soil-property variability while the statistical analysis quantifies it.

CONCLUSIONS

On the basis of results obtained in this study, the following conclusions can be drawn:

1. The statistical approach is useful in systematically organizing data.
2. In the sorting of data, histograms that exhibit more than one peak (multimodal) can indicate whether one or more populations are present.
3. Shear-strength characteristics exhibited the most variability. This is in agreement with other research.
4. Low coefficients of variation for the classification parameters may indicate whether one is dealing with the same soil type.
5. The beta distribution was found to model most soil properties investigated in this paper. In fact, due to its versatility, it could be expected to model most soil properties.
6. Whenever large amounts of data are available for a particular soil unit, a statistical treatment may provide better insights into the interrelations of the various soil properties and help the engineer to reduce the amount of judgment necessary in the selection of design parameters.
7. It is important to note that the statistical results presented in this paper apply only to clay material. If one were dealing with material similar in geologic origin and stress history, the results presented here could be of value.

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Use of Point Estimates for Probability Moments in Geotechnical Engineering

V. McGUFFEY, J. IORI, Z. KYFOR, AND D. ATHANASIOU-GRIVAS

In probabilistic geotechnical engineering, it is often necessary to obtain estimates of the mean and standard deviation of a function of one or more random variables. For this purpose, Rosenblueth first proposed the method of using point estimates for approximating probability moments. This method is advantageous in that it requires neither extensive computer capabilities nor complex mathematical derivations. The point-estimate method is described and compared with existing methods, and its usefulness is illustrated with examples of its application to common geotechnical functions.

Analytic expressions are available and can be used to evaluate the statistical values (mean, variance, higher moments) of soil properties with random variation, such as plasticity index, compression ratio, and undrained shear strength. Moreover, of equal importance in geotechnical practice is the determination of the statistical values of functions of soil properties.

As an example, consider the commonly used settlement equation expressed in the following form:

$$S = H \times CR \times \log(P_f/P_o) \tag{1}$$

where

S = total settlement within a soil layer,
H = thickness of the layer,
CR = compression ratio of the layer,
P_o = initial vertical stress within the soil layer, and
P_f = final vertical stress within the soil layer.

CR is a random variable. S is a function of this random variable, and therefore its statistical values depend on those of CR.

If the mean and standard deviation of S are known, one can infer something about the probability with which S receives values within certain limits. For example, by assuming a model for the probability density function (pdf) of S (e.g., normal), one can find (a) the probability that S lies below a certain value (S_1) by integrating under the pdf from $-\infty$ to S_1, (b) the probability that S lies above a certain value (S_2) by integrating under the pdf from S_2 to $+\infty$, and (c) the probability that S lies within a certain range (S_1 to S_2) by integrating the pdf from S_1 to S_2.

Approximate solutions to the problem of determining the statistical values of functions of random variables are usually possible (1). Currently, two approximate methods are widely used to do this: Monte Carlo simulation and Taylor series approximation. These two methods are briefly described in this paper. Since there are disadvantages to both of these methods, a third method, first proposed by Rosenblueth (2), is also described. It is shown that Rosenblueth's method overcomes the disadvantages of the other two methods. Its use is also illustrated with examples.

APPROXIMATE METHODS

Generally, if $Y = f(X_1, X_2, X_3, \ldots, X_n)$ and each of the variables X_i has a known mean and standard deviation, we wish to determine the mean and standard deviation of Y. For example, given Equation 1 and the means and standard deviation of CR, we wish to determine the mean and standard deviation of S.

Monte Carlo Simulation

The Monte Carlo simulation method determines the mean and standard deviation of a function of random variables by performing repeated computations by use of randomly selected point estimates for the component variables. The method is outlined as follows:

1. Select a random value for each of the component variables. The random values are selected to conform with the assumed distribution of each variable.
2. Using the randomly selected values of the component variables, compute the function.
3. Repeat steps 1 and 2 a large number of times. The number of times depends on the variability of the input and output parameters and the desired degree of accuracy.
4. Compute the mean and standard deviation of the function by using the data obtained from the simulation.

The Monte Carlo simulation requires a high-speed computer so that a large number of trials can be conducted. Furthermore, computer programs should exist that automatically make the necessary repetitions and accumulate values. Such programs, especially for very complicated functions, are very difficult to create.

