10. Decisions should be readily obtainable--management of owner and engineer should not be multilayered, no procrastination; and

11. Commitment by the utilities to cooperate with contractor.

CONCLUSION

There is no magic formula for success to a construction project. Common sense should govern all decisions. Among the secrets to improved contractual relationships and quality construction are creating a climate of respect and goodwill among the owner, engineer, and contractor; a willingness to adjust specifications to simplify construction while holding fast to end results; a willingness of the owner to assume risk for unforeseen conditions encountered and not let everything fall on the contractor's head.

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Analysis and Application of Correlated Compound Probabilities

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Many statistical applications require the calculation of compound probabilities and, frequently, the individual probabilities are not independent. The failure to recognize that correlation exists in cases such as these has resulted in numerous errors in the published literature. Although an exact analytical solution is not known, problems of this type can often be handled effectively by calculating lower and upper bounds for the desired compound probabilities. Bounds for both positively and negatively correlated cases are derived and then applied in the analysis of statistical acceptance procedures. The results of several computer simulation tests are presented to demonstrate the validity of the theoretically derived results.

The analysis of a variety of statistical acceptance procedures requires the calculation of compound probabilities. In many cases, the individual probabilities are correlated to some unknown degree, which precludes the direct calculation of the desired compound probability. However, lower and upper bounds for the desired probability can be calculated and, provided these bounds are not too far apart, this furnishes an interval estimate that is sufficiently precise for most practical purposes.

A previous paper (1) developed this approach for the case in which the individual probabilities are positively correlated. This paper repeats the derivation for positively correlated probabilities, develops the derivation for negatively correlated probabilities, applies these results to a simple sequential sampling scheme, and then derives the bounds for the probability of acceptance under a more complex acceptance procedure. This latter application is then checked by computer simulation.

BOUNDS FOR POSITIVELY CORRELATED PROBABILITIES

In accordance with a law of probability that is usually referred to as the general law of multiplication (2), the compound probability for the joint occurrence of event A and event B is given by Equation 1. Under this law, no assumption is made concerning the independence of these events, and they may be either positively or negatively correlated.

$$P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A) \quad (1)$$

When events A and B are correlated to some unknown degree, the values of $P(A|B)$ and $P(B|A)$ are not known and, consequently, $P(A \cap B)$ cannot be evaluated directly. However, when two events are positively correlated, the occurrence of one increases the likelihood of the occurrence of the other. This can be expressed in equation form as

$$P(A|B) \geq P(A) \quad (2)$$

which, when substituted into Equation 1, yields

$$P(A \cap B) \geq P(A) \cdot P(B) \quad (3)$$

as the lower bound for $P(A \cap B)$.

To obtain the upper bound, remember that any probability value is less than or equal to unity. Therefore, since $P(A|B)$ and $P(B|A)$ in Equation 1 both must be less than or equal to one,

$$P(A \cap B) \leq P(A) \quad (4)$$

$$P(A \cap B) \leq P(B) \quad (5)$$

and, from this,

$$P(A \cap B) \leq \mathrm{Min}[P(A), P(B)] \quad (6)$$

is derived as the upper bound.

Later on, it will be more convenient to designate these events numerically because the letters A and B will be used to refer to the first and second stages of a sequential sampling scheme. Equation 7 expresses both lower and upper bounds for the positively correlated case in this manner.

$$P_1P_2 \leq P_{pos} \leq \mathrm{Min}(P_1, P_2) \qquad (7)$$

where

P_1 = probability of occurrence of first event,
P_2 = probability of occurrence of second event, and
P_{pos} = probability of the joint occurrence of the two positively correlated events.

