Common Misunderstandings About the Internal-Rate-of-Return and Net Present Value Economic Analysis Methods

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Engineering economy and benefit/cost analysis manuals usually include the net present value and internal-rate-of-return methods for the analysis of mutually exclusive alternatives and, more times than not, contend that both methods, if properly applied, will invariably lead to the same economic decisions. However, it can be demonstrated that such a view is incorrect, as a general rule, and that use of the internal-rate-of-return method can lead to incorrect or ambiguous economic decisions. Accordingly, the purpose of this paper is to define the specific cases and situations in which application of the rate-of-return method will lead to incorrect or ambiguous economic decisions as well as to demonstrate why the net present value method is preferable and to explain the underlying reasons for the differences. Numerous examples will be employed to illustrate the various cases and underlying principles.

Among the more common methods of economic analysis used by engineers to judge the economic worth of mutually exclusive alternatives are net present value (NPV), benefit/cost ratio, and internal rate of return. Economists, however, have long warned about the dangers of using the internal-rate-of-return method for analyzing mutually exclusive alternatives. Specifically, use of the internal-rate-of-return method can lead to incorrect economic decisions when the alternatives are ranked in improper order or when multiple solutions (i.e., multiple internal rates of return) are encountered. Unfortunately, most engineering economy textbooks deemphasize these drawbacks and, as a consequence, practitioners have been misled about the desirability of employing the internal-rate-of-return method in the analysis of mutually exclusive alternatives. Therefore, a clarification of these and other related aspects is desirable.

THE INTERNAL-RATE-OF-RETURN METHOD AND RANKING CRITERIA

Once a set of mutually exclusive alternatives has been specified (to include, implicitly or explicitly, the null alternative), the stream of costs and benefits for each must be estimated year by year over a common analysis period or planning horizon. In turn, the alternatives must be ranked from lowest to highest. The usual (though not necessarily best) criterion is to rank them in ascending order with respect to the costs for the initial year; also, if the costs for the initial year of all (or some) alternatives are equal, then order those that have equal costs for the initial year in descending order with respect to the benefits in the following year.

A cutoff rate or minimum attractive rate of return (MARR) must be specified. This interest rate indicates the effective annual yield of the opportunities that will be foregone if the resources are used for one of the alternatives being analyzed. In essence, the analyst is merely trying to ensure that at least one of the alternatives being analyzed will provide a yield at least that high. Otherwise, other opportunities should not be foregone. Thus, the MARR can be regarded as the opportunity cost of capital for both borrowing and lending situations.

The analysis proceeds in stepwise fashion. We must first determine the lowest-ranked alternative that has an internal rate of return at least as high as the MARR. Thus, we determine the internal rate of return for alternative 1 (i.e., the lowest-ranked one) such that:

$$\sum_{t=0}^{n} B_{1,t}/(1+r_1)^t = \sum_{t=0}^{n} C_{1,t}/(1+r_1)^t \qquad (1)$$

where

$B_{1,t}$ = the benefits for alternative 1 during year t,
$C_{1,t}$ = the costs for alternative 1 during year t,
n = the number of years in the analysis period, and
r_1 = the internal rate of return.

It can be seen that the internal rate of return is simply the interest rate at which the NPV of alternative 1 is zero. (The above formulation implies that the null alternative has zero benefits and costs over the n-year analysis period.) If the internal rate of return (r_1) is equal to or greater than the MARR, then the alternative is regarded as acceptable. If not, it is rejected and the next-higher-ranked alternative is examined for its acceptability, and so forth, until the lowest-ranked acceptable alternative is identified.

After the lowest-ranked acceptable alternative is identified, then an incremental analysis is used to determine the acceptability of higher-ranked alternatives. Assuming (for simplicity) that alternative 1 is found to be acceptable, we then determine the incremental rate of return on the increments in benefit and cost between alternatives 1 and 2 ($r_{1/2}$ or Δ rate of return) such that

$$\sum_{t=0}^{n} (B_{2,t} - B_{1,t})/(1+r_{1/2})^t = \sum_{t=0}^{n} (C_{2,t} - C_{1,t})/(1+r_{1/2})^t \qquad (2)$$

where $B_{x,t}$ and $C_{x,t}$ are the benefits and costs for alternative x during year t. Rearrangement of the terms in Equation 2 shows that the incremental rate of return is simply the interest rate for which the NPV of alternative 1 is just equal to the NPV of alternative 2. If the incremental rate of return ($r_{1/2}$) is equal to or greater than the MARR, then the higher-ranked alternative is deemed to be better than the lower-ranked alternative (i.e., alternative 1 is rejected in favor of alternative 2). In turn, the incremental rate of return for the next-higher-ranked alternative as compared to alternative 2 ($r_{2/3}$) would be computed to determine which is preferable. However, if $r_{1/2}$ is less than the MARR, then alternative 2 would be rejected and the next paired analysis would be conducted between alternatives 1 and 3. That is, we would determine whether $r_{1/3}$ was equal to or greater than the MARR.

Analysis is continued in pairs for all higher-ranked alternatives until the highest-ranked alternative that has an incremental rate of return at least as high as the MARR is identified. That highest-ranked alternative will then be the best alternative, economically speaking.

MULTIPLE RATE PROBLEM FOR THE INTERNAL-RATE-OF-RETURN METHOD

Multiple solutions for the internal-rate-of-return method can arise in one of two ways. The first can occur when especially heavy costs are expected in the future (for example, rolling stock replacement or guideway resurfacing, rehabilitation, or restoration). The guideway resurfacing, rehabilitation, and restoration situation is especially pertinent for many existing roadways and bridges and provides a typical example in which multiple rates probably would occur; this is particularly true if roadway or bridge repairs cause some or all of the lanes to be closed during resurfacing, rehabilitation, or restoration. The example in Table 1 illustrates this first case.

The maximum number of internal rates of return can be determined from an inspection of the variation in the net cash-flow stream. The right-hand column in Table 1 shows the estimated year-by-year net benefits ($B_{1,t}$-$C_{1,t}$). Applying Descartes' rule of signs, the number of sign changes that occurs over the 30-year horizon indicates the maximum number of positive rates of return that can result. In this case, the net benefits changed signs three times, thus indicating that as many as three positive solutions or internal rates of return could occur.

The second and probably more frequent case in which multiple internal rates of return can occur is with incremental rate-of-return analysis for pairs of alternatives. This possibility is more common than we might be led to believe. It could apply, for instance, when higher initial outlays lead to different benefit-accrual patterns or when the future cost-outlay patterns for two alternatives are different. The example given in Table 2 illustrates the former situation and might be applicable if, say, a firm is deciding between two different oil pumps for the extraction of oil from a well. The more expensive pump would permit the oil to be extracted quicker and slightly increase the total amount of oil extracted. In this instance, there is a single internal rate of return for each alternative (analyzed separately), but there are two solutions, or internal rates of return, associated with the incremental costs and benefits between alternatives 1 and 2.

Table 1. Costs and benefits for two-stage improvement of an existing bridge.

End of Year t	$B_{1,t}$[a] ($000s)	$C_{1,t}$[b] ($000s)	$B_{1,t}$ - $C_{1,t}$ ($000s)
0	-	50	-50
1	61	55	+6
2	63	0	+63
3	65	0	+65
.	.	.	.
.	.	.	.
.	.	.	.
9	77	0	+77
10	79	705	-626
11	81	610	-529
12	83	495	-412
13	85	0	+85
.	.	.	.
.	.	.	.
.	.	.	.
29	117	0	+117
30	119	0	+119

[a]Benefits in year t, net of annual operating and maintenance costs.
[b]Nonrecurring capital outlays in year t.

Table 2. Rate-of-return analysis for two oil pump alternatives.

	Alternative 1 ($000s)		Alternative 2 ($000s)	
Year t	$B_{1,t}$	$C_{1,t}$	$B_{2,t}$	$C_{2,t}$
0	0	100	0	110
1	70	0	115	0
2	70	0	30	0

Note: The internal rate of return for alternative x (r_x) would be the interest rate at which the discounted benefits just equal the discounted costs; the formulation would be the same as that shown in Table 1. In alternative 1, r_1 = 25.69 percent; in alternative 2, r_2 = 26.16 percent. The internal rate of return for the incremental benefits and costs between alternatives 1 and 2 ($r_{1/2}$) would be the interest rate at which the NPV of alternative 1 is just equal to the NPV of alternative 2.

As a general proposition, both of these cases can and do arise. Yet, Grant and others argue (1, p. 560):

> It cannot be emphasized too strongly that [multiple solution] cases such as those illustrated in [our examples] are the exception rather than the rule. They occur chiefly in the mineral industries and the petroleum industry; even there they arise only in rather specialized circumstances.

Similarly, Winfrey assumes away the multiple-solution problem for the internal-rate-of-return method by saying (2, p. 161), "Since the situation of two or more rates of return is so infrequent, there is no need to outlaw the rate-of-return method, a highly useful and understandable method of analysis." Newnan echoes this view, saying (3, p. 138): "In certain rare situations we find that solution of a cash-flow equation results in more than one positive rate of return."

Such instances are not necessarily exceptional, infrequent, or rare. Rather, for highways or bridges, which may require heavy outlays for reconstruction or replacement in future years, as well as for transit systems, which may require costly rolling stock replacement or rehabilitation every 10-30 years, the possibility of multiple rates of return is high, if not the typical expectation.

Analysts tend to regard especially high or low internal-rate-of-return values as being unrealistic or inappropriate. For example, most of the advanced pocket calculators that are preprogrammed to calculate the internal rate of return for a cash-flow stream identify only the lowest positive internal rate of return and thus ignore all others and imply their irrelevance. To the contrary, all multiple rates are valid and should be considered.

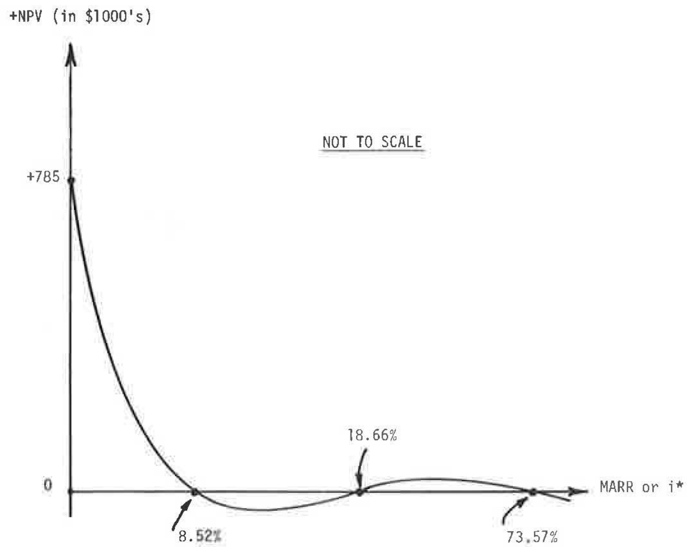
THE FALLACY IN MANY ENGINEERING ECONOMIC TEXTBOOKS

Many engineering economic textbooks incorrectly claim that all analysis methods (such as the NPV, benefit/cost ratio, and internal rate of return), when properly applied, lead to identical ranking of alternatives. For instance, Grant and others say (1, p. 117)

> Once a particular [MARR] is selected for the comparison of alternatives, a correct analysis of relevant rates of return will invariably lead to the same conclusion that will be obtained from a correct annual cost comparison or a correct present worth comparison.

Winfrey echoes the above position, saying (2, p. 123): "When properly applied in accordance with their limitations, each method will give a reliable result for economic evaluation and for project formulation." In a more recent article, he reiterates (4, p. 37): "All methods will give the identical selection of the alternative of greatest economy when the procedures of analysis are correctly chosen and properly used."

Figure 1. NPV at different MARR for the bridge alternative described in Table 1.



To explore this claim of engineering economists, let us consider two examples, the first given in Table 1 and the second in Table 2.