Taylor Series Expansion

Given a function $Y = f(X_1, X_2, X_3, \ldots, X_n)$, where each variable X_i is an independent random variable with known mean $\bar{X}_i$ and variance Var (X_i), expressions can be derived for the mean and variance of Y:

$$\bar{Y} = f(\bar{X}_1, \bar{X}_2, \bar{X}_3, \ldots \bar{X}_n) + 1/2 \sum_{i=1}^{n} (\partial 2y/\partial X_i^2)\ [\mathrm{Var}(X_i)] \quad (2)$$

$$\mathrm{Var}(Y) = \sum_{i=1}^{n} (\partial Y/\partial X_i)^2\ [\mathrm{Var}(X_i)] \quad (3)$$

The problem with this method is that partial differentiation, performed on even simple functions, may result in complex expressions. Furthermore, differentials of complex functions may not even exist.

Point-Estimate Method

A typical pdf of a random variable X is shown in Figure 1. Usually, only the first two or three moments of the distribution can be estimated accurately. Following a method first proposed by Rosenblueth (2), the pdf of X can be approximated by a two-point probability mass function. The mass function consists of concentrations P_+ and P_- at X_+ and X_-, respectively (see Figure 1). This is analogous to representing a distributed load on a beam by a force (or forces) acting through a point (or points).

If Y(X) is a function of X, a two-point approximation for the pdf of Y is obtained by evaluating the function Y(X) at X_+ and X_-:

$$Y_+ = Y(X_+) \quad (4)$$

$$Y_- = Y(X_-) \quad (5)$$

The concentration at Y_+ is the same as it is at X_+ (i.e., P_+). Similarly, the concentration at Y_- is P_-. This is shown schematically in Figure 2. The two-point approximation for the pdf of Y can then be used to determine the first two moments of Y:

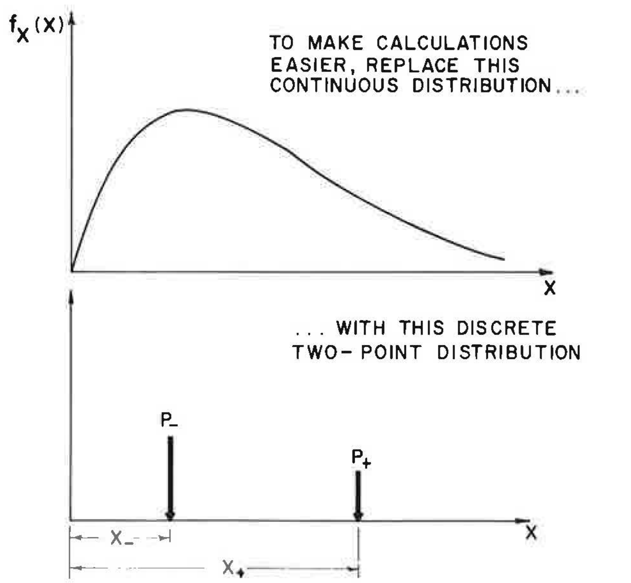
$$E(Y) = P_+Y_+ + P_-Y_- \quad (6)$$

$$E(Y^2) = P_+Y_+^2 + P_-Y_-^2 \quad (7)$$

$$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 \quad (8)$$

The total mass of the two-point approximation must be equal to that of the actual density function (i.e., must be equal to one). Also, the first three

Figure 1. Typical probability density function and two-point probability mass function of X.



moments of the two-point approximation must be equal to those of the actual density function. Thus, P_+, P_-, X_+, and X_- must satisfy the following simultaneous equations:

$$P_+ + P_- = 1 \quad (9)$$

$$P_+X_+ + P_+X_- = \bar{X} \quad (10)$$

$$P_+(X_+ - \bar{X})^2 + P_-(X_- - \bar{X})^2 = \sigma_x^2 \quad (11)$$

$$P_+(X_+ - \bar{X})^3 + P_-(X_- - \bar{X})^3 = \nu_x\sigma_x^3 \quad (12)$$

where

$\bar{X}$ = expected value of X,
σ_x = standard deviation of X, and
ν_x = skewness of X.

