Uncertainty Analysis of Settlement Rate

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The uncertainty associated with the prediction of the final settlement of structures built on compressible soils has been described by Corotis, El-Moursi, and Krizek (1). Another major problem is that of predicting the rate of settlement, which involves the uncertainties in the values assigned to both the total settlement (the soil-compressibility properties and applied loads) and the average degree of consolidation (the coefficient of consolidation). Consequently, any time versus settlement curve predicted on the basis of these parameters also has some degree of uncertainty, regardless of the accuracy of the equations that are used. In the analysis presented here, a simplified expression for the average degree of consolidation and a probabilistic approach to the time versus settlement relation are developed for a homogeneous clay stratum. The total settlement and the average degree of consolidation are considered as random variables, and the probability density function of the latter is combined with the probability distribution for the total settlement to develop the probability distribution of the time versus settlement curve. Although this study includes a large variety of different soils, the specific influences of the soil origin, the sampling techniques, and the detailed testing procedures on the data are not explicitly treated; it is fully recognized that these effects may be quite significant in some cases, and the results must be viewed within the context of these limitations.

DESCRIPTION OF PROBLEM

The time-dependent settlement [S(t)] associated with the primary consolidation of a homogeneous compressible soil stratum of thickness (H) underlying a foundation is usually determined from the relation

$$R(t) = S(t)/H = (S/H)U = RU \quad (1)$$

where

R(t) = settlement ratio at time (t),
S = total settlement, and
U = average degree of consolidation.

The settlement ratio (R) depends on the soil-compressibility properties and the applied loads and can be adequately expressed in a probabilistic design by a lognormal distribution (1). The average degree of consolidation depends on the coefficient of consolidation of the soil, the thickness of the soil stratum, and the nature of the drainage surfaces. Unfortunately, it is often impossible to determine these variables with complete certainty, and U should be considered a random variable in deriving a probability function for R(t). (In order to keep the problem tractable, the uncertainty associated with equation 1 itself will not be considered.)

RATE OF CONSOLIDATION

The basic theory of one-dimensional consolidation considers the time rate of compression of an idealized saturated soil. The magnitude of the settlement is determined by the compressibility of the soil structure alone, whereas the rate of settlement is controlled by the viscosity of the pore water, the geometry of the pore structure, and the compressibility of the soil structure. The differential equation describing the process of consolidation of a horizontal bed of clay is

$$\partial u/\partial t = c_v \, \partial^2 u/\partial z^2 \quad (2)$$

where

z = vertical coordinate,
c_v = average coefficient of consolidation, and
u = hydrostatic excess pressure.

The solution of equation 2 in terms of the transformed variable (U) gives

$$U = 1 - \sum_{i=0}^{i=\infty} (2/W_i^2) \exp(-W_i^2 T) \quad (3)$$

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where

$W_i = \pi(2i + 1)/2$ and
$T = c_v t/H^2$.

The complexity of handling equation 3, which is plotted in Figure 1, arises from the summation process, and it is desirable to either simplify it or to find an equivalent expression that overcomes this complexity within reasonable limits of accuracy.

Accordingly, if the summation term of equation 3 is set equal to $\exp(-\alpha T)$, the resulting expression for α becomes

$$\alpha = -(1/T)\ln \sum_{i=0}^{i=\infty} (2/W_i^2)\exp(-W_i^2 T) \tag{4}$$

and, for i = 0, equation 3 can be written as

$$U = 1 - [(8/\pi^2)\exp(-\pi^2 T/4)] \tag{5}$$

A plot of equation 5 is also shown in Figure 1, from which it can be seen that the average degree of consolidation can be adequately described by equation 5 for values of $T > 0.2$. Without significantly affecting the average degree of consolidation, equation 5 can be modified to satisfy the boundary conditions at T = 0 by approximating $8/\pi^2$ as unity, thereby allowing it to be simplified to

$$U = 1 - \exp(-\pi^2 T/4) \tag{6}$$

which is also shown in Figure 1. Finally, Figure 1 also illustrates a plot of U described by the one-constant hyperbolic model

$$U = T/(\Phi + T) \tag{7}$$

in which T is the time factor and Φ is a constant that was found to be 0.12 by minimizing the integrated squared difference between equation 7 and the exact expression for the average degree of consolidation given by equation 3. Such a method is especially useful here since the traditional approach of matching the slope at T = 0 cannot be used because equation 3 has an infinite slope at T = 0.