The data in Table 1 represent the expected year-by-year costs and benefits associated with undertaking a specified course of action over a 30-year analysis period or planning horizon. The numbers appear somewhat typical for previously built highways or bridges that now are in need of repair, restoration, or replacement, or they could apply to a transit system that plans to extend its lines in the future. In this case, assume that a community has an old bridge that is in imminent danger of collapse. In turn, the public works department was ordered by the city council to analyze the various repair strategies that would ensure safe operation of the bridge for the next 30 years and to evaluate the economic worth of each, relative to the null or abandonment alternative. Among the possibilities are (a) make minor repairs to the bridge now and a major overhaul 10 years later or (b) completely overhaul the bridge now. The appropriate benefit and cost data for the first of these two alternatives are given in Table 1. Presumably, the second alternative would have higher initial outlays and thus would be analyzed in terms of the incremental benefits and costs after the first alternative has been analyzed in terms of its acceptability.

Accordingly, the data in Table 1 represent the incremental costs and benefits for the first alternative relative to bridge abandonment. In turn, we can calculate the internal rate of return for this lowest-cost alternative. The discounted internal-rate-of-return method, properly applied, would yield three rates of return in this instance: 8.52, 18.66, and 73.57 percent. First, all of these solutions or rates are correct. Internal rates of return (r_1) = 8.52, 18.66, and 73.57 percent, where r_1 is the interest rate (or rates) that satisfies the following identity:

$$\sum_{t=0}^{30} B_{1,t}/(1 + r_1)^t = \sum_{t=0}^{30} C_{1,t}/(1 + r_1)^t \qquad (3a)$$

or

$$\sum_{t=0}^{30} (B_{1,t} - C_{1,t})/(1 + r_1)^t = 0 \qquad (3b)$$

That is, they represent the interest rates for which the NPV of this alternative is zero. Second, in the absence of any other information, how do we interpret these rates? Suppose, for instance, that the appropriate MARR is judged to be approximately 10 percent. Then, by using just the internal-rate-of-return figures, we will obtain either an ambiguous answer or an incorrect one. That is, we presumably would incorrectly regard the alternative as acceptable (since both 18.66 and 73.57 percent are higher than the MARR) or would incorrectly regard the decision as ambiguous.

By contrast, if we had simply computed the NPV (or discounted benefits minus discounted costs) for the stated MARR of 10 percent, we would have learned that the NPV was negative and thus that the minor bridge repair alternative was economically infeasible and should be rejected. Specifically, for a MARR of 10 percent, the NPV would be equal to -$14 140. In addition, the benefit/cost ratio for this alternative would be 0.981 (or less than 1.0, indicating rejection) at an interest rate of 10 percent. In sum, we see that all methods of analysis do not invariably provide either identical or sound conclusions. The NPV and benefit/cost ratio methods are conclusive and unambiguous; the internal-rate-of-return method is ambiguous and inconclusive.

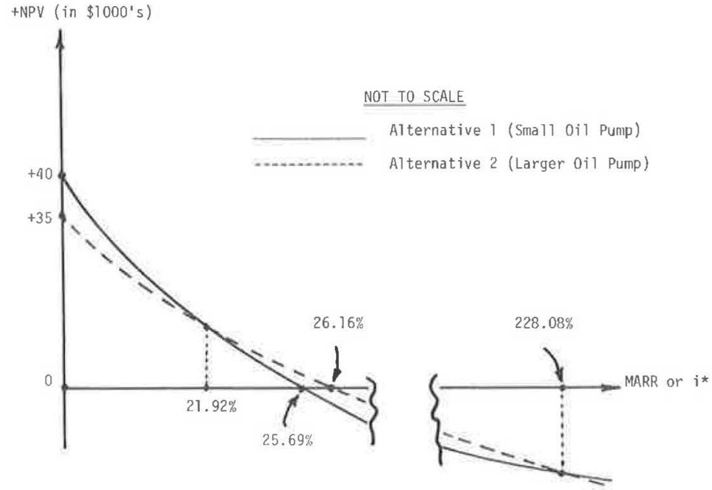
Moreover, in this situation the internal-rate-of-return method could have led us astray for a wide range of circumstances. In Figure 1, a plot of the appropriate NPVs versus the interest rate is shown for the full range of interest possibilities. Clearly, if the appropriate MARR was deemed to be between 8.52 and 18.66 percent, then an analysis that did not reject this alternative would be incorrect—a result that would not necessarily result from strict application of the internal-rate-of-return method.

A Simplified Example

Now, let us review the circumstances for another example situation, one involving overall analysis as well as incremental analysis between a pair of alternatives. The particular example was chosen for clarity and to minimize calculations and is given in Table 2. It deals with two investment alternatives, each having two-year cost and benefit streams, as shown. The additional initial investment (of alternative 2 over alternative 1) will permit earlier recovery of the overall gains, as well as lead to slightly higher two-year gains (measured in current or undiscounted dollars).

To apply the internal-rate-of-return method, we first

Figure 2. NPV at different MARR for the two (oil pump) investment alternatives described in Table 2.



need to specify the MARR. Let us assume it is 15 percent. Next, we calculate the internal rate of return for the lowest-cost alternative and then ask whether the rate is at least as high as the MARR. (This is the first step in answering the question, "Is any alternative worth undertaking?") Since the internal rate of return for alternative 1 (25.69 percent) is higher than the MARR, alternative 1 is judged to be acceptable, economically. In turn, we must calculate the incremental internal rate of return ($r_{1/2}$) associated with the incremental costs and benefits between alternatives 1 and 2, the latter being the higher initial-cost alternative. That is, $r_{1/2}$ is the discount rate that satisfies the following identities:

$$\sum_{t=0}^{2} (B_{2,t} - B_{1,t})/(1 + r_{1/2})^t = \sum_{t=0}^{2} (C_{2,t} - C_{1,t})/(1 + r_{1/2})^t \qquad (4)$$

or

$$\sum_{t=0}^{2} (B_{1,t} - C_{1,t})/(1 + r_{1/2})^t = \sum_{t=0}^{2} (B_{2,t} - C_{2,t})/(1 + r_{1/2})^t \qquad (5)$$

The internal rate of return for the incremental costs and benefits is not a single rate but two of them—21.92 and 228.08 percent. Since both of these rates (examined without any other information) are greater than the MARR, the analyst would presumably regard alternative 2 as being more attractive than alternative 1, economically speaking. Or, the decision would be regarded as ambiguous.

However, neither of the above conclusions would be correct. For instance, the NPV method will show that for a MARR of 15 percent the NPV of alternative 1 is $13 800 as compared to only $12 684 for alternative 2, thus unambiguously indicating the preferability of alternative 1. Similarly, if the benefit/cost ratio method had been used, the ratio for alternative 1 would have been 1.138 for an interest rate of 15 percent; thus, alternative 1 is acceptable. In turn, the incremental benefit/cost ratio for the increments in benefits and costs between alternatives 1 and 2 can be shown to be 0.884, or less than 1.0, thus indicating that alternative 2 should be rejected. Accordingly, it is obvious that the various analysis methods do not invariably lead to the same sound conclusions about which alternative is best. Moreover, the situation would be even more perplexing if the MARR were, for example, about 25 percent. In this instance, use of the internal-rate-of-return method would lead to the acceptance of alternative 1 (since r_1 is greater than the MARR) but would provide little guidance about the acceptability of the higher-cost alternative, since one of the incremental rates of return is less than the MARR and one is higher. By contrast, either the NPV or benefit/cost ratio methods would have shown that alternative 2 is unambiguously the best choice for a MARR of 25 percent (see Figure 2).

Reasons for Different Conclusions from Different Methods

We saw that the internal-rate-of-return and NPV methods sometimes lead to conflicting decisions about which alternative is best. In turn, we should ask, Which method gives correct results? Why can and do the results sometimes differ?

To begin, it seems appropriate to emphasize the objective of the analyst (or investor)—maximize the net gains or profits to be accrued over the analysis period or planning horizon. That is, we wish to identify which project will maximize the surplus that a firm or community will accrue over the analysis period. Accordingly, the economist has argued that the NPV (computed at the MARR) is a simple and unmistakable indicator of a project's profitability and that the project that has the highest positive NPV will be the best, economically speaking. Moreover, the internal rate of return will sometimes prove to be a misleading indicator of profitability.

Consider again the example in Table 2. Given these two alternatives (relative to investing in neither), which would accrue the highest profit or surplus by the end of year 2? The answer involves two aspects: (a) the MARR or opportunity cost of capital, which informs us about the yield possibilities that we must forego if we invest in alternative 1 or 2, and (b) the possible uses of any net revenues or benefits that are accrued prior to the end of the two-year analysis period. For the first aspect, and again assuming that the MARR is 15 percent, to invest $100 000 in alternative 1 would mean that we would forego the opportunity to accumulate $132 250 by the end of year 2. But, by foregoing this opportunity and investing in alternative 1, we would accrue annual net earnings of $70 000 at the end of years 1 and 2. Obviously, though, if we had invested in alternative 1, the first-year earnings of $70 000 would be reinvested during the second year rather than sit idle. A reasonable assumption is (as is implicit with the NPV method) that these early-year earnings would be reinvested at the MARR (which, after all, represents a best estimate of the potential yield of any outside opportunities). Accordingly, if the first-year earnings of \$70 000 were reinvested at 15 percent, one year later we would have accumulated \$80 500 (or \$70 000 plus \$10 500 in yield) plus of course the \$70 000 that was generated in the second year by the initial investment. Altogether then, an investment in alternative 1 would require us to forego \$132 250 and instead to accumulate \$150 500 during the same two-year period. The profit or net gains to be accumulated by the end of two years would be \$18 250 (or \$150 500 less \$132 250). A similar analysis can be carried out for alternative 2, again for a MARR of 15 percent. For an investment in alternative 2, we would accumulate total earnings of \$162 250 by the end of year 2 and would forego the opportunity to earn \$145 475. Thus, the profit accrued by the end of the two-year period would be \$16 775.

Both investment alternatives are profitable (relative to investing in neither); however, alternative 1 is more profitable than alternative 2. Moreover, this is the same result (i.e., decision) that was obtained from NPV analysis, a result that is hardly surprising since (in the parlance of engineering economy) the NPV is exactly equivalent to the net future worth when the latter is multiplied by the single-payment present-worth factor. That is,

NPV for alternative 1 (at 15 percent) = (net future worth) $\times$ (P | F, 15 percent, 2) = (\$18 250) $\times (1.15)^{-2}$ = \$13 800.
NPV for alternative 2 (at 15 percent) = (net future worth) $\times$ (P | F, 15 percent, 2) = (\$16 775) $\times (1.15)^{-2}$ = \$12 684.

where P | F = the present worth given the future value. That is, if a project has the highest positive net future worth, then it also will have the highest positive NPV. By contrast, for a MARR of 15 percent, the internal-rate-of-return method would lead to the conclusion that alternative 2 was the most profitable or that the choice was ambiguous. Moreover, when using the internal-rate-of-return method, such a confusing result would surely be forthcoming for any MARR value below 21.92 percent (and probably for any value up to 25.69 percent). All in all, the NPV method leads to correct results, whereas the internal-rate-of-return method sometimes provides incorrect or ambiguous ones.

The internal-rate-of-return method sometimes gives misleading results or ones that differ from those obtained by using either NPV or benefit/cost ratio calculations because of different assumptions about reinvestment of early-year benefits or revenues. [For extensive coverage of this point, see articles by Hirshleifer, Lorie and Savage, Renshaw, and Solomon (5).] To use the internal-rate-of-return method is to assume implicitly that earnings accrued prior to the end of the analysis period are reinvested at the internal rate of return for the remaining years. To use the NPV (or benefit/cost ratio) method is to assume implicitly that prior-year earnings are reinvested at the MARR (or opportunity cost of capital) for the remaining years. In the Table 2 example, for instance, use of the NPV method implies that the \$115 000 first-year earnings of alternative 2 were reinvested at the MARR (of 15 percent) during the second year; however, the internal-rate-of-return method implied that these same earnings were reinvested at a rate of 26.16 percent during the second year—no wonder the results were different in this case. Also, for this example, the internal-rate-of-return method would imply that the first-year earnings for alternative 1 would be reinvested at 25.69 percent, but those for alternative 2 would be reinvested at 26.16 percent. Such an assumption would be nonsensical on two grounds. For one, Why should the reinvestment rate differ from one alternative to another? Should they not be equal? For another, we should recognize that the MARR is the indicator of our other investment or reinvestment opportunities and that the MARR has no necessary relationship to the internal rate of return.