If X is symmetrically distributed, then ν_x = 0 and the above expressions yield

$$P_+ = P_- = 1/2 \quad (13)$$

$$X_+ = \bar{X} + \sigma_x \quad (14)$$

$$X_- = \bar{X} - \sigma_x \quad (15)$$

That is, the distribution of X is approximated by a two-point mass function, the mass of which is concentrated equally at one standard deviation above and below the mean value (see Figure 3).

If Y is a function of two random variables (X_1, X_2), then the joint distribution of the two random variables can be approximated with a four-point mass function (see Figure 4). In general, if Y is a function of N random variables, then 2^N points are needed to approximate the multivariate mass function. Hence, the weighing factor (P) for the case of uncorrelated random variables with symmetrical distributions is equal to $(1/2)^N$.

The determination of the first two moments of Y, when Y is a function of two uncorrelated, symmetrically distributed random variables, is given as follows:

$$Y_{++} = Y(X_{1+}, X_{2+}) \quad (16)$$

$$Y_{+-} = Y(X_{1+}, X_{2-}) \quad (17)$$

$$Y_{-+} = Y(X_{1-}, X_{2+}) \quad (18)$$

$$Y_{--} = Y(X_{1-}, X_{2-}) \quad (19)$$

where Y_{++} is the function y evaluated at (X_{1+}, X_{2+}), etc.:

$$E(Y) = 1/4(Y_{++} + Y_{+-} + Y_{-+}\ Y_{--}) \quad (20)$$

$$E(Y^2) = 1/4(Y_{++}^2 + Y_{+-}^2 + Y_{-+}^2 + Y_{--}^2) \quad (21)$$

$$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 \quad (22)$$

EXAMPLES

Settlement

A surcharge is to be placed on an area that is underlain by a 10-ft-thick layer of clay (see Figure 5). The initial vertical stress within the clay layer (P_o) is 300 lbf/ft², and the final vertical stress (P_f) will be 400 lbf/ft². CR is a random variable with a mean value of 0.20 and a standard deviation of 0.05. By using Rosenblueth's method, find the mean and standard deviation of S.

$S = H \times CR \times \log P_f/P_o = 1.25$ CR.
$CR_+ = 0.25$.
$CR_- = 0.15$.
$S_+ = 1.25\ CR_+ = 0.313$ ft.
$S_- = 1.25\ CR_- = 0.188$ ft.

$\bar{S} = E(S) = 1/2\ (S_+ + S_-) = 0.250$ ft.
$E(S^2) = 1/2\ (S_+^2 + S_-^2 = 0.067$.
$\mathrm{Var}(S) = E(S^2) - E(S)^2 = 0.004$.
$\sigma_S = 0.063$ ft.

Assuming that S is a normally distributed random variable, find the probability that the settlement will be (a) less than 0.10 ft, (b) more than 0.30 ft, and (c) between 0.10 and 0.30 ft.

The pdf of S is shown in Figure 6. The probabilities of the above three conditions are equal to the indicated area S: $P(S < 0.10) = 0.0087$, $P(S > 0.30) = 0.2148$, and $P(0.10 < S < 0.30) = 0.7765$.

Since settlement is a linear function of CR, Rosenblueth's method was actually not even required. However, Rosenblueth's method is valuable when one is dealing with complex nonlinear functions or functions of several random variables.

Bearing Capacity Factor

Lambe and Whitman (3) define the bearing capacity factor (N_γ) of a soil as a function of the friction angle ϕ.

Figure 2. Two-point probability mass function of Y.

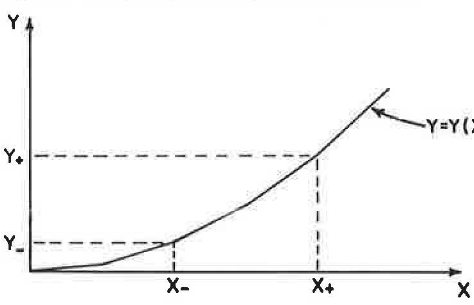


Figure 3. Symmetrical two-point mass function.

