A MODEL FOR INVESTIGATING THE EFFECTS OF SERVICE FREQUENCY AND RELIABILITY ON BUS PASSENGER WAITING TIMES

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A model of bus and passenger arrivals at a bus stop is proposed. Passengers are considered to be either random arrivals or non-random arrivals whose time of arrival is planned so as to insure a given probability of catching a selected bus. Buses are modeled as having a lognormal distribution of arrival times from day to day. The impacts on expected wait time of service frequency and reliability for both random and non-randomly arriving passengers are identified. The effects of frequency and reliability on the proportion of the user population who plan their arrival time are also explored through a small empirical study. The empirical results support the conceptual basis of the model, and indicate that it should be a useful tool for transit operators and planners.

The wait time experienced by transit users is one of the most important elements of the level of service provided by a transit system. For this reason, it is important to understand the effects on wait time of changes in basic service characteristics, such as frequency of service and schedule reliability. Such information is vital to the transit operator for evaluation of the costs and benefits of service changes.

The commonly used model which asserts that average wait is one-half the headway is true only under very special circumstances. This model requires that all passengers arrive at the bus stop at random and that headways be perfectly regular. These conditions are not generally met in the real world.

Several improvements to this simplistic model have been suggested by various authors (2,3,4,5,6,7), but with one exception have not formally treated non-random arrivals of passengers. The inclusion of this category of passenger arrivals is one major objective of the model presented here.

A second major objective of the current study is to incorporate the effect of service reliability on passenger wait time. By focusing clearly on this issue, a model can be formulated which will allow the transit operator to evaluate the impacts of operating changes designed to improve the reliability of service.

Previous Studies

Several previous studies have investigated the waiting times experienced by bus transit passengers. If passengers arrive at a bus stop at random, the average time they will have to wait before a bus comes is

$$E(w_r) = \frac{E(h)}{2}\left\{1 + \frac{V(h)}{[E(h)]^2}\right\} \quad (1)$$

where w_r = wait time for a randomly arriving passenger
h = headway between buses
$E(\cdot)$ = expected value of a random variable
$V(\cdot)$ = variance of a random variable

This expression has been derived by a number of authors including Welding (7), Holroyd and Scraggs (2), and Osuna and Newell (5). However, if buses tend to adhere to a fixed schedule and there are passengers who make the same trip frequently, it may be expected that some passengers will plan their arrival at the bus stop so as to be there just before the bus comes. In this case, we might expect average waiting time to be less than given by equation 1. O'Flaherty and Mangan (4) and Seddon and Day (6) have provided data and analysis which support this assertion. Both studies found average wait time to be considerably less than that predicted on the basis of randomly arriving passengers, and Seddon and Day used regression analysis to arrive at the following relationship:

$$E(w) = 1.71 + 0.57\, E(w_r) \quad (2)$$

where $E(w)$ = observed average waiting time.

More recently, Jolliffe and Hutchinson (3) proposed an improved model based upon considering passengers to be of three types: a proportion q, whose arrival time is causally coincident with the bus; a proportion (1-q)p, who arrive so as to minimize expected wait time; and a proportion (1-q)(1-p), who arrive at random. The proportion q, whose arrivals are coincident with the bus arrival, represent those people who run to the stop because they see the bus coming, and thus wait zero time.

The arrival time which minimizes expected waiting time is found in the following way. For times t (in one-minute increments) the waiting times to the next bus were found for each of several days of observations. These times were then averaged to obtain EW(t), the expected waiting time for a passenger arriving at t. By doing this for many values of t, a minimum of EW(t) was observed, and this waiting time was taken to be the average wait for the proportion (1-q)p who arrive so as to minimize expected wait. A simpler procedure based on a model of the distribution of bus arrival times was discussed in an appendix to the paper, but was not used in the empirical work.

The present paper represents a further modification of the Jolliffe and Hutchinson model which is different in three significant ways. First, arriving passengers are simply considered to be either random or non-random. This simplification is based on the premise that passengers who run and catch the bus because they see it coming are really either random arrivals or non-random ("planned") arrivals. These two groups make decisions as to arrival time which are clearly different. However, this is not true for passengers whose arrival is coincident with the bus. These people are really random or non-random arrivals whose original decision as to arrival time is modified slightly as a result of seeing the bus coming. They do not constitute a behaviorally distinct group, and thus should be included in the two larger groups which are distinguishable.

The second way in which this study differs from that of Jolliffe and Hutchinson is in the use of a theoretical model for the probability distribution of bus arrival times. The observed data on bus arrivals are used to estimate this distribution, and then the arrival times of non-random arrivals are derived from the estimated distribution. This eliminates considerable computation and also permits the exploration of decision rules other than the minimization of expected wait time.

The third major difference from the Jolliffe and Hutchinson work is that non-random, or planned, arrivals are assumed to minimize expected wait time subject to a constraint which results in a fixed (small) probability of missing the selected bus. The imposition of this constraint reflects more risk-averse behavior, and is more consistent with anticipated actions of people who must either reach their destination (e.g., work) on time, or make connections to other scheduled transit services.

