated by n_i/n, n_i being the number of vehicles having speed v_i and $n = \sum_1^C n_i$. Hence we can approximate $p_t(i)$ by

$$\hat{p}_t(i) = \frac{n_i}{n} = \frac{n_i/T}{n/T} = \frac{\hat{q}_i}{\hat{q}} \tag{6}$$

where T is the period of observation and $\hat{q}_i$ is the observed arrival rate per unit of time. For large T, we can assume that $\hat{q}_i$ and $\hat{q}$ are equivalent to the underlying corresponding traffic intensities q_i and q, so that we have

$$p_t(i) = q_i/q \qquad (i = 1, 2, \ldots, C) \tag{7}$$

upon applying the frequency interpretation for probabilities.

Now consider the expected value of V_t^{-1} given by

$$E\{1/V_t\} = \sum_1^C \frac{1}{v_i} p_t(i)$$

Upon substitution of Eqs. 7 and 2 in turn we get

$$E\{1/V_t\} = \sum_1^C \frac{1}{v_i}\frac{q_i}{q} = \frac{1}{q}\sum_1^C k_i = \frac{k}{q} \tag{8}$$

From Eq. 8 we learn that the reciprocal of an individual time speed is an unbiased estimate of k/q or

$$q = k\left[E(1/V_t)\right]^{-1} \tag{9}$$

in contrast to the relation involving q, k, and space mean speed $\overline{v}_s$.

As a practical consideration, no one would use a single observation on V to estimate the expected value of the population in this case. Any individual speed could only relate to one of C denumerable substreams. One then considers a random sample of n successive speeds passing a point denoted by $V_1, V_2, \ldots, V_n$. Employing stationarity, these speeds can be considered to be identically distributed in the multinomial population defined by Eq. 7 and in addition may be dependent on each other. These variables obey the law of large numbers under certain conditions that imply that the covariance between any two sample speeds V_i and V_{i+m} tends to zero as the lag m increases (14, chapter 10). The law of large numbers informs us that the sample mean approaches the population mean so that, for large n,

$$E\left\{\frac{1}{n}\sum_{1}^{n}\frac{1}{V_i}\right\} \approx \frac{k}{q} \tag{10}$$

But, from our sample,

$$(1/n)\sum_{1}^{N}(1/v_i) = \overline{v}_h^{-1}$$

where $\overline{v}_h$ is the harmonic mean of the time speeds. Hence the harmonic mean $\overline{v}_h$ of the time speeds can be employed as an asymptotic estimate in the homogeneous traffic relation, as follows:

$$q \approx k\,\overline{v}_h \tag{11}$$

in contrast to Eq. 1. By employing the harmonic mean of speeds obtained from the time process to estimate the mean speed in the space process, one can thus correctly formulate the fundamental relation in Eq. 1.

In the foregoing treatment, the symbols q and k were used to indicate population parameters, where q represented an underlying flow and k represented an underlying concentration. This was done in order to be consistent with their historical treatment in the literature. It is unfortunate that this same treatment has confused these symbols with their observed measurements. Thus, on presenting the following section on examples involving the harmonic mean, the quantitites q and k will be perceived to represent measured quantities in order to be consistent with another body of the literature on traffic measurements. It would have been preferable to use the symbols λ for the underlying flow (replacing q) and ψ for the underlying concentration (replacing k). It is hoped that this dual use of the symbols q and k will not prove to be confusing to the reader.

Examples Using Harmonic Mean

Example 1: Manual Volume Counts—Manual traffic counts are used to obtain flow in traffic surveys where perhaps it is desired to know only the volume of traffic that affects an intersection in order to establish a warrant for signalized control or redesign. This type of method is also employed when other mechanical equipment cannot be readily installed. It is customary to start the count at the start of an hour or the start of a 15-minute period. This is called asynchronous counting, relating to the fact that a vehicle may not be at the location at the start of the count and similarly the count does not end specifically at the instant of arrival of the last (uncounted) vehicle. Synchronous counting refers to initiating the time period at the arrival of a vehicle and terminating the count at the arrival of an uncounted vehicle (10). The asynchronous count data are typically easier to acquire and for large counts would closely approximate the synchronous method.