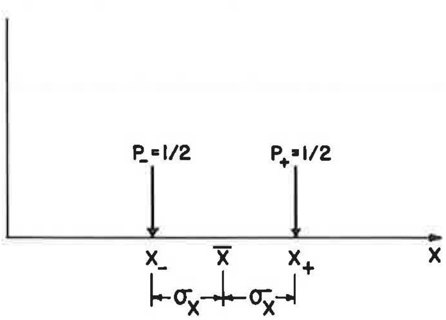
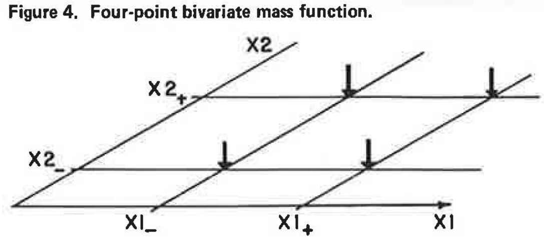


Figure 4. Four-point bivariate mass function.



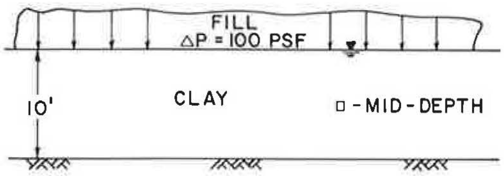


Figure 5. Settlement problem.

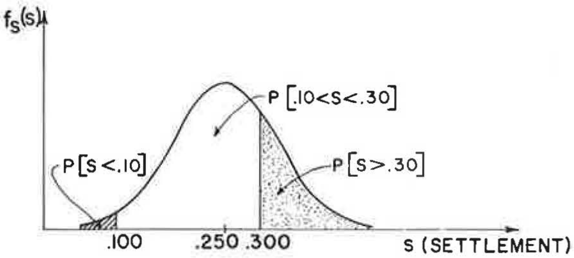


Figure 6. Probability density function of S.

$$N_\gamma = 1/2 \left\{[(1 + \sin\phi)/(1 - \sin\phi)]^{5/2} - [(1 + \sin\phi)/(1 - \sin\phi)]^{1/2}\right\} \quad (23)$$

The friction angle ϕ is a random variable with a mean value of 30° and a standard deviation of 3°. By using Rosenblueth's method, find the mean and standard deviation of N_γ.

$\phi_+ = 33°$.
$\phi_- = 27°$.
$N_{\gamma+} = 9.68$.
$N_{\gamma-} = 4.97$.

$\bar{N}_\gamma = E(N_\gamma) = 1/2\ (N_{\gamma+} = N_{\gamma-} = 7.33$.
$E(N_\gamma^2) = 1/2\ (N_{\gamma+}^2 + N_{\gamma-}^2) = 59.20$.
$Var(N_\gamma) = E(N_\gamma^2) - E(N_\gamma)^2 = 5.47$.
$\sigma_{N_\gamma} = 2.34$.

Shear Strength (Mohr's Theory)

Estimate the mean and variance of the strength (τ) of a soil in accordance with Mohr's theory:

$$\tau = \sigma \tan\phi + c \quad (24)$$

where σ, $\tan\phi$, and c are uncorrelated random variables with the following characteristics:

Variable	Mean	SD	+	−
σ (lbf/in²)	20	1.41	21.41	18.59
tan ϕ	0.55	0.24	0.79	0.31
c (lbf/in²)	7.94	1.47	9.41	6.47

Since τ is a function of three random variables, τ must be evaluated eight times:

$\tau_{+++} = 26.32$
$\tau_{++-} = 23.38$
$\tau_{+-+} = 16.05$
$\tau_{+--} = 13.11$
$\tau_{-++} = 24.10$
$\tau_{-+-} = 21.16$
$\tau_{--+} = 15.17$
$\tau_{---} = 12.23$

$\bar{\tau} = E(\tau) = 1/8\ \Sigma\tau = 18.94$ lbf/in².
$E(\tau^2) = 1/8\ \Sigma\tau^2 = 384.64$.
$Var(\tau) = E(\tau^2) - E(\tau) = 25.92$.
$\sigma_\tau = 5.09$ lbf/in².

This problem was solved by Harr (4) by using Taylor series approximation. The results are the same.

Factor of Safety (Infinite Slope)

One geotechnical function that is not often considered random is the factor of safety (FS). However, one must agree that, because the FS is a function of random variables, it is itself a random variable.