The Model

Passengers are considered to be either random arrivals or non-random arrivals. The observed average wait time of all passengers as a whole is then

$$E(w) = aE(w_n) + (1-a)\,E(w_r) \tag{3}$$

where $E(w_n)$ = expected wait time for non-random arrivals
$E(w_r)$ = expected wait time for random arrivals
a = proportion of non-random arrivals, $0 \le a \le 1$

The expected wait time for random arrivals is given by equation 1. The expected wait time for non-random arrivals can be developed using the following model.

Suppose we observe bus arrival times at a given stop over several successive days. It is likely that the potential reduction in waiting time resulting from planning one's arrival time at the bus stop arises from the ability to predict the time of arrival of a given bus on different days rather than the regularity of headways on a single day. It is thus of interest to construct a probability distribution of arrival times of a given bus on different days.

Such a distribution should reflect several facts about the service. First, there is a definite earliest time of arrival, dictated by the distance from the terminal to the stop and the speed limit in effect; thus, the distribution should be truncated to the left. Second, there is a finite probability of the bus being very late, or even cancelled; so the distribution should have a long tail to the right. Finally, one of the major sources of lateness in the bus arrival times is increased dwell time at stops further up the line because of late arrival, and hence larger boarding volumes than expected. Thus, once a bus is initially delayed, subsequent delays become longer and longer. If we argue that delay at a stop is proportional to lateness arriving at that stop, we obtain a model of lateness as the result of a series of multiplicative effects.

A probability distribution consistent with all of these characteristics is the lognormal, and this distribution will be used here. If the arrival time of a bus, t, is distributed lognormal, its density function may be expressed in terms of two parameters, μ and σ, as follows:

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma}(\ln t - u)\right]^2\right\} \tag{4}$$

The parameters μ and σ may be given intuitive meaning by noting that if t is lognormally distributed, $\ln t$ is normally distributed. We then have

$$\mu = E(\ln t) \tag{5a}$$

$$\sigma^2 = V(\ln t) \tag{5b}$$

The mean and variance of t may be expressed in terms of μ and σ as

$$E(t) = e^{\mu + \sigma^2/2} \tag{6a}$$

$$V(t) = [E(t)]^2\,[e^{\sigma^2} - 1] \tag{6b}$$

It will be assumed that non-random arrivals choose their time of arrival so as to insure that the probability of missing the bus is no greater than some value X. This time is found by utilizing the relationship between the cumulative distribution function (CDF) for the lognormal and that of the standard normal random variable. It can be shown (see, for example, (1)) that the following relationship holds.

$$F(t) = G_z\left[\frac{1}{\sigma}(\ln t - u)\right] \tag{7}$$

where $F(t)$ = CDF of a lognormal random variable with parameters μ and σ evaluated at t
$G_z(\cdot)$ = CDF of the standard normal random variable (i.e., N(0,1))

By setting the probability level, X, we can solve equation 7 to find the value of t, denoted t_a, at which the passenger should arrive. This is done by setting

$$\frac{1}{\sigma}(\ln t_a - \mu) = G_z^{-1}(X)$$

$$\text{or}\quad t_a = \exp\left[\sigma G_z^{-1}(X) + \mu\right] \tag{8}$$

The exact probability level chosen is somewhat arbitrary, but an appropriate value is likely to be in the range .005 - .05. For the empirical analysis in this paper, the value X = .01 has been used. Should another value be deemed more appropriate by a user of this model, the substitution in equation 8 is straightforward.

Given t_a, we must compute the expected wait time for a passenger arriving at that time. The wait time is

$$w_n = \begin{cases} t_1 - t_a, & \text{if } t_a \le t_1 \\ t_2 - t_a, & \text{if } t_a > t_1 \end{cases} \tag{9}$$

That is, if the passenger arrives before the desired bus arrives at t_1, he waits $t_1 - t_a$. If, however, he has missed the desired bus, he must wait for the succeeding bus which arrives at t_2. If the distributions of bus arrival times are assumed independent the expected wait is

$$E(w_n) = \int_0^{t_a} [E(t_2) - t_a]\, f(t_1)dt_1 + \int_{t_a}^{\infty} (t_1 - t_a) f(t_1)dt_1$$

$$= E(t_2)\, F(t_a) - t_a + \int_{t_a}^{\infty} t_1 f(t_1)dt_1$$

$$= E(t_2)\, F(t_a) - t_a$$

$$+ [\exp(\mu + \sigma^2/2)]\left[1 - G_z\left(\frac{\ln t_a - \mu}{\sigma} - \sigma\right)\right] \tag{10}$$

We are thus able to predict the mean waiting time of non-random arrivals in terms of the bus service characteristics. One important element of the model expressed in equation 3 is still missing, however. It is to be expected that the proportion, a, of all passengers who are non-random arrivals will also be a function of the service characteristics. For example, as headways become longer the potential benefit from learning the schedule and planning one's arrival time becomes larger; hence, we would expect a to increase. Likewise, as service becomes more dependable, planning one's arrival time becomes easier, and again we would expect a to increase.