Some engineering economists argue the inappropriateness of considering reinvestment possibilities for any revenues or benefits that are accrued prior to the end of the planning horizon (such as those accrued at the end of year 1 for alternatives 1 and 2 in Table 2). Winfrey, for example, says (2, pp. 162-163)

> It is most difficult to convince the layman that his rate of return on a given investment is dependent upon how he reinvests return from that investment; neither does it seem logical when comparing possible investment alternatives that the choice of investment could depend upon how the return from each alternative would be reinvested.

Accumulated profit or net gain over the entire investment period is clearly related to and thus dependent on reinvestment of revenues or benefits gained along the way; thus, no wise investor will choose to ignore them. Let us demonstrate the point by a somewhat contrived (yet appropriate) example. [Another interesting example and discussion of this same aspect appears in Mishan (6, p. 225).] Suppose, for instance, that we want to borrow \$70-90 now and that, in turn, we go to the ABC Loan Company to request one of the two following loans, as shown in Table 3:

1. A \$70 loan now to be paid back in two installments, the first one of \$75 one year from now and the second of \$70 ten years from now; or
2. A \$90 loan now to be paid back in two installments, the first one of \$115 one year from now and the second of \$5 ten years from now.

Assume that the ABC Loan Company estimates its MARR to be 8 percent and that the company wishes to know which loan plan (if any) would be most profitable. In turn, let us assume that the ABC Loan Company prefers to use the internal-rate-of-return method and calculates the various rates of return, as shown in Table 3. Accordingly, the company notes that loan plan 1 has an internal rate of return (r_1) of 22.84 percent and thus is acceptable (since it is larger than the MARR of 8 percent). Next, the company notes that the incremental rates of return ($r_{½}$) of 16.26 percent and 99.35 percent are both higher than its MARR, thus suggesting that loan plan 2 is better than loan plan 1 or that the choice is ambiguous. However, if the company had used either the NPV method or the benefit/cost ratio method, it would have discovered that loan plan 1 and not plan 2 is clearly best, as shown in the calculations below. In fact, the data in Figure 3 will show that loan plan 1 is better for the ABC Loan Company whenever its MARR value is below 16.26 percent.

NPV Method

Loan plan 1

$[NPV_{1,10}]_{8\%}$ = \$31.87.

Loan plan 2

$[NPV_{2,10}]_{8\%}$ = \$18.80.

Benefit/Cost Ratio Method

Loan plan 1

$[BCR_{1,10}]_{8\%}$ = 1.455.

Loan plan 2

$[BCR_{2,10}]_{8\%} = 1.209.$

Comparison of loan plans 1 and 2

$[BCR_{1/2,10}]_{8\%} = 0.347.$

Again, the differences between the NPV and internal-rate-of-return methods (in indicating the best alternative for a MARR of 8 percent) stem from the different assumptions with respect to reinvestment. First, the reinvestment possibilities for the payback amounts received in year 1 should not be ignored. The $75 received in year 1 for plan 1 or the $115 received for plan 2 would not be ignored or placed in a drawer for the remaining nine years. Rather, they would be reinvested in other investment opportunities or in early (rather than later) year enjoyment. The most reasonable assumption (in the absence of other information) is that these yearly earnings will be reinvested at the MARR.

Second, if the year-1 payback amount of $75 for plan 1 were to be reinvested at our assumed MARR of 8 percent for the remaining 9 years, then by the end of year 10 the ABC Company would accumulate $149.93, which can then be added to the 10th year payback amount of $70. Thus, the accumulated earnings will be $219.93 by the end of year 10 if the early-year earnings are reinvested at 8 percent. These accumulated earnings ($219.93) when discounted to their present value at 8 percent and balanced against the $70 initial investment will be exactly equal to the NPV of $31.87 shown in Table 3. This proves that the NPV method implicitly assumes that any early-year earnings are reinvested at the MARR. That is, if r is the reinvestment rate for the year-1 earnings, then at 8 percent the NPV of the accumulated 10-year earnings less initial costs would be as follows:

$$\{[\$75\,(1 + r)^9 + \$70]/(1.08)^{10} - \$70\} = \$31.87.$$

This identity would hold only for a reinvestment rate (r) equal to 8 percent.

Third, it can be shown that, for the internal-rate-of-return method, early-year earnings are assumed to be reinvested at the internal rate of return. If this is true, then for plan 1 the first-year payment of $75 is reinvested for the remaining 9 years at 22.84 percent, thus accumulating $477.66 by the end of 10 years, to be added to the 10th year payment of $70. The accumulated 10-year earnings will be $477.66 + $70 = $547.66 [e.g., $\$547.66 = \$75(1.2284)^9 + \$70$]. These accumulated earnings when discounted to their present value will be identical with the initial investment of $70 only for an interest rate of 22.84 percent, the internal rate of return. In short, this proves that the early-year earnings were assumed to be reinvested at the internal rate of return, an assumption that is clearly different from that used for the NPV method. Moreover, if the early-year earnings were reinvested at any reinvestment rate other than the internal rate of return, the 10-year accumulated earnings when discounted at the internal rate of return would not be equal to the discounted costs (which in this case were equal to the initial loan amount).

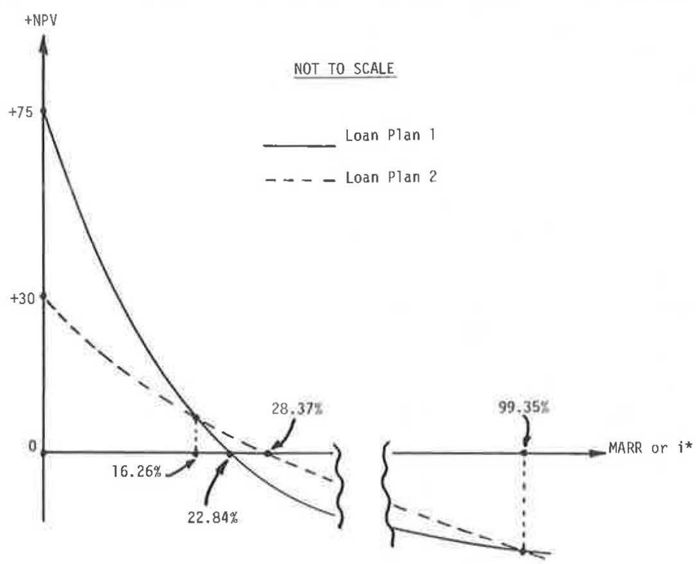
Given that we have proved that the internal-rate-of-return method uses the internal rate of return as the reinvestment rate and that the NPV method used the MARR as the reinvestment rate, we can be more explicit about the confusion in applying the internal-rate-of-return method to the selection of the best loan plan in Table 3. Note first that plan 1 has an internal rate of return of 22.84 percent, and plan 2 has a rate of 28.37 percent. As a consequence, the method assumes that the year-1 payment for plan 2 can be reinvested at a higher rate than can the year-1 payment for plan 1. What rationale is there for assuming different reinvestment rates for

Table 3. Two loan and payback possibilities from the ABC Loan Company's viewpoint.

	Loan Plan 1		Loan Plan 2	
Year t	Loan Amount ($)	Payback Amount ($)	Loan Amount ($)	Payback Amount ($)
0	-70		-90	
1	0	+75	0	+115
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
5	0	0	0	0
6	0	0	0	0
7	0	0	0	0
8	0	0	0	0
9	0	0	0	0
10	0	+70	0	+5

Note: For loan plan 1, r_1 = 22.84 percent; for loan plan 2, r_2 = 28.37 percent; $r_{1/2}$ = 16.26 and 99.35 percent for loan plans 1 and 2.

Figure 3. NPV at different MARRs for the two ABC Loan Company plans described in Table 3.



earnings that are accrued at the same point in time? None. By the same token, in the computation of the incremental rates of return, we assumed that the difference in first-year payments (or \$115 - \$75) was reinvested at either 16.26 percent or 99.35 percent–again, an assumption without rationale and one that is very different from that used to analyze separate projects.

Rate of Return for a Reinvestment Rate Equal to MARR

If identical assumptions are made about reinvestment for all methods, then invariably the same conclusions will result. As noted before, both the NPV and benefit/cost ratio methods assumed that the MARR would be the proper reinvestment rate. Thus, let us use the same reinvestment assumption for the rate-of-return method and then compare the results. That is, we will assume that early-year earnings are reinvested at the MARR and then determine the interest rate at which the discounted accumulated earnings just equal the discounted costs. However, the resultant interest rate, strictly speaking, will reflect more than internal earnings and thus will be designated by an R_x instead of an r_x. This R_x value will represent the effective yield to be obtained over the analysis period and will be equivalent to the internal rate of return only in exceptional cases. [This adjusted rate-of-return value has been termed the "equivalent rate of return" by Solomon (5, p. 74) and the "reinvestment-corrected internal rate of return" by Mishan (6, p. 228).]

Let us apply this procedure to the loan example in Table 3. Again let us assume that the MARR, and thus the reinvestment rate, for early-year earnings is 8 percent. Accordingly, the calculations would be as follows:

Effective Rate of Return (R_1) for Loan Plan 1

Accumulated 10-year earnings = $\$75(1.08)^9 + \$70 = \$219.93$.

In turn, find R_1 such that the discounted earnings are just equal to the discounted costs, or

$\$219.93/(1 + R_1)^{10} = \$70$;

thus, R_1 is equal to 0.1213 or 12.13 percent.

Effective Rate of Return (R_2) for Loan Plan 2

Accumulated 10-year earnings = $\$115(1.08)^9 + \$5 = \$234.89$.

In turn, find R_2 such that the discounted earnings are just equal to the discounted costs, or

$\$234.89/(1 + R_2)^{10} = \$90$;

thus, R_2 is equal to 0.1007 or 10.07 percent.

In this case, one in which the modified- or effective-rate-of-return method uses a reinvestment-rate assumption that is identical to that used in the NPV and benefit/cost ratio methods, the outcome and conclusions will be identical for all methods. That is, loan plan 1 provides an acceptable rate of return (i.e., one that is higher than 8 percent) and has a yield that is higher than that for plan 2. We also could have computed the modified or effective rate of return on the increments in costs and benefits between plans 1 and 2, although the step is unnecessary. The resultant incremental return figure would be -2.86 percent; this result is obvious when we note that the extra initial-year cost of \$20 led to extra accumulated 10-year earnings of only \$14.96.

OTHER CONFUSING ASPECTS ABOUT REINVESTMENT OF EARLY-YEAR BENEFITS

First, analysts have been troubled about benefits or gains that are accrued in nonmonetary rather than monetary terms. How does the concept of reinvestment apply, for example, to time savings accrued in a year prior to the end of the analysis period? The answer is simple and straightforward. Reinvestment principles (broadly construed) apply with equal validity to monetary and nonmonetary benefits that are accrued prior to the end of the analysis period because time savings accrued in earlier years are more valuable to people than the same amount of time savings accrued in a later year. Or, put somewhat differently, enjoyment (or consumption of earnings accrued earlier) is more highly valued than that accrued later. Also, the MARR, rather than the internal rate of return, reflects (in part) the strengths of people's tastes and preferences with respect to the importance of enjoyment now versus enjoyment later. Specifically, the MARR reflects the trade-off between people's time preferences and the rate of productivity of investments and thus the marginal rate of time preference is (roughly) equal to the marginal rate of productivity, both being equal to the MARR.

Second, Grant and others dealt extensively but confusingly with the matter of reinvestment (1, p. 563). In essence, they deal with the calculation and interpretation of an adjusted- or effective-rate-of-return figure.

They argue that such a method of computing an adjusted rate of return (or R_x) is fallacious, saying in part (1, pp. 563-565):

> Sometimes an analyst uses two or more interest rates because this method of analysis is required by company policy. Or he may mistakenly believe that this technique will give him useful conclusions. In either case, one aspect of his computational procedure will be the assumption of reinvestment at some stipulated interest rate. Various weaknesses in the reinvestment assumption are brought out in [the Table 4 example, to be discussed] and in several of the problems at the end of this appendix. . . .
>
> The fallacy in this type of analysis [i.e., that in which an adjusted or effective rate of return is calculated] may be even more evident if we apply the [adjusted-rate-of-return] method to the following estimates for another investment proposal [shown in Table 4].