If N are the number of vehicle counts in a time period T, then the flow (vehicles per unit time) is computed as $q = N/T$. Under light flow, the observed flow per unit time can be considered to have a Poisson distribution with arrival rate λ. Hence for time period T, the number of vehicles N has a Poisson distribution with mean λT. The observed ratio, q, has a mean equal to λ, since

$$E(q) = E(N/T) = \frac{1}{T}E(N) = \frac{\lambda T}{T} = \lambda$$

However, the value T may itself be considered to represent approximately the sum of N vehicle headways (times between front bumpers) so that $T = \sum_{i=1}^{N} h_i$. We may therefore write

$$q = \frac{N}{T} = \frac{N}{\sum_{1}^{N} h_i} = \left[\frac{1}{N}\sum_{1}^{N} h_i\right]^{-1} = \frac{1}{\bar{h}}$$

where $\bar{h}$ is the mean headway. This is only an approximation of the actual situation because, if N vehicles were actually counted, then the time interval T would represent the sum of N-1 headways and 2 partial headways.

Now consider the total observed flow q to consist of the sum of N individual flows, or let $q = \sum_{1}^{N} q_i$, where $q_i = 1/h_i$ in which q_i and h_i are respectively the instantaneous flow and headway associated with each vehicle.

Then we may write

$$q = 1/\bar{h} = \frac{1}{\frac{1}{N}\sum h_i} = \left(\frac{1}{N}\sum \frac{1}{q_i}\right)^{-1}$$

Thus the average flow, when computed from individual flows associated with each vehicle, is the harmonic mean of the individual flows.

Example 2: Detector Measurements—There are various methods of reducing traffic data from measurements taken from a detector or pair of detectors at a location.

Method 1: Pneumatic Tubes (BPR Traffic Analyzer)—A pair of pneumatic tubes are stretched across a given lane on the roadway; these tubes are usually separated by a distance d, 8 to 10 ft apart. When a vehicle's front tires cross over the first tube a signal is sent to a counter to register its arrival time and when the front tires cross the next tube another counter records another arrival time. The difference of these arrival times, t_i, represents the traversal time—the time it took the vehicle to traverse the known distance d. Let us assume that T is the entire period of observation while N is the total count (9). Then the flow, speed, and concentration may be obtained by means of the following formulas.

Individual speeds: $v_i = d/t_i \quad (i = 1, 2, \ldots, N)$

Space mean speed estimate:

$$\bar{v}_s \approx \frac{Nd}{\sum t_i} = \frac{N}{\sum t_i/d} = \left[\frac{1}{N}\sum \frac{1}{v_i}\right]^{-1}$$

$= \text{harmonic mean of the spot speeds}$

$= \bar{v}_h$

Flow: $q \approx N/T$

Concentration: $k = q/\bar{v}_s \approx \frac{N}{T} \cdot \frac{\sum t_i/d}{N} = \frac{1}{d}\frac{\sum t_i}{T}$

Method 2: Tape Switch or Occupancy Detector—Another type of method to directly measure flow q and speed $\bar{v}_s$ is from a simple occupancy detector or tape switch over a particular lane during a data sampling period of duration T. Let

N_T = number of vehicles that traversed the detector during interval T;
O_T = estimated portion of the time T that the axles of the vehicle were sensed by the detector (occupancy);
t_i = traversal time of the ith vehicle length sensed by the detector; and
L = average length of vehicles.

Then we can form the following computations.

Occupancy: $O_T = \sum t_i$

Individual speeds: $v_i = L/t_i$
Space mean speed estimate:

$$\bar{v}_s \approx N_T \frac{L}{O_T} = N_T \frac{L}{\sum t_i} = N_T \left(\sum t_i/L\right)^{-1}$$

$$= \left(\frac{1}{N_T} \sum_{1}^{N_T} \frac{1}{v_i}\right)^{-1} = \bar{v}_h$$

= harmonic mean of the spot speeds

Flow: $q \approx N_T/T$

Concentration: $k \approx q/\bar{v}_h = \frac{N_T}{T} \frac{\sum t_i/L}{N_T}$

$$= \frac{\sum t_i}{TL} = O_T/(TL)$$

Method 3: Detectors Involving Two Classes of Vehicles—Consider two classes of vehicles such as passenger and commercial vehicles that can be distinguished by height sensors installed under overpasses. Denote the measurements on occupancy, number of vehicles, and average vehicle length by O_i, n_i, and L_i, where $i = 1, 2$ designates passenger and commercial vehicles respectively.

