BEARING CAPACITY OF ANISOTROPIC AND NONHOMOGENEOUS CLAYS UNDER LONG FOOTINGS

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There is experimental evidence that saturated clays in undrained conditions have depth-dependent anisotropic strength properties. Hence there is a need for an appropriate strength law to determine the failure field for the medium in question. The critical bearing capacity of a footing in this medium was determined by a modified version of Hill's failure mechanism. According to this, in a nonhomogeneous medium where strength increases depth, failure takes place closer to the surface than it does according to Prandtl's mechanism. Anisotropy and nonhomogeneity must be taken into consideration in general.

•ANISOTROPIC or nonhomogeneous materials are the subject of numerous studies (1, 2, 3, 4, 5, 6, 7, 8). In engineering applications there is frequently a shear line along which the anisotropic strength is correlated (9, 10, 11, 12, 13). Livneh, Greenstein, and Shklarsky (14, 15, 16, 17) claim, however, that this approach is meaningless in an anisotropic medium and is thus inapplicable for practical purposes. The strength of saturated clay in undrained conditions is known to be anisotropic (2, 18) and to increase with depth (19). Therefore, to determine critical bearing capacity under long footings we have to formulate the strength law (yield function) and from it derive the field of failure (slip-line field) for the given medium. The notations used in the formulations in this paper are as follows:

- c = cohesion,
- c_o = surface value of cohesion,
- $\bar{C}$ = material constant in Hill's model,
- $D_1, D_{\pi/4}, D_2$ = material constants,
- $D(\psi, x, y)$ = strength factor,
- G = depth gradient of D,
- f = yield function,
- i, j = characteristic directions,
- J_2 = second invariant of stress deviator,
- k = material constant,
- N_c = critical bearing-capacity factor,
- p = mean stress in plane,
- q = critical bearing capacity,
- u, v = axial velocity components,
- x, y = coordinate system in plane,
- $\lambda_y, n_{..}$ = anisotropic functions,
- ξ = angle between x and major principal strain-rate directions,
- σ_1, σ_3 = principal components of stress tensor,
- $\sigma_x, \sigma_y, \tau_{xy}$ = components of stress tensor in (x, y) plane, and
- ψ = angle between x and major principal stress directions.

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YIELD FUNCTION

The principal mechanical property of a cohesive 2-dimensional medium such as the clay described previously is a yield function independent of the mean stress whose general form is

$$f = J_2^{1/2} - D = 0 \tag{1}$$

where D = strength factor, which is D (ψ, x, y). D also equals the radius of the Mohr circle at failure; in an anisotropic medium it is not identical with cohesion (14, 15, 16, 20). That is to say,

$$J_2 = \tfrac{1}{2} S_{ij} S_{ij} = \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 = \frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2 \tag{2}$$

For a homogeneous medium, the yield function reduces to a model such as that proposed by Hill (21) or Davis and Christian (2).

In this study, we present for practical purposes the following pattern for D:

$$D = D(\psi)_{y=0} + Gy \tag{3}$$

where

ψ = angle (positive counterclockwise) between x and σ_1 directions (Fig. 1), and
$G = \partial D/\partial y$, the depth gradient of D (Fig. 2).

The first term in Eq. 3 represents the surface variation (y = 0), as follows:

$$D(\psi)_{y=0} = D_1 - \left(D_1 - D_{\pi/4} + \frac{D_2 - D_1}{2^k}\right) \sin^{2k} 2\psi + (D_2 - D_1) \sin^{2k} \psi \tag{4}$$

where

D_1, $D_{\pi/4}$, and D_2 = ψ = 0, $\pi/4$, and $\pi/2$ respectively [Fig. 3 (2)], and
k = material constant.

This paper deals with cases of constant G (19).