For the cash flows shown in Table 4 (1, p. 565), the internal rate of return is 0 percent, thus indicating that the NPV is zero at 0 percent. Moreover, for any positive discount rate, the NPV is negative, thus indicating that the investment proposal is financially unattractive. In turn, Grant and others state that 10 percent is the "rate that the company is expected to make on other investments," thus indicating that 10 percent is the MARR and, therefore, the appropriate reinvestment rate for early-year gains (1, p. 565). Accordingly, they calculated the adjusted rate of return (R) and found it to be 6 percent. (Although I have some reservations about the way in which Grant and others calculated the adjusted rate of return, I will withhold discussion of those points until later.) In turn, they conclude (1, p. 565): "In effect, the investment proposal yielding 0 percent has been combined with the 10 percent assumed to be earned elsewhere in the enterprise to give the misleading conclusion that the proposal will yield 6 percent." To the contrary, the 6 percent is a true indicator of the effective yield to be anticipated from the cash flows shown in Table 4 since it properly reflects the reinvestment earnings of the positive cash flows. Moreover, the unattractiveness of the project is reflected in the fact that the overall yield is less than 10 percent, the effective yield that would be anticipated if the year 0 and year 1 funds were invested in other foregone opportunities. That is, if we were to invest in this plan for five years, then profits would be lost relative to other investment possibilities but not in an absolute sense.

Also, the folly of arguing that reinvestment should not be considered in calculation of the effective yield is obvious when we consider the circumstances for the cash flows as altered in Table 5. The cash flow for investment is the same for the Table 4 and 5 investments, and the internal rate of return is identical for the two (i.e., 0 percent). Even so, it should be apparent that the plan in Table 5 is much less attractive financially than that in Table 4. That is, we surely do care about when the earnings are received (i.e., early versus later) and what we do with them. The internal-rate-of-return figure of 0 percent would not reflect that fact, but the calculation of a modified or reinvestment-corrected rate of return would vividly demonstrate it. To be specific, and using the Grant procedure, the effective rate of return for the plan in Table 5 would be about 2 percent, thus indicating that the effective yield of this plan is far less than that for the plan in Table 4.

Also, the internal rate of return will be equal to the effective yield to be expected from a project in only two circumstances (both of which must be considered highly unlikely): (a) when all of the project earnings are accumulated solely at the end of the analysis period or (b) when the MARR is exactly equal to the internal rate of return. Only in these two cases will the NPV and internal-rate-of-return methods incorporate identical assumptions with respect to reinvestment and thus always provide identical decisions about acceptability and ranking.

Finally, note that the procedure to be used for calculating the adjusted or effective rate of return must be geared to the assumed financing plan and reinvestment strategy. Since either can vary, the procedure is somewhat arbitrary and the results will change accordingly. In the Table 4 example, Grant and others first discount the investment costs for years 0 and 1 to their present value at 10 percent; in effect this implies that $427 300 was borrowed in year 0 and that the balance between $427 300 and the $200 000 needed in year 0 was invested for one year at 10 percent and then used to pay the year 1 investment costs. This assumed financing policy, along with reinvestment of the $10 000 year 1 earnings, resulted in the maximum adjusted-rate-of-return value that could be achieved with a MARR of 10 percent. Alternatively, we could have simply adjusted the cash flows in the right- and left-hand columns, borrowed only $200 000 in year 0 and $120 000 in year 1, and then calculated the adjusted rate of return. For the latter set of assumptions, the adjusted-rate-of-return value for the Table 4 example would be 3.9 percent and that for Table 5 would be 0.85 percent. These effective yield values are lower than those obtained for the different financing and reinvestment strategies used by Grant and others, thus emphasizing that the yield is dependent on the reinvestment and financing plans and that it is necessary to be explicit about both.

Table 4. Five-year cash flows for an investment proposal.

Year	Cash Flow for Investment ($000s)	Cash Flow from Excess of Operating Receipts over Disbursements ($000s)
0	-200	
1	-250	+130
2		+110
3		+90
4		+70
5		+50
Total[a]	-450	+450

[a]Present worth at 0 percent.

Table 5. Five-year cash flows for an altered investment proposal.

Year	Cash Flow for Investment ($000s)	Cash Flow from Excess of Operating Receipts over Disbursements ($000s)
0	-200	
1	-250	+10
2		+20
3		+30
4		+40
5		+350
Total[a]	-450	+450

[a]Present worth at 0 percent.

SUMMARY AND CONCLUSIONS

With both public and private investment projects, multiple solutions can occur and thus can lead to ambiguous or incorrect investment decisions if one uses the internal-rate-of-return method. Even if this occurrence is rare (a fact that has yet to be established), its possibility alone should discourage even the most serious advocate of the internal-rate-of-return method. This method lacks generality.

ACKNOWLEDGMENT

I am grateful for the painstaking reviews and helpful criticism afforded by my colleagues Tung Au and Chris Hendrickson and by Robley Winfrey. The manuscript was markedly improved as a result of their collective effort, though I do not hold them accountable for any errors of fact or judgment, nor do I imply agreement on their part with my views.

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Discussion

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Wohl has proved nothing. He does not tell the whole story. He uses misleading illustrations. He does not point out the shortcomings of the NPV method. He does not prove the reinvestment assumption. He does not mention that industrial officials want to know the rate of earning and not their net dollar sum of present worth. He does not recognize that transportation officials (highway, mainly) prefer the benefit/cost ratio or the rate of return. He does not mention that the rate-of-return method has been used for years without the disastrous results he says could happen.

In the rate-of-return method, under conditions of two or more reversals of sign in the combined accumulated negative and positive cash flows, it is acknowledged that two or more rates of return may be found. But Wohl does not mention that the NPV method may also give two or more solutions. Which alternative gives the highest NPV depends on the choice of discount rate (MARR). For every rate of return isolated by the rate-of-return method in a multiple-rate example, there is likewise an NPV of zero. The in-between NPV solutions will be net negative or net positive. A data set can easily be formed in which a discount rate of 8 percent would indicate an NPV to support alternative A; however, a discount rate of 9 percent would support alternative B.

Wohl's statement and illustration that many roadways and bridges provide examples in which multiple rates would occur are far from reality and practice. His example shows that an author can find a set of figures to prove a hypothesis, no matter how unrealistic the figures used may be.

A highway engineer would not invest 50 and 55 units of cost at ages 0 and 1, and then at age 10 invest 705 units, followed in successive years by costs of 610 and 495 units. This cost cash flow is bad enough, but look at the annual benefits. They follow a straight-line increase of two units per year. There is no change in this benefit flow after the initial investment to reduce the "eminent danger of collapse" or after the three years of heavy investments. If these years of investment would not reduce costs or increase the benefits, then why make the capital expenditures? Another unrealistic feature of Table 1 is that the straight-line flow of benefits continues to age 30 years. Traffic over the bridge just could not produce such a constant rate of cost reduction.

It is useless and misleading in economic analysis to bring into the solution for today's choice of alternative those far-into-the-future needs for maintenance and repairs under conditions that, as of today, cannot be estimated under any acceptable probability of actual occurrence. Any such estimates destroy confidence in the analysis.

Engineers and the engineering economists agree that the objective of the analysis for the economy of proposed investments is to determine which alternative proposal will maximize the net dollar return—on a discounted basis, of course. We also take the position that all methods of analysis, when properly calculated, will arrive at the same alternative as the choice. Wohl's paper does not disprove this statement. Let him try some realistic situations in place of his custom-built unrealistic illustrations in Tables 1-3 and solve them by the methods of (a) equivalent uniform annual cost, (b) benefit/cost ratio, (c) rate of return, and (d) NPV. He can even try some realistic multiple-rate situations and, when properly handled, find agreement within the methods. Note that Tables 1-3 each have two or more solutions in the rate-of-return and NPV methods.

The rate-of-return solution could involve more calculations than would the NPV solution, but not extensively so when applied to realistic alternatives and when computers are used. But remember that the NPV solution should be made for a range of discount rates—particularly when there is evidence of two or more reversals of sign in the cash-flow sequence or when the NPV sums for a pair of alternatives have but a small difference.

The rate-of-return method is not ambiguous and inconclusive. It is the NPV and benefit/cost ratio methods that have these characteristics. In the example of Table 1 and Figure 1, the three rate-of-return answers are correct and the whole truth. Getting only one answer from the NPV and benefit/cost ratio methods leaves us in ignorance of the whole truth. For example, had a MARR of 8 percent or less been chosen, the project would have had a positive NPV. Also, a positive NPV would have been found at any MARR between 18.66 and 73.57 percent. Are these answers not facts the analyst should give to the decision maker?

The decision maker wants to know what each alternative will produce on its own: What is its rate of earning? What are the comparative earnings by pairs of the multiple alternatives under analysis (the differential solutions)? The analysis for economy in no way deals with the handling of the paid-back income. This matter is for the decision maker to evaluate on the basis of what is expected to be the future situation for investment when the project generates the incomes forecast.

Throughout the paper the economic analysis is confused with the decision on choice of alternative by the decision maker. These two items are distinctly separate. The findings of economic analysis are only guides to use in the decision-making process.

Under certain conditions, a choice of alternatives may be different, depending on what MARRs are used and whether there are two or more changes in sign. But such difference is not due to the reinvestment assumption. Any difference is due wholly to the results of the combinations of the three variable factors within the solution equations. The three factors are (a) the discount rate, (b) dollar amount of each cash flow, and (c) yearly time spacing of the cash flows. Thus, when the NPV method is used at a discount rate of 15 percent, both alternatives 1 and 2 in Table 2 would be favored. The basic characteristic of exponential mathematics results in multiple solutions whenever there are two or more reversals of sign. The factor $(1 + r)^n$ produces strange results with changes in the magnitude of the dollar sums in the cash flow.

Regardless of the reinvestment assumption and other factors, Tables 2 and 3 under the rate-of-return solution present the true rate of return for each alternative. Further, the NPV answers are fact for each alternative for the single discount rate used in the calculations.

Tables 1-3 each contains two or more reversals of sign, either in the prime statement of cash flows or in the differences in flow between the two alternatives. Could it be that it is only under conditions of two or more reversals of sign that the difference in choice of alternative is likely to differ between the rate-of-return and NPV methods? The paper does not answer this question.

Wohl has proved nothing about the reinvestment assumption. Wohl says that his calculations prove that the returns were reinvested. The basis for my statement is found in the mathematical equation used in calculating the rate-of-return solution. The equation can be expressed in terms of present worth or in terms of compounded amounts. The present-worth solution equation may be written for plan 1, Table 3:

$$0 = -70 + 75(PW - x\% - 1\ year) + 70(PW - x\% - 10\ years) \quad (6)$$

When this is solved by trial, the rate (x) will be found to be 22.84 percent. The proof solution is

$$0 = -70 + 75(0.814\ 067) + 70(0.127\ 821) = -70 + 61.05 + 8.95 = 0 \quad (7)$$

The compound-amount solution equation for the rate of return may be written

$$0 = -70(CA - x\% - 10\ years) + 75(CA - x\% - 9\ years) + 70 \quad (8)$$

Again, solving by trial produces 22.84 percent for the rate (x). The proof calculation is

$$0 = -70(7.823\ 446) + 75(6.368\ 810) + 70 \quad (9)$$

$$0 = (-547.66 + 477.66)/70 = 0 \quad (10)$$

Proof Equations 9 and 10, in terms of compounding, are what Wohl says prove the rate-of-return solution; Equation 6 assumes the reinvestment of the 75 payback at age 1. There is absolutely nothing in any of the above equations that supports the assumption of the reinvestment of the 75 payback. In fact, all that Wohl has proved is that, if $P(1 + r)^n = CA$, then $P = CA\ [1/(1 + r)]$ or that CA times the present-worth factor equals P.

Equation 6 simply finds the present worth of the cash-flow items that enter the equation as isolated items pertaining to the situation under analysis. They represent both outgo and income with no reference as to future disposition of the paybacks of the initial sum with interest earnings. Certainly, Equation 6 does not handle the 75 factor twice.

When Table 3 is analyzed by differences in cash flow, plan 1 is preferred at a MARR of 15 percent (NPV is -1.29), but plan 2 is preferred at a MARR of 17 percent (NPV is +0.67). The break-even MARR is 16.26 percent. This example is evidence that two or more solutions are possible with the NPV method, a fact that is not mentioned in the paper.