In order to use Eq. 11, it is necessary to obtain an estimate for

$$\bar{v}_h = \left[\frac{1}{\sum_{1}^{2} n_i} \sum_{i=1}^{2} \sum_{j=1}^{n_i} v_{ij}^{-1}\right]^{-1}$$

where v_{ij} is the jth measurement for the speed of a vehicle in the ith class. However, the quantity $\sum_{1}^{n_i} v_{ij}^{-1}$ is estimated by $\sum_{1}^{n_i} t_{ij}/L_i = O_i/L_i$, where t_{ij} is the time measurement for the jth vehicle in the ith class.

Hence the approximation for $\bar{v}_s$ is given by

$$\bar{v}_h = \sum_{1}^{2} n_i \left[\sum_{1}^{2} O_i/L_i\right]^{-1}$$

If we now write

$$O_i/L_i = \sum_{1}^{n_i} t_{ij}/L_i = \sum_{1}^{n_i} v_{ij}^{-1} = n_i \, (\bar{v}_{h,i})^{-1}$$

where $\bar{v}_{h,i}$ is the harmonic mean speed in category i, we have for the combined harmonic mean

$$\bar{v}_h = \sum_{1}^{2} n_i \left[\sum_{1}^{2} n_i/\bar{v}_{h,i}\right]^{-1}$$

as the estimate of the space mean speed. This relation is easily extended to apply to several classes of vehicle lengths instead of only two. Reference is made to the corresponding formula found in Weinberg et al. (18), which differs from the above.

The calculations for flow and concentration are $q = \sum_{1}^{2} n_i / T$ and $k = q/\bar{v}_h$ respectively, where T is the total observation time.

Relationships in Wardrop's Discrete Model

One can find basic relationships that relate time and space distributional properties. The first to consider is that

$$p_s(i) = \frac{\bar{v}_s}{v_i} p_t(i) \qquad i = 1, 2, \ldots, C \tag{12}$$

where $p_s(i)$ and $p_t(i)$ are the corresponding space and time probabilities for Wardrop's substream or discrete model. This is the discrete analogue for the corresponding relation found in the continuous case by Haight (10) and by Breiman (3). Equation 12 is directly obtained by using Eqs. 2 and 4 in the definition of $p_s(i)$; i.e.,

$$p_s(i) = k_i/k = \frac{q_i/v_i}{q/\bar{v}_s} = \frac{\bar{v}_s}{v_i} p_t(i)$$

Another important relationship is that found by Wardrop:

$$\sigma_s^2 = \bar{v}_s(\bar{v}_t - \bar{v}_s) \tag{13}$$

where σ_s^2, $\bar{v}_s$, and $\bar{v}_t$ are respectively the space speed variance, space mean speed, and time mean speed. This is proved by employing the definitions of the variance and mean, as follows:

$$\sigma_s^2 = \sum_{i=1}^{C} (v_i - \bar{v}_s)^2 p_s(i) = \sum_{i=1}^{C} v_i^2 p_s(i) - \bar{v}_s^2 \tag{14}$$

where $\bar{v}_s = \sum_{1}^{C} v_i p_s(i)$ is the expectation of V_s.

By using Eq. 12, the summation term on the right reduces to

$$\sum_{i=1}^{C} v_i^2 \left(\frac{\bar{v}_s}{v_i} p_t(i) \right) = \bar{v}_s \sum_{1}^{C} v_i \frac{q_i}{q} = \bar{v}_s \bar{v}_t$$

which when substituted in Eq. 14 yields Eq. 13. It may be seen that more general relations involving the moments of the space and time speed distributions can be derived from Eq. 12. Thus, if $\mu_s'^{(r)}$ and $\mu_t'^{(r)}$ designate the corresponding r th moments of the space and time distributions about the origin, we have

$$\mu_s'^{(r+1)} = \bar{v}_s \, \mu_t'^{(r)} \tag{15}$$

For the corresponding moments about the mean (central moments) between the time $(\mu_t^{(r)})$ and space $(\mu_s^{(r)})$ distributions, the following formula can be established:

$$\mu_t^{(r)} = \sum_{j=0}^{r} \binom{r}{j} \left(\frac{\mu_s^{(j+1)}}{\bar{v}_s} + \mu_s^{(j)} \right) (\bar{v}_s - \bar{v}_t)^{r-j} \tag{16}$$