FIELD OF FAILURE IN ANISOTROPIC INHOMOGENEOUS CLAY

To determine the stress field in an anisotropic clay medium, the components of the stress tensor at failure should be formulated as follows:

$$\sigma_{x,y} = p \pm D \cos 2\psi$$
$$\tau_{xy} = D \sin 2\psi \tag{5}$$

When Eq. 5 is substituted, the equilibrium equations read:

$$\frac{\partial p}{\partial x} + 2D(-\sin 2\psi + n_o \cos 2\psi) \frac{\partial \psi}{\partial x} + 2D(n_o \sin 2\psi + \cos 2\psi) \frac{\partial \psi}{\partial y} + G \sin 2\psi = 0 \tag{6}$$

$$\frac{\partial p}{\partial y} + 2D(n_o \sin 2\psi + \cos 2\psi) \frac{\partial \psi}{\partial x} + 2D(\sin 2\psi - n_o \cos 2\psi) \frac{\partial \psi}{\partial y} - G \cos 2\psi = 0 \tag{7}$$

where

$$n_o = \frac{\frac{\partial D}{\partial \psi}}{2D} = \tfrac{1}{2} \frac{\partial}{\partial \psi} \ln[D(\psi)_{y=0} + Gy] \tag{8}$$

Figure 1. Description of the coordinate system.

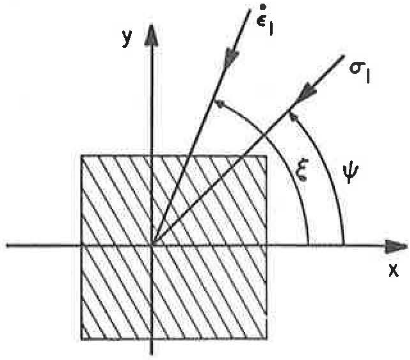


Figure 2. Variation of the strength factor, D, with depth.

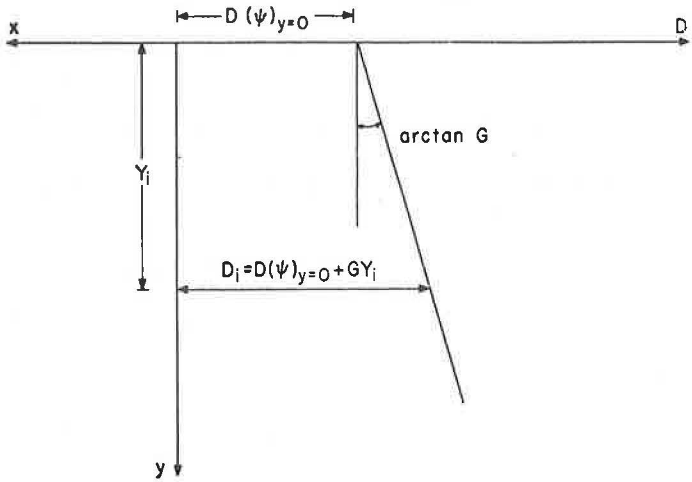
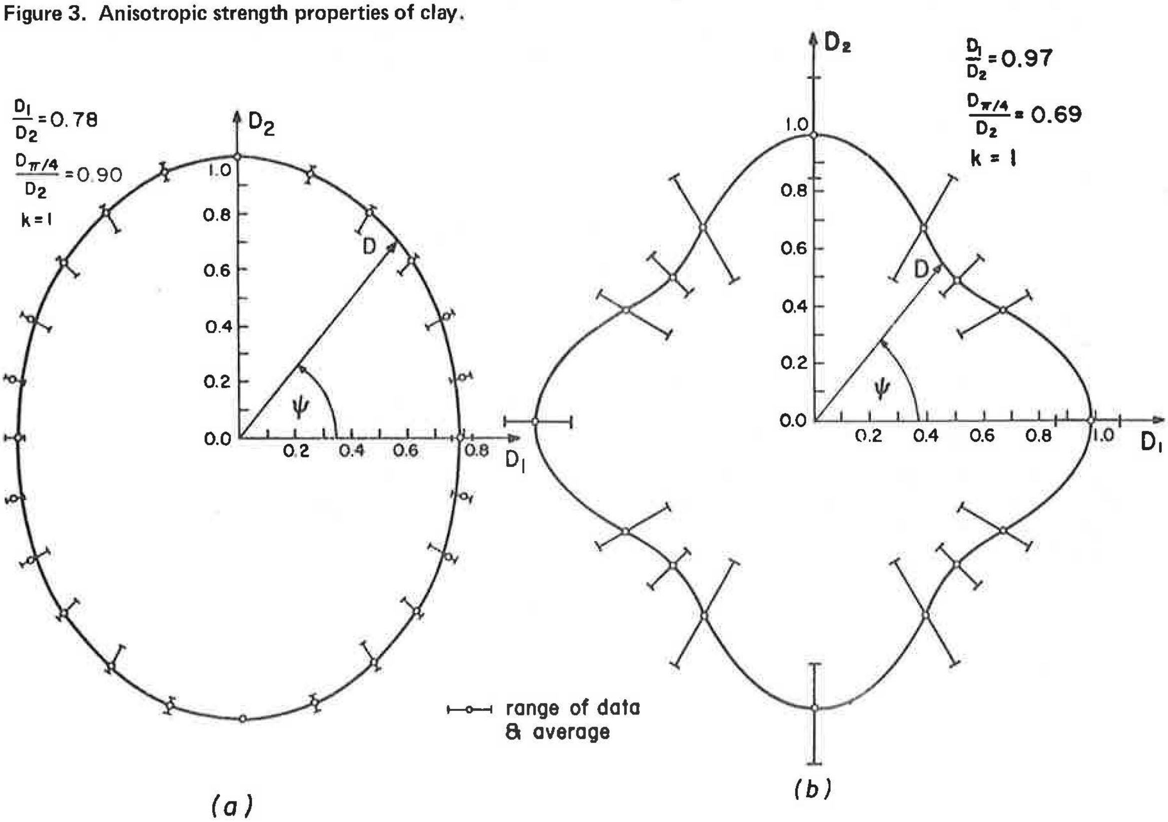