The infinite sum in equation 3 makes the classical consolidation equation untractable for use in deriving the probability distribution of U. From Figure 1 it is seen that equation 5 approximates equation 3 very accurately for all except very small values of the time factor. However, equation 5 was not used for further analysis because it violates the initial condition ($U \neq 0$ at T = 0), and its seemingly simple form becomes cumbersome in deriving a closed-form expression for the probability distribution of U. Instead, the approximation given by equation 6 was used because it has the correct limiting values (U = 0 at T = 0 and U = 1 at $T = \infty$), provides a simple mathematical form, and compares reasonably well with the solution obtained from classical consolidation theory. The hyperbolic model was not used because it has less theoretical justification than the exponential model and the resulting approximation is no better than that given by equation 6. The analysis developed here quantifies the uncertainty in the settlement rate on the basis that equation 6 is an acceptable approximation to the actual situation (i.e., equation error is neglected), and the results must therefore be viewed in this context. Results closer to those predicted by classical theory can be obtained by using equation 3 (with its associated increase in complexity) or by applying a correction factor to equation 6. The use of equation 6 is justified on the basis that (a) it is simple and leads to readily tractable expressions in the derivation, (b) there are insufficient detailed data to support the use of equation 3 with its associated complexity, and (c) the application of the suggested method of analysis (the derived distributions) is adequately demonstrated by the simplified approach.

APPLICATION OF PROBABILITY THEORY

Soil Data Reduction

The probabilistic approach adopted in this study was applied to data from over 700 consolidation tests on undisturbed soils of alluvial, marine, aeolian, and residual origin. About three-quarters of these data were obtained from Greece and its environs, and the rest were obtained from different parts of the United States; the same test procedure and specimen size were used in all cases.

A summary of the statistical parameters for U obtained at time factors of 0.075, 0.75, and 3.72 (corresponding to elapsed times of 1, 10, and 50 min respectively) and c_v at a consolidation pressure equal to 1200 kPa is given in Table 1. In an attempt to obtain soil groups having similar compressibility properties, the soil samples were divided into three main groups (A for $\gamma_d \leq 1.50$; B for $1.50 < \gamma_d \leq 1.75$; and C for $1.75 < \gamma_d$) based on dry density, which is known to be highly correlated with compressibility. For the purpose of determining the settlement ratio, group A was further subdivided into three subgroups (A-1 for $\gamma_d \leq 1.00$, A-2 for $1.00 < \gamma_d \leq 1.25$, and A-3 for $1.25 < \gamma_d \leq 1.50$). The results of a Kolmogorov-Smirnov one-sample test indicate that the null hypothesis that c_v satisfies the normal probability density function can be accepted beyond the 15 percent significance level. Consequently, the normal distribution will be used to describe the inherent variability of c_v in the probabilistic analysis of the time versus settlement prediction.

Probability Density Function

Since c_v is assumed to follow a normal distribution, U will follow a reverse lognormal distribution with a location parameter equal to one. Figure 2 shows a plot corresponding to the mean value of U and one standard deviation above and below the mean. Histograms at three different times are superimposed to indicate the amount of dispersion; these histograms show a skewed character for different values of elapsed time. The results of a Kolmogorov-Smirnov one-sample test indicate that U satisfies the reverse lognormal density function up to the 15 percent significance level. Although the derived distribution should theoretically be truncated at a minimum value of U = 0, this truncation will be significant only for small values of time (when the consolidation is small) and is therefore not of practical concern.