CONTINUOUS SPACE AND TIME SPEED DISTRIBUTIONS

The speed of a vehicle is generally considered to obey some unknown continuous distribution such as a Gaussian or gamma distribution instead of the discrete distribution considered by Wardrop in his substream model. In Haight (10) there is introduced a basic relation that connects space and time distribution of speeds for which an intuitive argument was provided. If one lets the space and time distribution of speeds be represented by the corresponding probability density functions $f_s(v)$ and $f_t(v)$, then, analogous to Eq. 12, the following is obtained:

$$f_s(v) = \frac{\overline{V}_s}{v} f_t(v) \qquad 0 \leq v < \infty \tag{17}$$

where $f_s(v)$ and $f_t(v)$ are identically zero for $v < 0$.

Breiman (3) provides a rigorous proof for Eq. 17 that involves an analysis of the time-space process of speeds, and in fact his result is applicable to a more general distribution function that may involve discontinuities. Wardrop's substream model, involving a completely discrete or discontinuous set of probabilities, is in fact a special case of Breiman's result.

A heuristic proof of Eq. 17 may be developed from the discrete relation of Eq. 12. This development follows.

Let the range of V be finite, with minimum and maximum values of 0 and M respectively. Now partition the closed intervals (0, M) into n subintervals defined by (v_{i-1}, v_i) for $i = 1, 2, \ldots, n$, where $v_0 = 0$, $v_{i-1} < v_i$ and $v_n = M$. We can designate this partition by $I_n = \{v_{i-1}, v_i\}$.

If the randon variable V belongs to the ith interval, we can arbitrarily assign the value v_i to V. Thus for the partition of n intervals we can associate the probability that V assumes the value v_i by means of the probability $p(v_i)$. If the random variable is in a space process, we designate the probability by $p_s(v_i)$, and if it is in the time process, we designate the probability by $p_t(v_i)$.

Thus for the partition I_n we know from Eq. 12 that $p_s(v_i) = \overline{v}_s p_t(v_i)/v_i$.

Let the number of subdivisions be increased, with each interval being made sufficiently small so that with good approximation we have

$$p_s(v_i) \approx f_s(v)\,\Delta v \text{ and } p_t(v_i) \approx f_t(v)\,\Delta v$$

where

$$\Delta v = v_i - v_{i-1} \text{ and } v_i \approx v.$$

For any such fine partition we then have, applying Eq. 12,

$$f_s(v)\,\Delta v \approx \frac{\overline{V}_s}{v} f_t(v)\,\Delta v$$

which, upon division of both sides by Δv, completes our proof.

BREIMAN'S FUNDAMENTAL THEOREM

Previous results have provided us with an essentially unbiased expression for the mean space speed $\overline{v}_s$. Thus the harmonic mean $\overline{v}_h$ of the synchronous time speeds at a given point on the road is used to estimate $\overline{v}_s$. This may be put in the form (see Eq. 9)

$$\overline{v}_s = E_s(V) = \left[E_t(1/V)\right]^{-1} = \left[E_{sy}(1/V_0)\right]^{-1} \tag{18}$$

wherein the subscripts t and sy on the right side have almost identical meanings. Although the subscript t was previously used to indicate the synchronous speed of the vehicle or "sy" as it passed a ground detector, it could have been applied to the asynchronous time case discussed previously under Example 1: Manual Volume Counts. Similarly, the quantity V_0 represents the observed speed of a car, C_0, when it reaches a designated location L_o. It would be equivalent to the quantity V in the above expression under E_t.

The estimate of the right side of Eq. 18 is the harmonic mean $\overline{v}_h$ of the synchronous speeds at L_o, or

$$\overline{v}_s \approx \left\{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{v_{oi}}\right\}^{-1} = \overline{v}_h \tag{19}$$

where $v_{o1}, v_{o2}, \ldots, v_{on}$ represent the set of observed speeds of each successive car, C_{oi}, ascertained when it reaches L_o.

Breiman's theorem (4) allows us to form the unbiased estimate of any function of speed and headway in the space process in terms of a similar function in the synchronous time process. It therefore allows us to develop valid analyses of traffic data reduced from aerial photographs. From aerial data, not only joint speed characteristics but also headway measurements of successive vehicles are obtained.