BOUNDS FOR NEGATIVELY CORRELATED PROBABILITIES

For the negatively correlated case, another law of probability, the general law of addition (2), is useful. This is given in its basic form by Equation 8 and in a transposed form by Equation 9.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad (8)$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \qquad (9)$$

Because the maximum value for the term $P(A \cup B)$ in Equation 9 is unity, this leads to the expression for the lower bound given by Equation 10. The maximum operator is required to ensure that the expression will not produce a value less than zero.

$$P(A \cap B) \geq \mathrm{Max}[0, P(A) + P(B) - 1] \qquad (10)$$

When events A and B are negatively correlated, the occurrence of one decreases the likelihood of the occurrence of the other. This can be expressed in equation form as

$$P(A \mid B) \leq P(A) \qquad (11)$$

which, when substituted into Equation 1, yields

$$P(A \cap B) \leq P(A) \cdot P(B) \qquad (12)$$

as the upper bound for $P(A \cap B)$.

As with the positively correlated case, it will be convenient to designate the events numerically. Equation 13 expresses both lower and upper bounds for the negatively correlated case in this manner:

$$\mathrm{Max}(0, P_1 + P_2 - 1) \leq P_{neg} \leq P_1P_2 \qquad (13)$$

where

P_1 = probability of occurrence of first event,
P_2 = probability of occurrence of second event, and
P_{neg} = probability of the joint occurrence of the two negatively correlated events.

ANALYSIS OF A SEQUENTIAL ACCEPTANCE PROCEDURE

A fairly common sequential acceptance procedure requires that, when the tests on the initial sample indicate a deficiency, a second sample be taken and combined with the first to make the final assessment of compliance. Because the first sample is incorporated into the final sample, the probabilities of acceptance by the two stages of this procedure are not independent. For example, when a group of unusually low test values decreases the probability of acceptance by the first stage of the procedure, the presence of these same low values tends to reduce the probability of acceptance by the second stage of the procedure. Consequently, the two probabilities are positively correlated to some degree.

Although it might seem that this would be a likely application of the bounds for positively correlated probabilities, it will be seen that the bounds for negatively correlated probabilities are appropriate in this case. If A and B are defined as the events of passing the first and second stages of the sequential acceptance procedure, respectively, and if A' represents the complement of A (i.e., the event of not passing the first stage of the acceptance procedure), the probability of acceptance at either stage A or stage B is given by Equation 14.

$$P(\mathrm{Accept}) = P(A) + P(B \mid A') \cdot P(A') \qquad (14)$$

Because events A and B are not independent, the value of $P(B \mid A')$ is not known and the second term in the equation for P(Accept) cannot be evaluated directly. However, lower and upper bounds for this term can be calculated that, when added to P(A), will determine lower and upper bounds for P(Accept). To accomplish this, observe that, because A and B are positively correlated events, A' and B are negatively correlated. Therefore, by incorporating the bounds for negatively correlated probabilities from Equation 13 into Equation 14, Equation 15 can be derived.

$$P(A) + \mathrm{Max}[0, P(A') + P(B) - 1] \leq P(\mathrm{Accept}) \leq P(A) + P(B) \cdot P(A') \qquad (15)$$

Then, by noting that

$$P(A') = 1 - P(A) \qquad (16)$$

and by using slightly more convenient nomenclature, Equation 17 can be derived to give the bounds for the probability of acceptance under this type of sequential acceptance procedure.

$$\mathrm{Max}(P_A, P_B) \leq P \leq P_A + (1 - P_A)P_B \qquad (17)$$

where

P_A = probability of acceptance at stage A,
P_B = probability of acceptance at stage B, and
P = probability of acceptance at either stage A or stage B.

ANALYSIS OF A COMPLEX ACCEPTANCE PROCEDURE

The acceptance procedure used in this example is described in a recent National Cooperative Highway Research Program (NCHRP) report (3, p. 29) and was selected because it is sufficiently complex to require all of the theory developed in this paper. It is part of a specification for pavement thickness that uses both dual requirements and a sequential acceptance provision. The pavement thickness is considered acceptable if, based on five randomly selected cores, both of the following conditions are met:

1. The average length of the cores is equal to or greater than the specified thickness and
2. No more than 20 percent of the pavement (as estimated from the sample) has a thickness less than 85 percent of the specified thickness.

If either of these conditions is not met, an additional 10 randomly located cores are taken and combined with the original five cores. Then, in order for the pavement thickness to be judged acceptable, the same dual requirements must both be satisfied.