It has been shown (1) that the inherent uncertainty associated with the determination of R can be adequately described in a rational design by a lognormal distribution, and the above analysis indicates that U can be described by a reverse lognormal distribution. Thus, if R and U are assumed to be independent of one another, the time versus settlement ratio defined by equation 1 may be shown to follow a lognormal distribution with

$$\check{m}_{R(t)} = \check{m}_R \check{m}_U \tag{8}$$

and

$$\sigma^2_{\ln R(t)} = \sigma^2_{\ln R} + \sigma^2_{\ln U} \tag{9}$$

Figure 1. Degree of consolidation versus time factor.

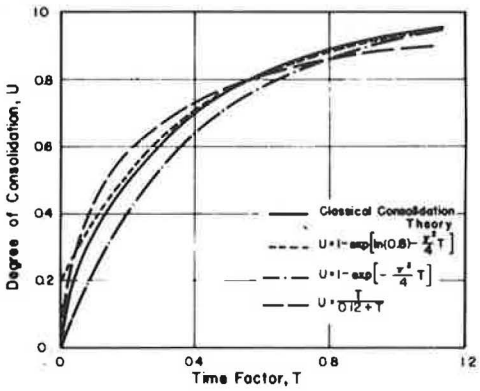


Figure 2. Degree of consolidation versus logarithm of time factor.

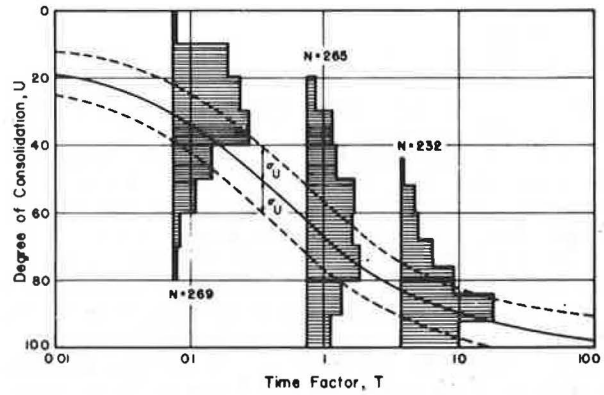
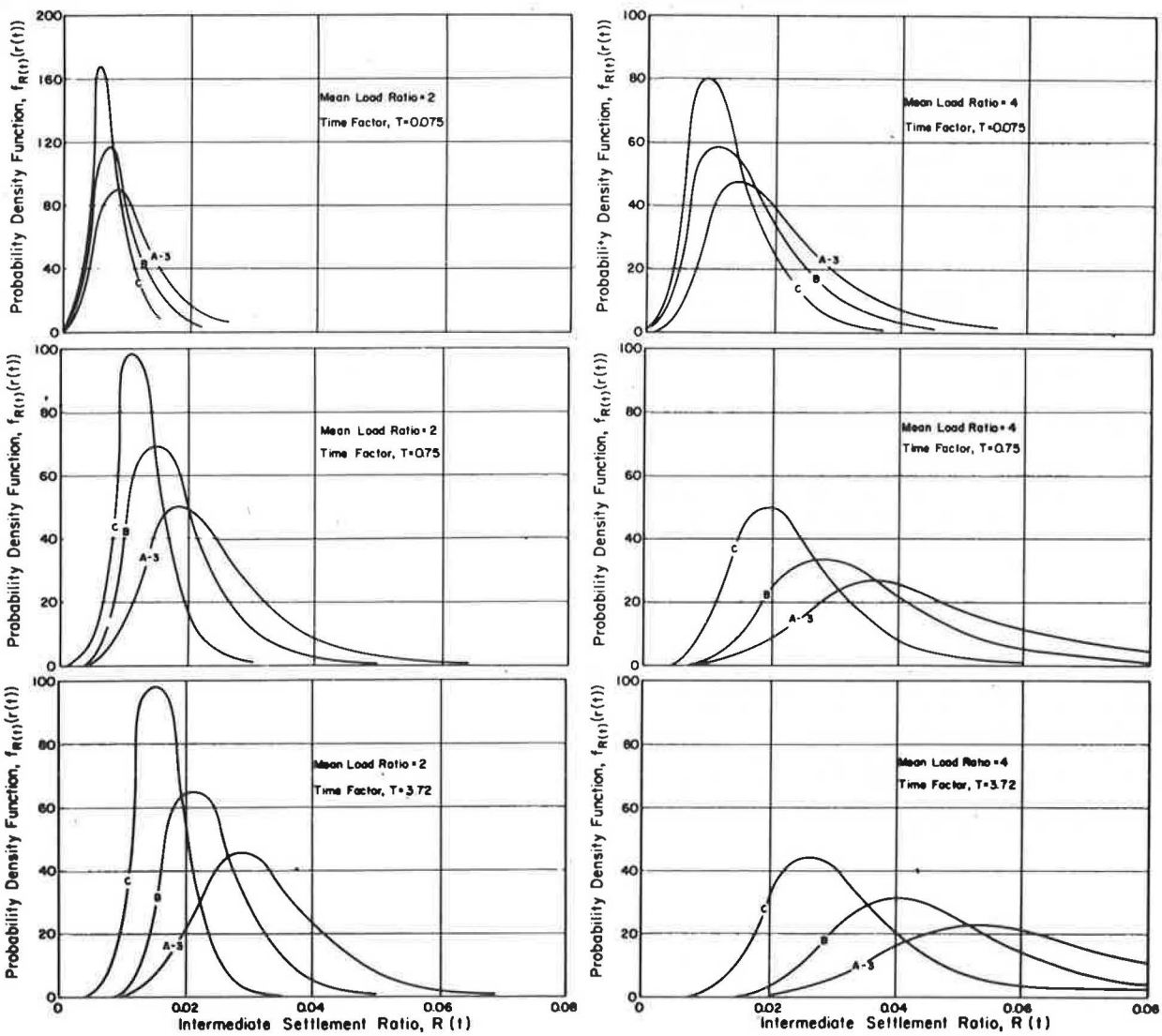