Figure 3. Anisotropic strength properties of clay.



This is not exclusive for a strength factor that obeys Eq. 4, but it is valid for any plausible continuous pattern, such as Hill's or Davis' models. Equations 7 and 8 are hyperbolic and yield

$$\begin{vmatrix} 1 & 0 & 2D(-\sin 2\psi + n_o\cos 2\psi) & 2D(n_o\sin 2\psi + \cos 2\psi) \\ 0 & 1 & 2D(n_o\sin 2\psi + \cos 2\psi) & 2D(\sin 2\psi - n_o\cos 2\psi) \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{vmatrix} = 0 \tag{9}$$

which in turn has 2 roots

$$\frac{dy}{dx} = \frac{\sin 2\psi - n_o\cos 2\psi \pm \sqrt{1 + n_o^2}}{\cos 2\psi + n_o\sin 2\psi} = y'_{i,j} \tag{10}$$

which represent the slopes of the characteristic lines in the stress field. These lines, in the anisotropic-inhomogeneous cohesive media, meet at right angles.

The compatibility equations of the stress field that are satisfied along characteristic lines are as follows:

$$dp + \bar{\lambda}_y d\psi + G(\sin 2\psi - \cos 2\psi\, y'_i)dx = 0 \quad \text{along i line} \tag{11}$$

$$dp - \bar{\lambda}_y d\psi + G(\sin 2\psi - \cos 2\psi\, y'_j)dx = 0 \quad \text{along j line} \tag{12}$$

where

$$\bar{\lambda}_y = 2[D(\psi)_{y=0} + Gy]\sqrt{1 + n_o^2} \tag{13}$$

Equations 11 and 12 are the equilibrium equations and reduce to those of Hill and Davis for their particular models in a homogeneous medium and to those of Hencky in an isotropic-homogeneous one, where D represents cohesion.

The stress field is identical with the velocity characteristics of the following model, which represents the particular case of displacement under plastic deformation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{14}$$

$$\cot 2\xi = \frac{\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}}{\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}} = \frac{1 + n_o\tan 2\psi}{\tan 2\psi - n_o} \tag{15}$$

where

u and v = axial velocity components, and

ξ = the angle (positive counterclockwise) between the x and major principal strain-rate directions.

Equation 14 indicates that the plastic deformation involves no change of volume; Eq. 15 indicates that in an anisotropic medium, the directions of the principal stresses and strain rates are not coincidental. Such is the case, for example, with Hill's model (21), for which Eq. 15 reduces to

$$\tan 2\xi = (1 - \bar{C})\tan 2\psi \tag{16}$$

In an isotropic medium ($n_o = 0$), $\psi = \xi$, and the directions coincide.