Although this may be an effective specification from the standpoint of accepting good workmanship and rejecting bad workmanship, it is somewhat of an analyst's nightmare to determine the operating characteristic (OC) curves for a plan of this type. Previously, because the various steps of this procedure are correlated in a complex manner, the only practical way to develop these curves would have been by computer simulation. However, by applying a combination of the techniques derived in this paper, it is possible to develop bounds for the probability of acceptance under this specification. Depending on the actual values used in the calculations, these bounds may be sufficiently close together to be of practical use. If not, it will be necessary to resort to computer simulation.

Since this is a sequential acceptance procedure, the bounds given by Equation 17 will be appropriate except that a further development is required. Whereas the probabilities P_A and P_B are fixed and known in Equation 17, they are not known in the case of the pavement thickness specification. In this case, P_A and P_B represent the probabilities of passing the dual requirements of stages A and B, respectively, of the sequential procedure. Since the dual requirements are always applied to the same sample, the probabilities of passing the dual requirements separately are positively correlated. Thus, the values P_A and P_B cannot be calculated directly but must themselves be defined by bounds in accordance with Equation 7.

What remains is to find a new minimum and a new maximum for the left and right sides, respectively, of Equation 17, both in terms of the bounds on P_A and P_B. The expression on the left will have its lowest value when P_A and P_B are at their minimums. When this expression is set equal to a new variable L and the appropriate bounds from Equation 7 are applied, the new lower bound is given by Equation 18.

$$L \geqslant \mathrm{Max}(P_{1A}P_{2A}, P_{1B}P_{2B}) \tag{18}$$

where

- L = the value of the left side of Equation 17 when applied to the pavement thickness specification;
- P_{1A}, P_{2A} = the probabilities of passing the first and second requirements, respectively, of stage A of the pavement thickness specification; and
- P_{1B}, P_{2B} = the probabilities of passing the first and second requirements, respectively, of stage B of the pavement thickness specification.

In order to determine the maximum of the expression on the right side of Equation 17, it will be convenient to set this expression equal to a new variable R and then recombine terms as shown in Equation 19:

$$R = P_A + (1 - P_A)P_B = P_B + (1 - P_B)P_A \tag{19}$$

Although it is possible to derive the maximum of R more formally by using calculus, it can be deduced quite readily by inspection of Equation 19. In the first arrangement of terms, R is at its maximum for any given value of P_A when P_B is at its maximum. Similarly, in the second arrangement of terms, R is at its maximum value for any given value of P_B when P_A is at its maximum. Therefore, R is at its maximum when P_A and P_B are both at their maximum values. By substituting the upper bounds for P_A and P_B given by Equation 7 into Equation 19, the upper bound for this expression becomes

$$R \leqslant \mathrm{Min}(P_{1A}, P_{2A}) + [1 - \mathrm{Min}(P_{1A}, P_{2A})] \cdot \mathrm{Min}(P_{1B}, P_{2B}) \tag{20}$$

Finally, Equations 18 and 20 are combined and rearranged slightly to yield

$$\mathrm{Max}(P_{1A}P_{2A}, P_{1B}P_{2B}) \leqslant P \leqslant \mathrm{Min}(P_{1A}, P_{2A}) + \mathrm{Min}(P_{1B}, P_{2B}) - \mathrm{Min}(P_{1A}, P_{2A}) \cdot \mathrm{Min}(P_{1B}, P_{2B}) \tag{21}$$

where P is the overall probability of acceptance under the pavement thickness specification. Note that Equation 21 was developed in a general way and is appliable for any acceptance procedure of the same form as the pavement thickness specification.

COMPUTER SIMULATION TESTS

In order to check the theoretical bounds given by Equation 21, several tests were made by using computer simulation. To simplify the presentation, the specific thickness units have not been identified. In this example, the standard deviation is 2.5 percent of the specified thickness, which is typical for concrete pavement. Each simulation result is the average of a minimum of 2000 replications of the sampling procedure. The theoretical bounds are calculated by using conventional normal distribution theory for the first of the dual requirements and the noncentral-t distribution (4,5) for the second requirement. [The first reference on noncentral-t is more instructive; the second provides more complete tables. As an alternate method, slightly less precise results can be obtained by interpolating between the OC curves of Military Standard 414 (6).] The results of these tests are listed in Table 1 and plotted in Figure 1. Note that, in every case, the simulation results fall within the theoretically predicted bounds.