There is some logic to the fact that compound interest factors and their reciprocals (the present-worth factors) retain in their calculations the interest earning for each time period. This truth is well known and is inherently involved in all compound-interest calculations. But this is far different from stating that the compound-interest factors assume the reinvestment or retention of principal repayments as well as the interest earnings from their data of cash flow.

Obviously, under the rate-of-return procedure the project must earn at the rate solved for. That answer is what the rate-of-return method is supposed to produce. As compared to NPV, the important fact is that, when the MARR is less than the rate-of-return solution, the project earnings above MARR (the NPV dollars) have to be earned at a rate above MARR. That rate of earning is given by the rate-of-return solution. Thus, the two methods are consistent. All three methods (including the benefit/cost ratio) use the identical input data and the same compound-interest theory. Their answers must agree in result or be convertible to each other.

Perhaps Wohl is not referring to this retention and compounding of interest earnings within the mathematical system. But what does he have in mind about reinvestment? His paper does not prove the correctness of the reinvestment assumption. If this reinvestment theory is true, then such reinvestment assumption applies to every possible application of compound-interest mathematics that involves outgo and income cash flows.

Wohl states that the rate-of-return method assumes a reinvestment rate equal to the rate given by the solution and that the NPV method assumes a reinvestment rate equal to the MARR rate used in calculating the NPV. My conclusion is that neither method assumes a reinvestment of payback sums, and nowhere in the total use of compound interest can it be found that such reinvestment is included.

Acceptable managerial procedure is to give consideration to the timing and dollar amount of each payback cash flow. This consideration is not and should not be a part of the calculations to determine which of a pair of proposals has the highest rate of return or NPV. All such considerations for disposal of paybacks reflect judgments based on current positions of the owner and of the community.

Author's Closure

Winfrey refuses to recognize the problems created by use of the internal-rate-of-return method and by failure to account for reinvestment of any early-year gains that are accrued by a project. [Fortunately, other engineering economists understand these points and do take account of reinvestment in their engineering economy texts; among them would be Newnan (3, Appendix 7-A) and White and others (7).] Space does not permit me to retrace all the arguments that underlie these concepts or to respond to all of Winfrey's discussion. Rather, I will restrict my closure to the following points.

Winfrey begins by noting my failure to mention that industrial officials want to know the rate of earnings and that transportation officials prefer the benefit/cost ratio or internal-rate-of-return method. Sadly enough, this may be true since they probably were incorrectly taught to believe in the sanctity of the internal-rate-of-return method. More importantly, Winfrey proclaims that "the [internal] rate-of-return method has been used for years without the disastrous results [that Wohl said] could happen." But how can a project, once it is built, possibly provide any information to suggest or prove that some other rejected alternative was more preferable? After all, no project will automatically signal that the wrong alternative was chosen. Only the analyst can prevent that, before the fact.

Winfrey says, "In the rate-of-return method... it is acknowledged that two or more rates of return may be found. But Wohl does not mention that the NPV method may also give two or more solutions." On the contrary, for any given MARR value, the NPV method will give only one solution, but the rate-of-return method can easily give more than one solution. To indicate otherwise is to be misleading. (Uncertainty with respect to determining which MARR should be used is a very different matter and applies equally to all methods, not just to the NPV method.)

Winfrey is obviously disturbed by my example in Table 1, which assumes heavy capital outlays in future years and indicates a straight-line growth in benefits (net of operating costs) over a 30-year period. To make matters worse, Winfrey says, "It is useless and misleading in economic analysis to bring into the solution for today's choice of alternative those far-into-the-future needs for maintenance and repairs under conditions that, as of today, cannot be estimated under any acceptable probability of actual occurrence." Foolish or not, traffic and transportation engineers commonly use a 25- to 35-year analysis period and they commonly assume a straight-line growth in net revenues or benefits. As one pertinent example, only 3 years ago De Leuw, Cather and Company conducted a 30-year economic analysis of four transit alternatives for Pittsburgh (8, Chapter X). They estimated the federal share (or 80 percent) of the year-by-year capital outlays for the four alternatives [transit expressway revenue line (TERL), light rail transit (LRT), rapid rail transit (RRT), and express bus transit (EBT)] as shown in Table 6 (8, p. XI-18); it is obvious that both the LRT and EBT alternatives will have heavy capital outlays some considerable years in the future. Moreover, the consultants assumed that both the operating costs and benefits would grow linearly up to year 2005 (8, pp. X-7, X-12, X-17, and X-18). Thus, here is a recent and actual example of exactly the situation that Winfrey feels is unrealistic. Fortunately, the consultants did not use the internal-rate-of-return method to select the best alternative because, for the capital outlay patterns shown in Table 6 and for the assumed conditions for benefits and operating costs, there is a wide range of benefit levels that will produce multiple rates of return.

Winfrey comments: "Under certain conditions, a choice of alternatives may be different, depending on what MARRs are used and whether there are two or more changes in sign. But such difference is not due to the reinvestment assumption.... Regardless of the reinvestment assumption and other factors, Tables 2 and 3 under the rate-of-return solution present the true rate of return for each alternative." While Winfrey admits that problems can arise with multiple sign changes, he totally confuses the issue by incorrectly stating that the differences are not due to different reinvestment assumptions and that the internal rate of return is the true rate of return. Consider the example in Table 7. If Winfrey were correct in saying that the internal rate of return was the true rate of return, then presumably he would be indifferent between alternatives 1 and 2 since (according to him) the true yield of each is 20 percent. In fact, for a MARR value below 20 percent, alternative 1 is clearly better than alternative 2 and will have a higher true yield than will alternative 2. Simply stated, alternative 2 will have a true yield of 20 percent if and only if the $10 000 earned at the end of year 1 can be reinvested for the remaining year at a rate of exactly 20 percent. But if the $10 000 is reinvested at any other rate, the true yield for alternative 2 will not be equal to its internal rate of return. However, the internal rate of return for alternative 1 is equal to its true yield since its earnings are accrued entirely at the end of the two-year analysis period.

As cited before, Winfrey admits that multiple internal rates of return could lead to different economic choices than would result from the NPV method, but he insists that "such difference is not due to the reinvestment assumption." The example in Table 7 is instructive. First, with the internal-rate-of-return method and for a MARR less than 20 percent, alternative 2 would be selected as acceptable and the best (or the choice would be ambiguous because of the multiple incremental internal rates of return). By contrast, the NPV or benefit/cost ratio methods would both show that, for a MARR below 20 percent, alternative 1 is unambiguously acceptable and the best choice. Second, and despite Winfrey's assertion that the above economic choice difference is not due to the reinvestment assumption, there can be nothing other than a difference with respect to reinvestment that can result in alternative 2 being wrongly chosen by the internal-rate-of-return method. That is, the economic yield from alternative 1 is in no way affected by reinvestment since all the earnings are accrued at the end of the two-year analysis period. But with alternative 2, the issue must be, What happens to the $10 000 that is earned at the end of the first year? Certainly, these first-year earnings will not be ignored or placed in a safety deposit box. Rather, these funds either will be spent on consumption and thus provide extra enjoyment for the remaining year (i.e., they will be reinvested on early- rather than later-year enjoyment) or they will be reinvested in the best foregone investment and thus provide some yield for the last year. If we assume (as is normally done) that (a) people's rate of time preference is equal to the rate of productivity, (b) both are equal to the MARR, and (c) the borrowing rate is less than or equal to the MARR, then the only reinvestment rate for the first-year earnings that will result in the effective yield being equal to the internal rate of return for the two-year period is 20 percent. That is, if r is the reinvestment rate and r_2 is the internal rate of return for alternative 2, then r is the discount rate that satisfies the following identity:

$$0 = -101\,000 + [10\,000/(1 + r_2)] - [10\,000/(1 + r_2)] + [10\,000(1 + r)/(1 + r_2)^2] + [133\,440/(1 + r_2)^2] \quad (11)$$

If r_2 is 20 percent (or 0.20), then the only reinvestment rate (r) that can satisfy this identity is also 20 percent.

All in all, Winfrey seems to think that there is something inherently different about being explicit with respect to reinvestment as opposed to being implicit. For instance, in Winfrey's discussion of loan plan 1 for the example in Table 3, he says that the internal rate of return is the interest rate (or x percent) that satisfies Equation 8, or

Table 6. Annual federal shares of total system capital costs for four transit alternatives.

Year	Alternative 1, TERL ($000 000s)	Alternative 2, LRT ($000 000s)	Alternative 3, RRT ($000 000s)	Alternative 4, EBT ($000 000s)
1976	-	-	-	-
1977	72.5	16.8	22.2	6.7
1978	77.7	73.5	76.7	8.1
1979	75.6	64.2	76.4	57.1
1980	73.6	65.7	72.0	62.2
1981	77.6	69.1	90.9	63.2
1982	1.7	1.5	1.6	3.4
1983	1.8	1.6	1.8	3.7
1984	1.9	1.7	1.9	4.2
1985	2.0	1.8	2.0	4.2
1986	2.1	1.9	2.1	4.4
1987	2.3	2.0	2.3	10.3
1988	2.4	2.2	2.4	11.2
1989	2.6	2.3	2.6	18.1
1990	23.2	21.2	22.0	22.1
1991	2.9	4.0	2.9	5.8
1992	3.0	13.4	3.0	6.2
1993	3.2	28.7	3.2	6.5
1994	3.4	30.7	3.4	6.9
1995	3.6	3.3	3.6	7.4
1996	3.8	3.5	3.8	7.8
1997	4.1	3.7	4.1	8.2
1998	4.3	3.9	4.3	8.8
1999	4.6	4.1	4.6	9.3
2000	4.8	4.4	4.8	9.8
Total	454.7	452.2	414.6	355.7

Table 7. Net annual cash flows for two alternatives.

Year	Alternative 1 ($000s)	Alternative 2 ($000s)	Δ (Alternative 2 - Alternative 1) ($000s)
0	-100	-101	-1.00
1	0	+10	+10.00
2	+144	+133	-10.56

Note: r_1 = 20 percent; r_2 = 20 percent; and Δr for alternatives 1 and 2 = 20 and 780 percent.

$$0 = -70(1 + 0.01x)^{10} + 75(1 + 0.01x)^9 + 70 \qquad (12)$$

In turn, x = 22.84 percent. But Winfrey incorrectly says, "There is absolutely nothing in any of the above equations that supports the assumption of the reinvestment of the 75 payback." Winfrey has compounded the $75 year 1 payback amount for nine years at 22.84 percent and thus has implicitly assumed reinvestment at that amount for the nine remaining years. After all, what other explanation can there be?

Winfrey asks, "Could it be that it is only under conditions of two or more reversals of sign that the difference in choice of alternative is likely to differ between the rate-of-return and NPV methods?" The answer to this question is usually. That is, problems can also arise (even when there is a unique internal rate of return) if the order in which mutually exclusive alternatives are ranked is changed. This could occur when two or more alternatives have identical annual cash flows during the initial time period.

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Discussion

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My intention is neither to defend nor to condemn the use of any of the accepted and proven valid methods for analyzing investment proposals. In our discussion, Reversals of Sign in Cash Flow Series (1, p. 556), we discuss the "cult of NPV" in some detail. Surely Wohl falls into this category of economist.

In teaching students, however, it is necessary that the major techniques actually used be presented in such a fashion that they will not be used incorrectly when the student applies them in the real world of decision making. Further, the results of a survey of the 1971 Fortune 500 firms (9) indicate that some 39-43 percent of the firms that responded use the rate-of-return criterion as their primary evaluation technique. Present worth ranked about second, along with payback period, as the primary techniques. Hence, a student should be taught the strengths and weaknesses of each of the major techniques of capital expenditure analysis.

Wohl is so intent on convincing us of the superiority of the present-worth criterion that, in the latter pages of his paper, he uses the future-worth criterion to demonstrate the validity of his arguments. Other authorities, who argue the superiority of future worth as the criterion, would appreciate his support.

Since the bulk of the discussion revolves around the multiple rate problem (what my coauthors and I call the problem of reversals of sign in cash-flow series), this is the specific point that I will address. Wohl makes the argument that each of the solving rates of return is exactly correct. This statement, however, requires a very specific qualification. Each is correct if the MARR that a firm requires from its investments equals the interest rate at which it acquires investment funds (the so-called borrowing rate). If this is in fact the case, then the firm is acting primarily as a money changer. There is little, if any, allowance in its financing-investment structure for productivity improvement.