BEARING CAPACITY OF FOOTING IN ANISOTROPIC, NONHOMOGENEOUS CLAY MEDIUM

Figure 4 shows that the case in question is symmetric, and that along the edge in fields 1 and 2, $\psi = 0$ and $\pi/2$ respectively; for both of which, $\partial D/\partial\psi = 0$ (D obeys Eq. 4). Hence, in these zones, $n_o = 0$, and the compatibility Eqs. 11 and 12 reduce to

$$dp - G\cos 2\psi \; y'_{i,j} \; dx = 0 \qquad \text{along i, j line} \tag{17}$$

If we substitute $y'_{i,j}$, we have

$$dp \pm G dx = 0 \qquad \text{along i, j line} \tag{18}$$

Equation 10 shows that along the edge in fields 1 and 2 $y'_i = -y'_j$ $(dx_i = -dx_j)$. Accordingly, in the symmetric case with a constant mean stress, the compatibility equations are satisfied by straight lines. In other words, in an anisotropic, nonhomogeneous medium when D increases only vertically, each failure field consists of 2 families of straight lines that meet at right angles.

In the homogeneous and weightless case (anisotropic or isotropic), the critical bearing capacity according to Hill is identical with its Prandtl counterpart. However, for the case under discussion in this paper, when D increases with depth, the critical bearing capacity according to Hill is lower. This could be expected because the material undergoes shear closer to the surface. By symmetry, point E in Figure 4 must be the midspan of the footing; this agrees with Hill's mechanism. The assumption for the fan-shaped field 3 is that it is rigid and that the shear line j (BC in Fig. 4) is circular. This assumption signifies that the characteristic line i passes through the singularity S (along BC, the j line, $y'_i = y/x$). In these circumstances it can be shown that the following set of equations is satisfied along the j line:

$$\left.\begin{aligned}
\frac{dx}{d\psi} &= x\frac{\dfrac{dy'_i}{d\psi}}{y'_j - y'_i}\\
\frac{dy}{d\psi} &= xy'_j\frac{\dfrac{dy'_i}{d\psi}}{y'_j - y'_i}\\
\frac{dp}{d\psi} &= \bar{\lambda}_y - G(\sin 2\psi - \cos 2\psi \; y'_j)\frac{dx}{d\psi}
\end{aligned}\right\} \tag{19}$$

Equation 14 was shown by Ince (22) to have a unique solution for given initial conditions, in this case, for point A—$p_A = D_1$, $y_A = 0$, $x_A = B/2$, and $\psi_A = 0$. This solution represents an approximate upper bound for the critical bearing capacity as follows:

$$N_c = \frac{q}{D_2} = \frac{q}{D_E} = \frac{p_E}{D_2} + 1 \tag{20}$$

where D_E and p_E refer to point E, at which $\psi = \pi/2$.

Figure 5 shows N_c plotted against D_1/D_2 [with $(G/D_2)(B/10)$ and $D_{\pi/4}/D_2$ as parameters] where the dimension of B is identical to that of D_2/G. For the total error because of noninclusion of the anisotropy and nonhomogeneity, it is seen that when $D_1/D_2 = 1.25$, $D_{\pi/4}/D_2 = 1.0$, and $(G/D_2)(B/10) = 0.1$, the exact value is about 50 percent higher than its isotropic counterpart, $N_c = \pi + 2$. The comparison refers to an undisturbed surface sample tested under routine laboratory conditions ($\psi = \pi/2$). In certain circumstances, 1 component of the error cancels the other out, in which case [$(G/D_2)(B/10) = 0.05$, $D_1/D_2 = 0.6$, and $D_{\pi/4}/D_2 = 0.75$] the isotropic value may be used.

In an isotropic medium where $c = c_o + Gy$, the compatibility Eqs. 11 and 12 reduce to

Figure 4. Rupture mechanism for bearing capacity determination.