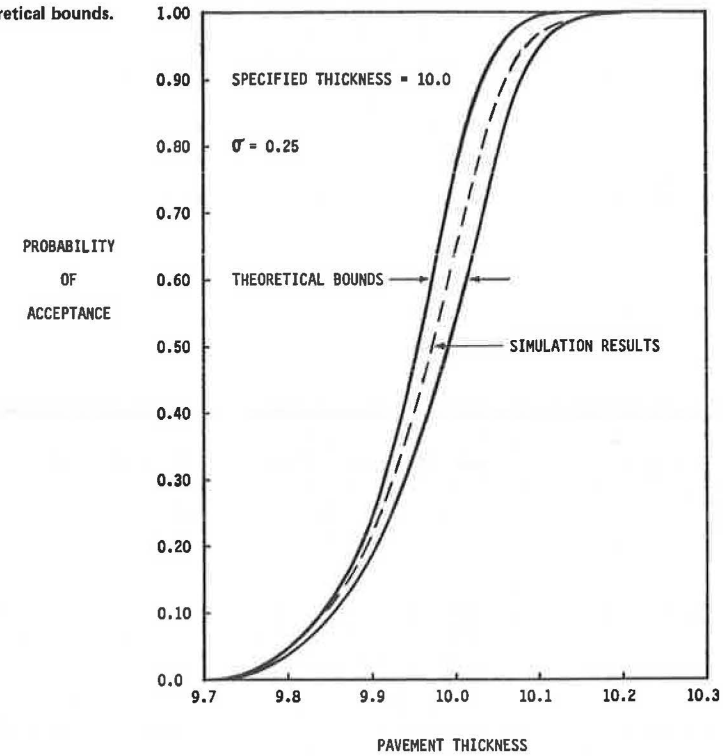
Although the interval estimates for the probability of acceptance provided by the theoretical bounds are not precise for some values of pavement thickness, they may still be useful, particularly if computer simulation is not readily available to provide better estimates. For example, the upper OC curve in Figure 1 indicates that, if the mean of the pavement thickness population is 9.85, the maximum probability of acceptance is 10.0 percent. This may be sufficient to convince the developers of this acceptance procedure that it will provide ample protection against accepting pavement that is deficient in thickness. Similarly, if a minimum probability of acceptance of 95.0 percent is desired, the lower OC curve indicates that a population mean of at least 10.11 must be obtained. This information would be extremely helpful to a contractor during both the bidding and construction stages of a project governed by this acceptance procedure.

Table 1. Computer simulation tests.

Population Mean	Estimated Probability of Acceptance	95.0% Confidence Interval	Theoretical Bounds for Probability of Acceptance
9.7	0.00	0.00-0.00	0.00-0.00
9.8	0.04	0.03-0.04	0.04-0.04
9.9	0.22	0.20-0.24	0.19-0.24
10.0	0.63	0.60-0.65	0.50-0.75
10.1	0.96	0.95-0.97	0.94-0.99
10.2	1.00	1.00-1.00	1.00-1.00

Note: Specified thickness = 10.0; σ = 0.25.

Figure 1. Comparison of simulation results with theoretical bounds.



SUMMARY AND CLOSING REMARKS

Many errors in the literature have resulted from the failure to recognize the existence of correlation in a variety of applications of compound probabilities. Although exact analytical solutions are not known, lower and upper bounds for the desired compound probabilities can be readily calculated. Bounds for both positively and negatively correlated cases were derived and applied to a complex acceptance procedure. Although the interval estimates provided by these bounds were not always precise, they can still be of considerable practical value, both to the specification writer in developing the acceptance procedure and to the contractor in determining the appropriate target value to meet it. Finally, several tests were made by computer simulation, all of which produced results that fell within the theoretically predicted bounds.

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