Table 1. Summary of statistical parameters for degree of consolidation and coefficient of consolidation.

Soil Property	Mean	Median	Standard Deviation	Standard Deviation of Mean	Coefficient of Variation	Skewness	Kurtosis	Number of Samples
Degree of consolidation at t = 1 min	31.52	29.03	13.35	0.838	0.424	0.787	3.383	269
Degree of consolidation at t = 10 min	62.50	60.10	17.90	1.132	0.286	-0.115	2.009	265
Degree of consolidation at t = 50 min	83.55	81.30	11.63	0.789	0.140	-0.889	3.087	232
Coefficient of consolidation	0.003	0.002	0.001	0.0001	0.400	0.682	2.836	221

Figure 3. Probability density functions of the intermediate settlement ratio for various times and mean loads.



where

$\check{m}$ = median and
σ_{ln} = standard deviation of the natural logarithm.

Figure 3 shows the probability density functions of the time-dependent settlement ratio for soil groups A-3, B, and C at mean load ratios of 2 and 4 and time factors of 0.075, 0.75, and 3.72. As the time factor and mean load ratio increase and as the dry unit weight of the soil decreases, the mode of the frequency curve is shifted to the right and the mean and the standard deviation of R(t) increase.

In the time versus settlement prediction problem, the variability of the average properties beneath a certain foundation rather than the point-to-point variability with the soil mass is needed. The variability of the average is a function of the spatial correlation structure and the number and location of the samples (2). As the number of samples (n) increases, the uncertainty associated with the predicted value of R(t) decreases. If n is reasonably large, the distribution of the time rate of settlement will be approximately normal with a mean of $m_{R(t)}$ and a standard deviation of $\sigma_{R(t)}/n^{1/2}$.

CONCLUSIONS

Based on the probabilistic analysis reported here, the following conclusions can be advanced:

1. The uncertainty associated with the evaluation of the coefficient of consolidation at a certain consolidation pressure can be adequately described by a normal distribution.
2. The reverse lognormal distribution with a location parameter equal to one appears to satisfactorily describe the average degree of consolidation.
3. The time versus settlement relation can be expressed in a probabilistic design by a lognormal distribution.
4. If the number of samples is reasonably large, the distribution of the time-dependent settlement ratio will be approximately normal, with the standard deviation being inversely proportional to the equivalent number of independent soil samples.

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