As mentioned earlier, one wants to compile synchronous time data. On each frame a particular location on the roadway is referenced, say L_o. If one is interested in a particular lane, then we first see what the traffic looks like at the instant of time when the front part of the vehicle passes directly over that location. This vehicle is labeled C_o, and its downstream predecessors are $C_1, C_2, \ldots$, and its followers are $C_{-1}, C_{-2}, \ldots$.

At the instant of time when C_o reaches location L_o, say, the following joint set of synchronous time measurements is simultaneously obtained (provided of course that they appear on the same frame):

Symbol Identification	←Upstream	At L_o	Downstream→
Vehicle:	$\ldots, C_{-2}, C_{-1},$	$C_o,$	$C_1, C_2, C_3, \ldots$
Location:	$\ldots, L_{-2}, L_{-1},$	$L_o,$	$L_1, L_2, L_3, \ldots$
Space headway (in ft) or gap:	$\ldots, X_{-1}, X_{-1},$	$X_o,$	$X_1, X_2, X_3, \ldots$
Speed (ft/sec):	$\ldots, V_{-2}, V_{-1},$	$V_o,$	$V_1, V_2, V_3, \ldots$

In practice, the front bumper (location) of C_o may not be at L_o for any frame. In general, its location is ascertained by the linearly interpolated distance between two successive frames. This interpolation is similarly performed for the other vehicles. The speeds can be obtained by simply dividing the distance moved for each vehicle from one frame to the next by the frame lapsed time. Each time that vehicle C_{-1} reaches L_o, then C_{-1} is redesignated C_o, and all other vehicles are similarly relabeled.

By applying Breiman's powerful fundamental theorem on synchronous data (4), any function of the space headway and speed process, say $\phi(X, V) = \phi(X_1, \ldots, X_n; V_1, \ldots, V_n)$, may be estimated by

$$E_s\, \phi(X, V) = \overline{v}_s\, E_{sy}\left[\phi(X, V)/V_o\right] \tag{20}$$

where the left side represents the average of any arbitrary function ϕ of the space headway and speed process while the expectation E_{sy} on the right represents the average of the same function of the synchronous (time) headway and speed process, each divided by the synchronous speed V_o at L_o. Some examples for the use of Eq. 20 follow.

Example 1: Equation 18

Let ϕ be identically equal to 1 in Eq. 20. This is allowed because ϕ is arbitrary. Then, since on the left the expected value of a constant equals that constant, we have

$$1 = \overline{v}_s\, E_{sy}\,(1/V_o)$$

which is the well-known result that $\overline{v}_s$ is the harmonic mean of the speeds at a fixed spot and is estimated by Eq. 19 in terms of the sample harmonic mean $\overline{v}_h$.

Example 2: Equation 17

In Eq. 20, let $\phi(X, V) = \phi(V)$, which is the outcome of successive values of V_o or speeds when cars pass the origin, L_o. The expectation on the right side of Eq. 20 is the expectation of $\phi(V)/V$ under the time distribution of speeds while the left side is the expectation of $\phi(V)$ under the space distribution of speeds. Thus we write Eq. 20 as

$$\int \phi(v) f_s(v)\, dv = \overline{v}_s \int \frac{\phi(v)}{v} f(v)\, dv$$

Since this holds for all functions $\phi(v)$, it certainly holds for

$$\phi(v) = \begin{cases} 1 & \text{where v is included in the interval } (v', v' + \Delta v') \\ 0 & \text{where v is not included in the interval } (v', v' + \Delta v') \end{cases}$$

From this we obtain for any v

$$f_s(v') = \overline{v}_s \frac{1}{v'} f_t(v')$$

which gives an alternate proof for relation 17.

Example 3: Variance of the Space Speed (σ_s^2)

By letting $\phi(V) = V^2$ in Eq. 20, we have

$$E_s V^2 = \overline{v}_s E_t(V) = \overline{v}_s \overline{v}_t$$

from which we obtain

$$\sigma_s^2 = E_s V^2 - \overline{v}_s^2 = \overline{v}_s (\overline{v}_t - \overline{v}_s)$$

which is another well-known result.