Most firms other than regulated utilities have many more profitable ventures from among which to choose than their available capital will support. As the result, the MARR is usually substantially greater than the interest cost of capital. Once the constraint that cost of capital must equal the MARR is removed, it becomes necessary to interpret the meaning of the cash flows.

The two basic series of cash flows are pure investment series and pure borrowing series. The cash-flow pattern in Figure 4 represents an investment. At time 0, $1000 is expended to produce positive cash flows of $500 at the end of each of three periods. A plot of NPV is a function of interest rate (i) for a pure investment situation and will always be positive and equal to the algebraic sum of the cash-flow series (+$500) when i = 0, sweep in a downward direction to the right, and become asymptotic to the initial investment amount (-$1000) as i increases without bound. The value of i at which NPV(i) intersects the zero axis on the NPV scale is the rate of return (i*) on the particular cash flow. The NPV(i) plot for a pure investment situation will always exhibit this appearance. If the value of i* is greater than or equal to the MARR, the cash-flow series is preferable to alternative investment at the MARR.

The cash-flow series in Figure 5 clearly represents borrowing. At time 0, $1000 is received for which payments of $500 are made at the end of each of three periods. A plot of the NPV as a function of i for a pure borrowing situation will always be negative and equal to the algebraic sum of the cash-flow series (-$500) when i = 0, sweep in an upward direction to the right, and become asymptotic to the initial amount borrowed ($1000) as i increases without bound. The value of i at which NPV(i) intersects the zero axis on the NPV scale is the interest rate (i*) paid on the particular cash flow. The NPV(i) plot for a borrowing situation will always exhibit this appearance. If the value of i* is less than or equal to the firm's cost of capital from other sources, then this cash-flow borrowing series is preferable to acquiring capital from those other sources.

With these fundamental characteristics of pure investment situations and pure borrowing situations in mind, let us now turn our attention to Wohl's examples. In the interest of brevity, I will concentrate on the simplest cash-flow series of his several examples, Table 2. The solving rates of return for alternatives 1 and 2 compared to doing nothing are 25.69 and 26.16 percent, respectively. Plots of NPV(i) for each of these independent alternatives are shown in Figure 2. The independent cash flows are given in Table 8.

Having calculated the i*'s for each alternative independently and found alternative 2 to have a higher i* than alternative 1, casual observation of the cash flows suggests that there must be an unusual difference between them. Alternative 2 requires an additional $10 000 investment and nets $5 000 less return.

If alternatives 1 and 2 are mutually exclusive, it then becomes necessary to analyze this difference by first looking at and interpreting the difference between the two cash-flow series. This is indicated in Table 8 as Δ(Alternative 2 - Alternative 1).

The cash-flow series and its attendant NPV(i) plot in Figure 6 clearly do not match either the pure investment or pure borrowing situations. It must, therefore, be a mix of the two. Since the sum of the cash flows is negative (-$5000) it looks more like borrowing than anything else. However, the NPV(i) plot bends back down and becomes asymptotic to -$10 000, as would be the case for an investment project. The zero NPV axis is intersected twice.

The -$10 000 at time 0 clearly is an investment that can only be recovered from the +$45 000 at time 1. The -$40 000 at time 2 must be the repayment of a loan that can only come from the +$45 000 at time 1. Ignoring the firm's MARR, assume that the firm can acquire investment capital at a cost of 10 percent/period. Then the implied amount of the loan at time 1 would be $40 000 (1/1.10) = $36 360. This would leave ($45 000 - $36 360) = +$8640 to return the -$10 000 investment at time 0. This may be illustrated as in Table 9 (columns 1-3).

Since the initial investment of -$10 000 is not returned, the rate of return on the investment is negative and the present worth at any positive interest rate is negative. Thus, by either the rate-of-return or present-worth methods, the conclusion would be reached that alternative 2 is not preferable to alternative 1.

Assume, now, that the firm has a MARR of 25 percent. We could then evaluate the remaining balance of the time 1 cash flow by representing the loan as $45 000 - $10 000(1.25) = $32 500. The breakdown of the cash-flow series would then be as given in Table 9, columns 1, 4, and 5. The loan cost, in this case, is 23.08 percent. If the firm can borrow from any source at less than this figure, then alternative 2 is not preferable to alternative 1.

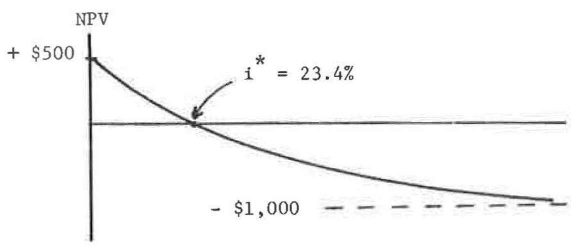
Suppose, however, that the analyst used the 25 percent MARR to evaluate the NPV of the entire cash-flow series Δ(Alternative 2 - Alternative 1) as Wohl suggests. This NPV is +$0.4 and alternative 2 is preferable to alternative 1. (In fact, for any chosen MARR between 21.92 and 228.08 percent the NPV will be positive.) This solution assumes that the cost of investment funds to the firm (the borrowing rate) is 25 percent and equals the rate required from investments (the MARR). If the borrowing rate does equal the MARR, then, of course, this result is correct and both the NPV and rate-of-return methods, properly applied, will show it (Table 9, columns 1, 6, and 7). The rate of return on the investment portion is 30 percent, in excess of the required MARR of 25 percent. Correspondingly, the present worth at 25 percent is positive and equal to 0.4.

Three basic points have been demonstrated:

1. The claim that present worth always leads to the correct choice requires the assumption that the borrowing rate to the firm always equals the MARR at which it invests funds. This is not the case with the vast majority of firms.

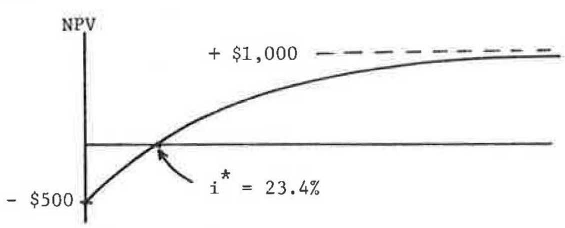
2. When a cash-flow series does not match the typical pure investment or pure borrowing models, that is, when it is a mix of borrowing and investing, it is necessary to divide the series into its investment and borrowing components.

Figure 4. Cash flow and NPV as a function of i, pure investment.



0	- $1,000
1	+ 500
2	+ 500
3	+ 500
∑	+ $ 500

Figure 5. Cash flow and NPV as a function of i, pure borrowing.



0	+ $1,000
1	- 500
2	- 500
3	- 500
∑	- $ 500

Table 8. Independent cash flows for the Table 2 example.

Year	Alternative 1 ($000s)	Alternative 2 ($000s)	Δ (Alternative 2 - Alternative 1) ($000s)
0	-100	-110	-10
1	+70	+115	+45
2	+70	+30	-40
Total	+40	+35	-5

Figure 6. Cash flow and NPV as a function of i, mixed borrowing and investment.

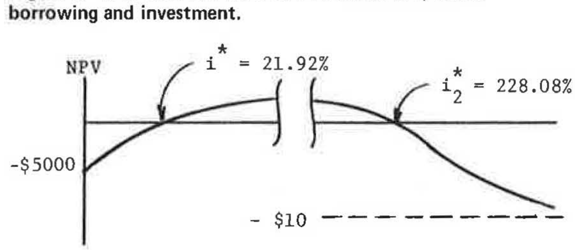


Table 9. The cash-flow series for the Table 2 example at different MARRs.

Year	Δ (Alternative 2 - Alternative 1) ($000s)	Loan Portion, 10% ($000s)	Investment Portion ($000s)	Loan Portion ($000s)	Investment Portion, 25% ($000s)	Loan Portion, 25% ($000s)	Investment Portion ($000s)
0	-10	0	-10.00	0	-10.0	0	-10
1	+45	+36.36	+8.64	+32.5	+12.5	+32	+13
2	-40	-40.00		-40.0	0	-40	0
Total	-5	-3.64	-1.36	-7.5	+2.5	-8	+3

3. Both the rate-of-return and present-worth methods will lead to correct choices when applied correctly. And both methods can lead to incorrect choices if applied incorrectly.

One additional caution should be added at this point. At no time have I used two interest rates that span the same time periods. One rate was used for cash flows in periods 0 and 1 and the other for periods 1 and 2. At no point are both interest rates used to transform cash flows or portions of cash flows over periods 0, 1, and 2. It is possible to show that $100 today is equivalent to $200 today if two interest rates are used over the same time periods. As the consequence, any method that allows for such equivalence transformations can lead to very strange results indeed.

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Author's Closure

Leavenworth's discussion has confused the issues and introduced a confusing terminology. Let me attempt to unravel his complicated machinations.

Leavenworth says that "Wohl is so intent on convincing us of the superiority of the present-worth criterion, that, in the latter pages of his paper, he uses the future-worth criterion to demonstrate the validity of his arguments." Of course—any student of cash-flow equivalence knows that present worth and future worth are interrelated by a single-payment compound factor (or its inverse), that if one value is positive so is the other, and that, if the present worth for project A is larger than that for project B, then the future worth for project A will always be larger than that for project B. (This point was made in the paper.) In short, present worth is not a superior criterion to future worth, and vice versa; they are identical. Both are correct and consistent, and both are superior to using the internal-rate-of-return method. (Also, the benefit/cost ratio method is as valid as either present-worth or future-worth methods.)

Leavenworth then launches into lecture 1 of engineering economics, carefully explaining the mechanics of pure borrowing and pure investment situations. Then, turning to my Table 2 example, he shows that the net cash flows between alternatives 2 and 1 lead to incremental cash flows of -$10 000 in year 0, +$45 000 in year 1, and -$40 000 in year 2. Leavenworth then adds: "The cash-flow series and its attendant NPV(i) plot clearly does not match either the pure investment or pure borrowing situation. It must, therefore, be a mix of the two." Such an interpretation is outrageous. The situation is simple. The firm (in this example) has two pure investment choices (as well as the do-nothing alternative): to invest $100 000 in alternative 1 or to invest $110 000 in alternative 2. Or does it invest in neither? Moreover, if it invests in alternative 2 instead of 1, it simply must invest an extra $10 000 at time 0, then as a result earn $45 000 more at time 1 but $40 000 less at time 2. Thus, when deciding between alternatives 1 and 2 (and aside from the question of the acceptability of either), the question is whether the time 1 and time 2 amounts merit the extra investment of $10 000 at time 0. That is a pure investment question. By contrast, Leavenworth says, "Ignoring the firm's MARR, assume that the firm can acquire investment capital [i.e., borrow] at a cost of 10 percent/period. Then the implied amount of the loan at time 1 would be $40 000 (1/1.10) = $36 360." On the contrary, no loan is implied and no money needs to be borrowed at time 1, but only at time 0; that is, we must acquire the capital to make the extra $10 000 investment at time 0 if, in turn, we are to receive $45 000 more at time 1 but $40 000 less one year later. The real question is, Can I reinvest the $45 000 extra accrued at year 1 so as to recoup enough to offset both the $40 000 loss at time 2 as well as the foregone earnings on the initial investment? Moreover, if we ignore the firm's MARR (as Leavenworth suggested) and only consider the borrowing rate, then we overlook the most basic principle in engineering economics—to guarantee that the yield from additional investment is at least as large as that of our foregone opportunities.

Leavenworth's next example is also absurd. Therein, he assumes that the firm has a MARR of 25 percent and, thus, that the extra $10 000 in time 0 can be invested for one year, accumulating a total of $12 500 and thereby reducing the amount needed for a loan at time 1 to $32 500. Again, such calculations are mere fiction and fantasy. First and foremost, $10 000 is not available at time 0 for investment at the MARR of 25 percent. Rather, that amount of investment is required in order to gain an extra $45 000 at time 1 and $40 000 less at time 2; nor is a loan required or implied at time 1.