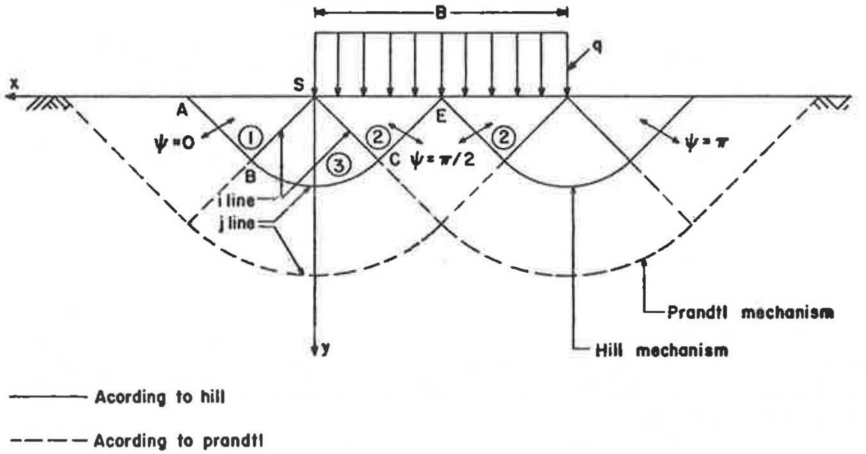
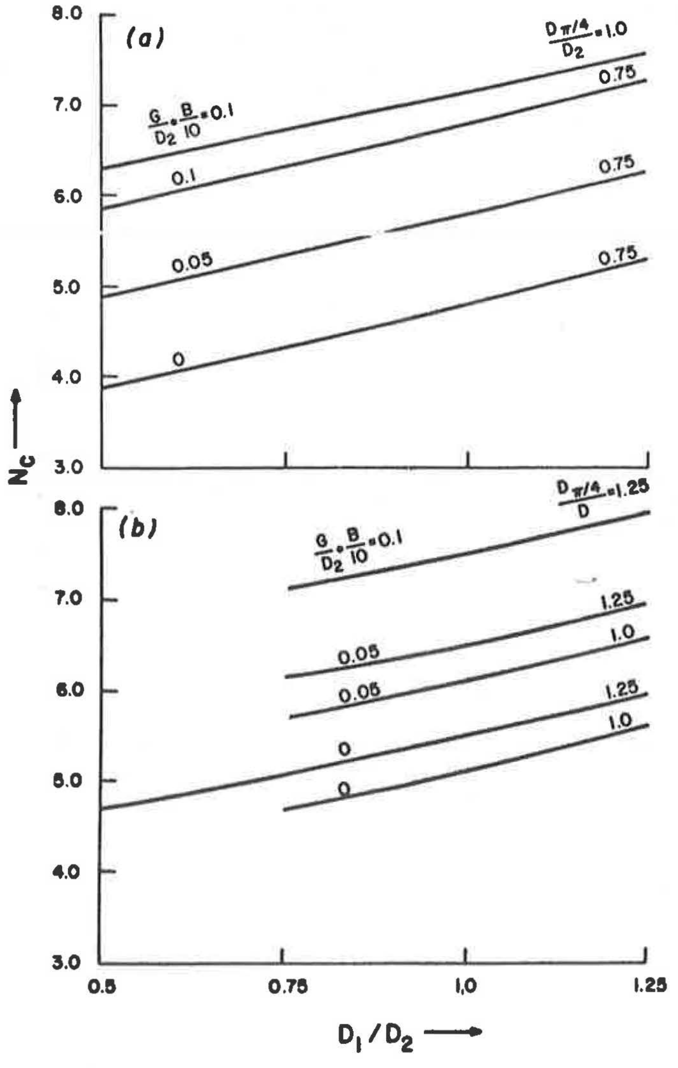


Figure 5. Effect of anisotropy and nonhomogeneity on N_c.



$$\begin{aligned} dp + 2c\,d\psi - G\,dx &= 0 \qquad \text{along i line} \\ dp + 2c\,d\psi + G\,dx &= 0 \qquad \text{along j line} \end{aligned} \tag{21}$$

According to Livneh and Greenstein (23), integration of the above yields the upper bound

$$q = (\pi + 2)\, c_{0.4B} \tag{22}$$

where $c_{0.4B}$ = cohesion at depth y = 0.4B.

CONCLUSIONS

In an anisotropic medium where the strength factor D increases with depth (as in the symmetric problem in Fig. 4), there are 2 fields of failure consisting of straight lines that meet at right angles.

Critical bearing capacity is determined by Hill's failure mechanism, as modified, in which the material is sheared closer to the surface than in Prandtl's model. Critical bearing capacity N_c is determined from a set of equations with a unique solution for given initial conditions, namely $P_A = D_1$, $Y_A = 0$, and $x_A = B/2$ (Fig. 4).

Anisotropy and nonhomogeneity may sometimes have a considerable effect on N_c compared to the isotropic value, $\pi + 2$, although in certain cases their contributions may cancel each other out. In an isotropic medium ($n_o = 0$) with depth-dependent strength, critical bearing capacity is obtained by multiplying the cohesion at depth 0.4B by the isotropic N_c.

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