Example 4: Expectation of Headway Distances in Space

Let $\phi = X_o$ and apply the fact that $E_s X_o = 1/k$. Equation 20 then becomes

$$\frac{1}{k} = \overline{v}_s E_{sy}(X_o V_o)$$

or

$$\frac{1}{q} = E_{sy}(X_o/V_o) \approx \frac{1}{n} \sum_{1}^{n} X_{oi}/V_{oi}$$

which is similar to one of Edie's formulas (7, Table 1) for measurements at a point. That is, let X_{oi} approximate $1/k_i$ while we let $\frac{1}{V_{oi}}$ approximate k_i/q_i (using Wardrop's Eq. 2.2). Then we obtain the corresponding relation,

$$\frac{1}{q} = \frac{1}{n} \sum_{1}^{n} (1/q_i)$$

Example 5: Expression as an Arithmetic "Mean"

In general, for any function ϕ, Eq. 20 provides an operational method of estimating the space expectation of any function of speeds and headways. For large N, we can write the theorem as

$$E_s\, \phi(X, V) \approx \overline{v}_h \cdot \frac{1}{N} \sum_{i=1}^{N} \phi \left\{ \left(X^{(i)}, V^{(i)} \right) \middle/ V_o^{(i)} \right\}$$

where $\overline{v}_h = \left\{ \frac{1}{N} \sum_{1}^{N} \frac{1}{V_o^{(i)}} \right\}^{-1}$

Thus Breiman's recipe for estimation is

1. Look at those time instants at which C_o passes L_o.
2. At those instants, calculate speeds and headways counting upstream and downstream.
3. Find the value of the function ϕ for these measurements; i.e., if $\phi = (X_o + X_1)\, V_o$, then the succession of measurements is represented by the vector

$$\{\phi_i\} = \left[\frac{X_{o1} + X_{11}}{V_{o1}}, \frac{X_{o2} + X_{12}}{V_{o2}}, \ldots, \frac{X_{oN} + X_{1N}}{V_{oN}} \right]$$

where, at the first time instant, X_{o1} and X_{11} are headways of C_o and C_1 while V_{o1} is C_o's speed, and so on.

4. Divide each ϕ by the corresponding value V_{oi}, the speed of the car at the origin, L_o.
5. Take the arithmetic mean of ϕ_i/V_{oi} and multiply by the harmonic mean of the V_{oi}; i.e., the space distribution estimate is

$$E_s \left\{ \frac{X_o + X_1}{V_o} \right\} \approx \left[\frac{1}{N} \sum_{1}^{N} \frac{1}{V_{oi}} \right]^{-1} \left[\frac{1}{N} \sum_{1}^{N} \frac{X_{oi} + X_{1i}}{V_{oi}^2} \right]$$

This example is given only for purposes of illustrating Eq. 20.

Many more illustrative examples can be formulated for the application of Breiman's theorem. One can obtain useful formulas for the dependence of successive headways or speeds of various order lags. It should be noted that the headways in this section were expressed in units of distance. Breiman calls these space headways as distinguished from time headways, which are expressed in units of time. Conventional nomenclature by highway engineers refers to Breiman's space and time headways as gaps and headways respectively. It could cause some confusion to discuss the space and synchronous distributions of "space headways", so it would perhaps be preferable to refer to Breiman's result as "the relation between the space and synchronous time distributions of gaps and speeds".

However, this relation also holds between the space and synchronous time distributions of headways and speeds, since a gap can always be expressed as a headway by simply dividing it by the speed, e.g.,

$$\phi = \frac{X_o^2}{V_o} = \frac{X_o^2}{V_o^2} \cdot V_o = H_o^2 \cdot V_o$$

where H_o is the corresponding headway for car C_o.

It should be stressed that the function $\phi(\ )$ does not have to involve V. In fact, it may be deduced from the derivation of Eq. 20 that any function of traffic involved in the carrier space process of speeds could have been substituted for gap (X) or headway (H). Thus, any of the characteristic traffic variables mentioned in the introduction could be substituted for X in order to obtain an unbiased estimate of its space mean. For example, one could obtain an unbiased estimate of the average number of vehicle occupants by using Eq. 20. This has been examined for several extreme cases as well as for an intermediate joint set of speeds and number of car occupants where the true space mean speed and mean number of occupants were known. It was ascertained that the usual method of the arithmetic mean number of occupants (ignoring speeds) may produce a slight bias for the intermediate case but could present a large bias in the extreme cases. In every instance, however, it was shown that the synchronous method of Breiman produced an unbiased estimate. This indicates the utility of traffic flow theory in allowing one to examine the validity of alternative methods as well as to provide an unbiased method of estimating traffic characteristics. This type of analysis is applicable to other measurement variables such as energy, age or make of vehicle, and proportion of heavy vehicles.

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