Last, it is not necessary (for his assumptions) to break the cash-flow stream into investment and loan portions and to carry out such arduous calculations. A simple NPV or net future worth calculation at the appropriate MARR will suffice. Of more importance, such machinations bear no resemblance to the internal-rate-of-return method, but are the heart of the ad hoc rate-of-return procedure of Grant, Ireson, and Leavenworth (1, Appendix B) to avoid the problems created by the internal-rate-of-return method.

My paper explored the use of the internal-rate-of-return and NPV methods under a rather strict and explicit set of assumptions about the borrowing and lending rates and about the time value of early (rather than late) consumption by people. Moreover, I explicitly noted that the use of different rates, consumption preferences, or different financing and investment plans could affect the effective yield that will be forthcoming from one project or another. In a roundabout fashion, Leavenworth is compelled to agree (as does Bergmann in a later discussion) that my conclusions are entirely correct for the assumptions that I made. His discussion then turned to the circumstances when the borrowing rate is different from the MARR. Unfortunately, his first two examples (which dealt with the incremental cash flows for the Table 2 example) were explicit only about one of these two rates (the borrowing rate in the first and the MARR in the second) but not about both. Then, later in his discussion, he explicitly assumes the borrowing rate and MARR to be 25 percent and concludes that "the rate of return on the investment portion is 30 percent, in excess of the required MARR of 25 percent." Presumably, then, an investor would conclude from Leavenworth's remarks that, all things considered, the effective annual yield (from investing in alternative 2 rather than 1) will be 30 percent for the two-year period, as compared with the MARR of 25 percent. This would suggest that, relative to investing in some foregone investment opportunity, the investor would accrue a profit of $1275 by the end of two years if the extra $10 000 were invested in alternative 2 rather than 1. But such a conclusion would be incorrect. Rather, for any borrowing and investment plan that can be devised (given Leavenworth's assumed borrowing rate and MARR), it can easily be shown that only a $625 profit and not $1275 can be accrued by the end of two years. In short, the effective yield from the two-year investment will be 27.48 percent (rounded off) rather than 30 percent, as indicated by Leavenworth.

The more important issue is, How should the analysis be done (especially for cash flows with two or more sign changes, as in the previous example) when the borrowing rate and MARR are not equal? First, we usually assume that the MARR is at least as large as the borrowing rate. Second, it also should be apparent that for an NPV analysis we do not apply the borrowing rate to any capital loans and then use the MARR value to discount the cash flows. Rather, we simply apply the MARR to determine the NPV. Remember, the choice is a simple one: Either acquire the investment capital for the project being analyzed and then accrue any and all future-year cash flows as a result thereof or acquire the same amount of investment capital and instead invest it in the best (otherwise) foregone investment opportunity that, by definition, will yield the MARR. The finance charges will be identical for both of the investment choices and thus can be disregarded in deciding whether to accept some project or do nothing (that is, or invest in the best foregone alternative). In short, it is neither necessary nor desirable to carry out the ad hoc rate-of-return procedure described by Leavenworth and others, at least not for these assumptions.

In the next to last paragraph of his discussion, Leavenworth claims to have demonstrated that "both the rate-of-return and NPV methods will lead to correct choices when applied correctly." On the contrary, what he did demonstrate was that the internal-rate-of-return method does not apply when multiple rates occur. He showed that an ad hoc rate-of-return method must be used in order to get around the problems created by multiple internal rates of return.

Discussion

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Wohl has taken a number of positions in his paper with which I disagree. I propose, however, to discuss only one area of disagreement—the reinvestment issue.

The question of the reinvestment assumption in the use of the internal-rate-of-return criteria for project analysis has become an issue of major importance. Some authors side with Wohl [White and others, for example (10, pp. 148-149)]; some do not [such as Grant and others (1, pp. 563-571)]. The issue is one of fundamental economic theory and definitions.

To illustrate the economic theory and resulting definition behind the concept of rate of return, I refer to Samuelson (11, pp. 599-601). Consider a perpetual investment (P) from which an amount (A) will be thrown off each year at interest rate (i). The interest rate is defined then as

$$i = A/P \tag{13}$$

The NPV of such an investment is P since

$$P = A/i \tag{14}$$

The internal rate of return of such an investment is i, since P = the discounted present worth of the costs, A(1/i) = the discounted present worth of the benefits, and Equation 14 shows them as equal. This is the definition of interest, present worth, and internal rate of return.

Under such a definition, is it necessary to invest A at i for Equation 14 to be true? The answer is obviously no. Yet the belief that such is necessary forms the basis of Wohl's contention that reinvestment of the benefits of an investment must be made, or be capable of being made, for internal-rate-of-return analysis to be valid.

The example of a perpetual investment illustrates this clearly. But what is at issue here is actually the fundamental definition of interest—in fact, of what return on investment means in the most basic sense. Investment returns may be invested (the reinvestment of Wohl) or consumed. Thus a definition of internal rate of return that requires reinvestment of the returns violates a fundamental economic concept, recognized by authorities such as Keynes (12, pp. 135-146) and Samuelson (11, pp. 599-601). A moment's thought brings to mind many examples where reinvestment of total product would be disastrous for society—agricultural production for example. If all the returns of agriculture were invested as seed, following the usual economic example of consumption and investment, nothing would be left for food.

I am not suggesting that adoption of Wohl's startling definition of internal rate of return is likely to lead the world to starvation. The world will continue to follow economic rules in spite of what academicians say. However, academicians should remain as much in touch with economic reality as they possibly can. Therefore, I urge Wohl to return to the fundamentals from which he has strayed so far.

The table that follows illustrates the induction of the formula for the future worth of an investment. A close examination of it may help to dispel the doubts of those who still cling to the reinvestment fallacy.

N	P ($)	Payment at Interest i ($ at 10%)	F_N($)
0	1000		
1		100	1100
2		110	1210
3		121	1331

$$F_1 = P + Pi = P(1 + i)^1 \tag{15}$$

$$F_2 = P(1 + i)^1 + iP(1 + i)^1 = P(1 + i)^2 \tag{16}$$

$$F_3 = P(1 + i)^2 + iP(1 + i)^2 = P(1 + i)^3 \tag{17}$$

Note that it was not necessary to invest the sum at the end of the third year ($1331) in order to induce our formula $F_N = P(1 + i)^N$. We merely had to reinvest the original investment itself and its interest. But this is the very meaning of investment. By extension, any cash flow of benefits that results from any investment may be considered single B_Ns thrown off because of each one's part of the original investment. Of course, future worth in all formulas implies reinvestment of the positive cash flows. But this is no more and no less than the meaning of future worth.

It is possible to specify that all returns of a given alternative must be reinvested at the opportunity cost of capital or at any other interest rate. But this reinvestment is merely a characteristic of the alternative and does not imply anything about the definition of the internal rate of return. Accordingly, I urge the logic of the discussion, so often mentioned by Wohl, of Grant and others (1, pp. 563-571).

Many authors also make Wohl's error. This has caused efforts to reconcile the internal-rate-of-return method with the other methods when, in fact, they were already reconciled. What is at stake in this controversy is Wohl's unacceptable definition of interest or internal rate of return. Return on capital may be consumed or reinvested, in whole or in part, but whether or not it is is a separate decision.

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Author's Closure

Steiner has distorted the intent of my paper. To begin, Steiner correctly shows that the internal rate of return is equal to the interest rate at which the discounted benefits are equal to the discounted costs; this definition is identical to that which I incorporated in my paper and in Equation 1. (There are, of course, other correct ways of defining the same internal rate of return; e.g., it is the interest rate at which the future worth of the benefits is just equal to the future worth of the costs, or it is the interest rate at which the equivalent uniform annual benefits are just equal to the equivalent uniform annual costs.) But, then, he thoroughly distorts the matter by insisting (to paraphrase his words) that my startling definition of internal rate of return requires reinvestment of the returns and violates a fundamental economic concept, recognized by authorities such as Keynes and Samuelson. By contrast, examination of Equation 1 and the remarks that immediately follow will show that my definition of the internal rate of return is consistent and identical with those of every author cited by Steiner. In no way have either I or others implied or said that the definition of the internal rate of return required reinvestment of the returns during the analysis period. Rather, what I and many others have maintained is that use of the internal rate of return as the sole indicator of the effective yield from some investment program over the planning horizon or as the criterion for making economic choices can lead to incorrect or ambiguous answers. Also, to use the internal rate of return as the sole indicator of the yield to be anticipated from a project implies that the yield from either reinvestment or consumption of any early-year returns is equal to the internal rate of return.

Moreover, Steiner even agrees to the above conclusion by saying (in the last sentence of his discussion): "Return on capital may be consumed or reinvested, in whole or in part, but whether or not it is is a separate decision." Even so, Steiner misses the essence of the problem. And that is, whether a separate decision or not, the early-year returns will clearly be consumed or reinvested in something during the remaining years and thus the rate of reinvestment of earnings can and will affect the overall earnings, effective yield, and decision for projects. As a consequence, to simply use the internal rate of return as a guide to economic decision making is to open the door to incorrect or ambiguous economic choices by implying that the reinvestment rate is equal to the internal rate of return.

In my paper I indicated how we can compute the effective rate of return for a project, a yield figure that will properly reflect not only the internal earnings but also the external gains from consumption or reinvestment of early-year gains. I also later indicated that the effective rate of return will be equal to the internal rate of return only when all earnings are accumulated solely at the end of the analysis period or when the MARR (or reinvestment rate) is exactly equal to the internal rate of return.

Discussion

Dietrich R. Bergmann, St. Clair Shores, Michigan

The paper is a partial repetition and extension of Wohl's past criticisms of the internal-rate-of-return method for comparing mutually exclusive alternatives. His initial critique (13) used illustrations in which the solution for the internal rate of return on either the base or incremental investment was unique. Those criticisms and similar criticisms of other authors were discussed by me in 1973 (14).

Wohl's 1975 paper (15) continued his criticism of the internal-rate-of-return method and devoted a part of it to situations where the solution for the internal rate of return is not unique. Many of the points in his 1975 paper are repeated in the present paper. Consequently much of my reaction to the paper corresponds to the viewpoints expressed in my discussion (16), which was published with that paper. In the interest of brevity, I refer the reader to that discussion.

Several additional points are appropriate on this occasion. Wohl states that "...the MARR can be regarded as the opportunity cost of capital for both borrowing and lending situations." Such could be the case and would be true when there is no limit on an organization's borrowing; however, more often than not the interest rate that applies to an organization's borrowing will differ from the MARR for that organization's investment opportunities. His analysis of the illustrations presented in Tables 1-3 implicitly assumes that these two rates are identical to each other. For the instances that they are, I have no quarrel with Wohl's designation of the preferred project.

Consider, though, the general case where the rate of interest on money borrowed by the investing organization is different from the investing organization's MARR. Teichroew, Robichek, and Montalbano (17) have shown that, for this more general case, the NPV and the internal-rate-of-return methods must both be refined if the projects under consideration are mixed projects (i.e., projects that combine the features of both investment and lending situations). In each of the three illustrations summarized in Tables 1-3, either the base or incremental cash-benefit flow falls into the mixed project category as defined by Teichroew and others (17).

The specific approach to be used in analyzing a mixed project is dependent on the investment situation, of which the following three are envisioned:

1. The investing organization is also the organization that receives the benefits that accrue from the investment;

2. The investing organization is not the recipient of the benefits that accrue from the investment; instead, the benefits accrue directly and entirely to the public; and

3. The benefits generated by the investment accrue partly to the investing organization and partly to the general public.

Wohl's illustrations summarized in Tables 2 and 3 typify investment situation 1. His remaining illustration (summarized in Table 1) appears to be an example of investment situation 2.

When analyzing mixed projects that reflect investment situation 1, the cash-flow stream must be divided into a borrowing portion, for which the interest rate paid on the borrowing is specified, and an investment portion, for which the MARR is specified. After this division is completed, the analyst must evaluate the investment portion of the cash-flow stream. The project is deemed to be acceptable either if the internal rate of return for the investment portion of the cash-flow stream is in excess of the MARR or if the NPV of the investment portion is greater than zero.