In order to explore the effects on a of changes in service characteristics, a small empirical study was undertaken of several services in Chicago with varying headways and reliability.

The Empirical Study

Four different services were observed for eight days each during the morning peak period in spring and summer of 1977. Data were collected on bus arrival times and passenger wait times. From observed bus arrival times, μ and σ were estimated for each service using equations 5a and 5b. Equation 8 then allowed the value of t_a to be determined, and the associated mean waiting time was found from equation 10. The expected wait for random arrivals was found by estimating the mean and variance of the headway distribution between successive buses, and applying equation 1.

Because passenger wait times were observed simultaneously with the bus arrivals, equation 3 may be rewritten to solve for a, the proportion of non-random arrivals:

$$a = \frac{E(w_r) - E(w)}{E(w_r) - E(w_n)} \tag{11}$$

Summary statistics for the observed services are shown in Table 1. The values of a observed range from .49 to .74.

If the model suggested in this paper is to be a useful tool for predicting the effects on waiting time of changes in service characteristics, it is necessary that we be able to predict the value of a as a function of these service characteristics. As discussed above, it may be expected that increasing headways would lead to increasing values of a, as the potential gain from planning one's arrival time is larger. Also, as service becomes more reliable from day to day, the task of planning one's arrival to correspond to the arrival time of the bus becomes easier, and we may expect a to increase. One useful measure of the day-to-day reliability of the service is the coefficient of variation in the bus arrival time distribution. For the lognormal distribution, the coefficient of variation has a simple mathematical expression:

$$CV = \frac{\sqrt{V(t)}}{E(t)} = (e^{\sigma^2} - 1)^{\frac{1}{2}} \tag{12}$$

The coefficient of variation is preferred to the standard deviation as a measure of reliability, as it also incorporates information regarding the mean of the distribution. In a skewed distribution like the lognormal, this is advantageous.

A simple model for a might be proposed as follows:

$$a = b_0 + b_1\, E(H) + b_2\, CV + e \tag{13}$$

where $E(H)$ = expected headway
e = error term
b_0, b_1, b_2 = constants.

We would expect to find $b_1 > 0$ and $b_2 < 0$.

The data on E(H), CV and a from the four services observed are summarized in Table 2. These observations are clearly insufficient for reliable statistical inference; however, a regression using these four points results in the model

Table 1. Summary statistics for observed bus services.

Service Number	Mean Headway (minutes)	Headway Variance	μ	σ	E(w) (minutes)	$E(w_r)$ (minutes)	$E(w_n)$ (minutes)	a
1	10.09	.165	-.467	.834	3.02	5.06	0.90	.49
2	11.85	.478	-.267	.920	3.33	5.94	1.20	.55
3	7.75	2.68	.946	.387	2.38	4.05	1.79	.74
4	9.06	3.41	.041	.767	2.50	4.72	1.31	.65

$$\hat{a} = .602 + .023\ E(H) - .27\ CV \qquad (14)$$

Thus, the data observed to date are at least consistent with a priori expectations about the model form. Additional data are being collected from other services in the Chicago area, but are not yet available at the time of this writing.

Table 2. Data on E(H), CV and a for services observed.

Service Number	E(H) (minutes)	CV	a
1	10.09	1.00	.49
2	11.85	1.33	.55
3	7.75	0.16	.74
4	9.06	0.80	.65

Application of the Model

The ability to predict the proportion, a, of non-random arrivals, as well as the expected waiting times for both random and non-random arrivals as functions of basic service characteristics allows the model to be easily applied in evaluation of proposed service changes. It has significant advantages over many previously available models as a result of incorporating non-randomly as well as randomly arriving passengers. As a result, the model proposed here is much more reflective of reality, and should provide a much better estimate of the impact of service changes. It also represents a significant improvement over previous models by providing a mechanism for evaluating the effects of reliability improvements.

Additional data are being collected to further verify the model empirically. These additional data will also provide an opportunity to test the sensitivity of the model results to the assumption of the probability level governing the arrival of non-random passengers, and to the assumption of a lognormal distribution for bus arrivals.

The major use of additional data, however, is to provide a firmer basis for estimating the proportion a, of non-random arrivals as a function of basic service parameters.

Conclusions

A model has been proposed to predict average passenger wait time at bus stops as a function of the headway distribution between successive buses and the arrival time distribution of a given bus from day to day. The model considers both random and non-random passenger arrivals. Its emphasis on the influence of service reliability on wait time sets it apart from previous models. For the first time, it provides the transit operator with the ability to predict the impact of changes in operations designed to improve reliability of service.

A very limited empirical study has produced results consistent with theoretical expectations, and has provided the motivation for further development and empirical verification of the model. This work is continuing, and further results are anticipated in the near future.

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