Wohl and I would probably agree that the analysis of mixed projects, reflecting investment situation 2, has been inadequately treated in the literature. There are at least two unique features here, the first of which is that there is no borrowing by the investing organization from the project's beneficiaries. The second is that the benefited members of the public do not reinvest their benefits at the agency's MARR. Instead, those benefits are either consumed or reinvested at a rate that economists occasionally have referred to as the social rate of discount. Consequently the analytical problem presented by investment situation 2 is as evident in applying the customary NPV method as it is when applying Wohl's version of the internal-rate-of-return method.

One way to resolve the analytical problems posed by mixed projects involving investment situation 2 would be to apply the social rate of discount to transform all the project's public benefits to their future value at the end of the planning horizon and to accept the project if, for the modified cash-benefit flow stream, the present worth exceeds zero or the internal rate of return exceeds the MARR. I might add that, even for pure investment projects characterized by investment situation 1, there is a good basis to support the contention that public benefits should be transformed to their future value at the planning horizon by using the social rate of discount before the project's NPV or internal rate of return is calculated for the transformed cash-benefit flow stream.

Obviously, mixed projects, which are characterized by investment situation 3, require an analytical approach that combines the features described above for investment situations 1 and 2.

In conclusion,

1. It appears to me that Wohl has endorsed the NPV approach for evaluating mutually exclusive alternatives chiefly because, using his words, it is "conclusive and unambiguous." I agree that the NPV method is conclusive and unambiguous; however, I must add that application of the method in the way Wohl suggests makes invisible to the analyst those investment situations where his version of the method should be refined, as suggested in the preceding paragraphs.

2. It appears that Wohl would have us never calculate a rate of return on an investment proposal. If so, how are we to determine a value for the MARR when the investment opportunities are far in excess of the available budget? I can think of one way for doing that, but it is no less computationally tedious than calculating a rate of return.

3. Use of the internal-rate-of-return method does not imply, as Wohl and others suggest it does, that the positive cash flows from an investment alternative are reinvested at the internal rate of return. When we calculate the internal rate of return we are merely developing an ordinal ranking statistic that is compared with the MARR, just as the value of the NPV is compared with zero in order to identify the preferred project. No more and no less is either explicitly stipulated or implied by the two methods.

4. The problems that Wohl describes result partly from a disagreement about the definitions of the NPV and internal-rate-of-return methods for evaluating mutually exclusive investment alternatives. He is critical of the internal-rate-of-return method because he has defined it in a manner that involves the shortcomings he alleges it to have. Hopefully, future papers on this subject will strive to identify commonly accepted definitions for both methods.

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This discussion has benefited from conversations with Henry M. Steiner and Richard S. Leavenworth. However, the views expressed are my own; they do not necessarily represent those of other individuals, my employer, or other organizations with which I am affiliated.

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Author's Closure

Bergmann concedes that he has no quarrel with my designation of the preferred project for the situation explored in my paper, in which the MARR can be regarded as the opportunity cost of capital for both borrowing and lending situations. However, his arguments deal mainly with an entirely different situation, that in which the borrowing rate and lending rate (that is, the yield of available investments outside the firm or MARR) are unequal. Accordingly, let me also address this different set of conditions as they relate to use of the NPV method, or such others as may be appropriate.

Bergmann argues that mixed projects are those that combine the features of both investment and lending situations; the examples in Tables 1-3 of my paper fall into the mixed-project category; and whenever the borrowing rate and MARR are equal, "the NPV and internal-rate-of-return methods must both be refined if the projects under consideration are mixed projects." Also, he says that, for projects such as those in Tables 2 and 3, "...the cash-flow stream must be divided into a borrowing portion, for which the interest rate paid on the borrowing is specified, and an investment portion, for which the MARR is specified." To the contrary, and notwithstanding the work of Teichroew (17) and Grant (1), it seldom is necessary to overcomplicate the analysis in this fashion. Put simply, the mere existence of a mixed project and of unequal borrowing and lending rates is not a sufficient condition for abandoning a simple NPV or net future value calculation.

Let me demonstrate this conclusion under the following conditions: (a) the lending rate (or rate at which an agency or firm is willing to loan or invest its assets) is its MARR, (b) the borrowing rate (BORR or rate at which investment capital can be acquired from outside sources) is unequal to the MARR, and (c) for public projects people's rate of time preference can be regarded as being equal to the MARR (more will be said about this assumption later). Under these conditions, an agency or firm when evaluating any project (or the difference between two competing projects) is faced with three options:

1. Acquire no capital and invest in none (i.e., the null alternative);
2. Acquire capital from liquidated assets, working capital, or borrowing and invest it in the best (otherwise) foregone investment opportunity; and
3. Acquire capital and invest it in the proposed project.

How, then, do we evaluate these options when MARR > BORR or MARR < BORR?

CASE 1: LET MARR > BORR

In this case, option 2 is always preferable to doing nothing. Thus, the choice is simply between a foregone alternative and the project in question. The investment capital to be acquired for each is identical, as will be the financing costs; therefore, we need only to compute the NPV (or, equivalently, the net future worth) of each proposed project with MARR as the discount rate. If the NPV of a proposed project is positive, it is acceptable; otherwise, invest in our best foregone investment alternative. Among a set of competing projects, the one having the highest positive NPV is the best. No other analysis is necessary for this case (nor for that in which MARR = BORR), and the complicated procedures of Teichroew and Grant can be ignored. Turning to the example in the table below, or to those in Tables 1-3, a straightforward calculation of the NPV at any given MARR value will always correctly identify a project's acceptability, as well as the best one when there is a competing choice. (Such calculations, if carried out for the example in the table below, will show that project X is unambiguously the best option whenever the MARR is between 30 and 40 percent. Also, the internal rate of return in the table is 0, 30, and 40 percent.)

Year	Net Annual Cash Flows ($)
0	-1000
1	+3700
2	-4520
3	+1820

CASE 2: MARR < BORR

At first glance, it may seem that, since the borrowing rate is greater than the MARR, a firm would always find option 2 unattractive. But that would not necessarily be correct. Simply stated, whenever a firm can acquire the necessary capital for investment in either the proposed project or a foregone alternative by liquidating some of its assets or by using working capital rather than by borrowing from outside sources, the financing costs would be represented by the MARR rather than by the BORR. Clearly a firm faced with these borrowing and lending rate conditions would avoid, so far as possible, borrowing from outside sources. Accordingly, if all the required capital can be acquired by using the agency or firm's assets, the appropriate discount rate is the MARR and the economic choice boils down to one between options 2 and 3, as before. And again, in this instance, a simple NPV or net future worth calculation at the MARR will indicate which choice is superior.

Second, whenever a firm or agency can act not only as an investor but also as a lender (a situation in which numerous private firms find themselves), and whenever the borrowing rate is larger than MARR, a firm would then regard the borrowing rate as its MARR and, accordingly, would simply carry out a straightforward NPV or net future worth analysis with the BORR (rather than MARR) as the appropriate discount rate. So once again, a complicated analysis is not necessary, even though the BORR > MARR.

Third, if and only if a firm can acquire the necessary investment capital only from outside sources, cannot become a lender (as well as investor), and the BORR > MARR, only then must we refine our techniques for evaluating mixed projects. (Frankly, I suspect that we seldom confront this rather special situation.) But even in this case, a simple and straightforward procedure will suffice to properly evaluate any proposal. [The one to be described is akin to that outlined in Teichroew (17); also, it is much simpler than that described by Grant (1).]

The underlying principle is to maximize the net future value and thus to minimize the amount of borrowing over the years. In short, borrow as little as possible and pay back borrowed funds as soon as possible. [Also, it should be evident that for this special case (i.e., capital can be acquired only by borrowing and the firm cannot be a lender) doing nothing is always preferable to acquiring capital and investing in a foregone alternative. Thus, the economic choice is simply between doing nothing and investing in a proposed project.] Accordingly, the procedure is as follows: Carry out a net future value analysis, compounding year by year over the n-year planning horizon. Whenever the accumulated net future value (in years 0 through n - 1) is negative, compound the balance to the next year at the BORR; if the accumulated balance is positive, compound the balance to the next year at the MARR. If the net future value at the end of n years is positive, the project will be acceptable; if not, the project is rejected. For a set of mutually exclusive projects, the one having the highest positive net future value (over the same planning horizon) will be the best.

As an example of the above procedure, let us carry out the net future value analysis for the cash flows shown in the preceding table, for a BORR of 35 percent and for a MARR of 32 percent (see Table 10). Since the net future value at the end of three years is negative (or -94.30), project X will be unacceptable and thus doing nothing will be best.

In summary, a simple NPV analysis with a discount rate of the MARR will always suffice so long as the BORR is equal to or less than the MARR. When the BORR is larger than the MARR, a simple NPV analysis with a discount rate of MARR will also lead to the correct economic choice if the firm can use its own assets to acquire the necessary capital. When the required capital can only be acquired from outside sources, and the firm can be a lender as well as investor, a simple NPV analysis with a discount rate of BORR will lead to correct economic choices. Finally, a more complicated type of

Table 10. Net future value analysis.

End of Year t	Cash Flow in Year t($)	Prior-Year Net Future Value Compounded to Year t ($)	Net Future Value at End of Year t ($)
0	-1000	-	-1000, compound at 35 percent
1	+3700	-1000(1.35) = -1350	+2350, compound at 32 percent
2	-4520	+2350(1.32) = +3102	-1418, compound at 35 percent
3	+1820	-1418(1.35) = -1914.3	-94.30

analysis (of the sort alluded to by Bergmann and described in the previous two paragraphs) must be used only when a firm cannot be a lender, cannot acquire capital except from outside sources, and has a borrowing rate greater than its MARR.

Bergmann also maintains that any benefits received by the public are consumed or reinvested at the social rate of discount. Many other economists argue differently and with persuasion. Not only is this a murky subject, but it also is one that is fraught with difficulty when we attempt to determine the appropriate social rate of discount. In sum, there is only an arbitrary basis for deciding on the propriety of its use as well as its value. [See Margolis's (18) review of this subject.]

Two final comments: One, Bergman is wrong in saying "...Wohl would have us never calculate a rate of return on an investment proposal." To the contrary, and as noted in my paper, if a rate-of-return figure is necessary (say, because of budget constraints) before the fact, then an effective rate of return should be calculated rather than the internal rate of return. Moreover, it is perfectly obvious that we need the actual effective yield that is being obtained from other ongoing (or past) projects. Two, Bergmann is clearly wrong in saying that: "Use of the internal-rate-of-return method does not imply, as Wohl and others suggest it does, that the positive cash flows from an investment alternative are reinvested at the internal rate of return." On the contrary, if the internal rates of return are used as the sole guide to make economic choices among mutually exclusive choices, then certainly he is wrong. Assume for the Table 7 example that the MARR is 20 percent (and equal to the borrowing rate). Then, what else other than an assumed reinvestment of the $10 000 in year 1 at 20 percent could have caused the two alternatives to be exactly equivalent for the two-year period? In a word, nothing.

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Potential of Pricing Solutions for Urban Transportation Problems: An Empirical Assessment

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This paper surveys the available empirical evidence on the elasticity of travel demand to assess the potential of pricing policies to alter travel behavior and thereby to solve various urban transportation problems. The first set of studies considers the responsiveness of fuel use to changes in gasoline price. The second set, econometric models of urban travel demand, estimates the direct and cross-price elasticities of the use of different modes with respect to different components of trip cost. The third set of evidence is composed of arc elasticity estimates based on the impacts on travel behavior of actual changes in the levels of roadway user charges and transit fares. For each study dealt with, the paper briefly summarizes its methodology, data base, and findings and subjects these to critical evaluation. The paper concludes with an evaluation of the body of results for the usefulness of pricing policies in urban transportation.

Economists have criticized perverse pricing as the crux of the urban transportation problem and, thus, have regarded corrective pricing policy as the key to the solution. The theoretical basis for such alternative pricing involves the need to internalize the often significant external social costs associated with urban travel (such as congestion, air pollution, and noise), particularly as these vary with respect to time of day, route, and mode of travel.

The objective of this paper is to determine how responsive urban travel behavior would be to various corrective pricing strategies:

1. To what extent, for example, could peak-period pricing alleviate congestion by diverting automobile drivers to other modes or to use of their automobiles at less-congested times or on less-congested routes?

2. To what degree might higher gasoline prices encourage motorists to economize on fuel by driving fewer kilometers or by purchasing smaller, more fuel-efficient automobiles? and