The FS of an infinite slope (see Figure 7) can be expressed as

$$FS = \tan\phi/\tan\beta \quad (25)$$

where ϕ is the friction angle of the slope soil and β is the slope angle.

The friction angle ϕ is a random variable with a mean value of 30° and a standard deviation of 3°. By using Rosenblueth's method, find the mean and standard deviation of FS.

$\phi_+ = 33°$.
$\phi_- = 27°$.
$FS_+ = 1.393$.
$FS_- = 1.093$.

$\overline{FS} = E(FS) = 1/2\ (FS_+ + FS_-) = 1.243$.
$E(FS^2) = 1/2\ (FS_+^2 + FS_-^2) = 1.568$.
$Var(FS) = E(FS^2) - E(FS)^2 = 0.023$.
$\sigma_{FS} = 0.151$.

Assuming the FS is a normally distributed random variable, find the probability that FS will be less than one.

The pdf of the FS is shown in Figure 8. The probability that the FS is less than one is indicated by the shaded region: $P(FS < 1) = 0.0537$.

To say that the FS is less than one is to say that the slope will fail. Therefore, the probability that the FS is less than one is the probability of failure (P_f). The P_f has been proposed as an alternative to the FS as a measure of safety.

CONCLUSIONS

The method of point estimates for probability moments has been presented and compared with the Monte Carlo simulation technique and the Taylor series approximation. Unlike the Monte Carlo simulation, the point-estimate method requires no extensive computer capabilities. Unlike the Taylor series

Figure 7. Infinite slope problem.

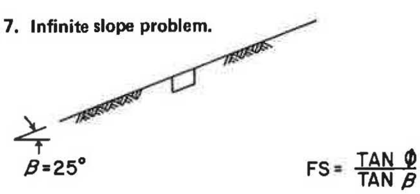
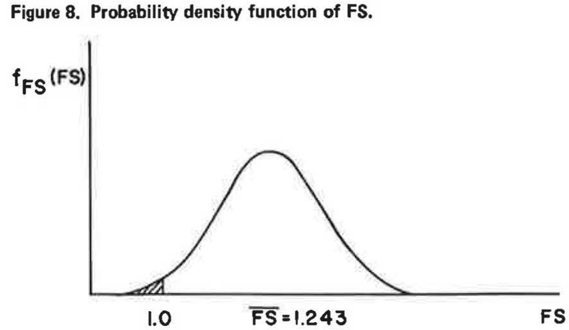


Figure 8. Probability density function of FS.



approximation, the point-estimate method requires no complex derivations. Yet the point-estimate method is as accurate as the Taylor series approximation.

The normal distribution is not the only type of distribution that can be assumed. Since most geotechnical properties can never take on negative numbers, the lognormal distribution may be a more appropriate model. Another suggestion is to use a symmetrical beta distribution, which is bounded by zero and twice the mean.

In this paper, it was assumed that input variables were symmetrically distributed and, in the case of two or more variables, uncorrelated. However, Rosenblueth's method is not limited by these conditions.

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Short-Term Reliability of Slopes Under Static and Seismic Conditions

AKIRA ASAOKA AND DIMITRI ATHANASIOU-GRIVAS

A simplified probabilistic approach to the determination of the short-term ("$\phi_u = 0$") reliability of clayey slopes under static and seismic conditions is presented. The uncertainties associated with (a) the undrained strength of soil and its spatial variation and (b) the analytic procedure used to assess the safety of the slope are considered, and probabilistic tools are introduced for their description and amelioration. The probability of the failure of a slope under static loading is first determined. The effect of an earthquake on the slope is introduced through an equivalent horizontal peak acceleration (deterministic), and the new probability of failure is obtained by using Bayes' theorem. Finally, the developed procedure is illustrated in an example, the results of which are presented and discussed.

The factor of safety F_s of a slope of cohesive soil under undrained ("$\phi_u = 0$") conditions, determined from equilibrium of moments around the center of a circular failure surface (see Figure 1), is given as

$$F_s = R \int_L c_u \, dL / aW \tag{1}$$

where

R = radius of the circular failure surface,
L = length of the failure surface,
c_u = undrained shear strength of soil,
a = distance between W and the center of the circle, and
W = weight of the sliding soil mass.

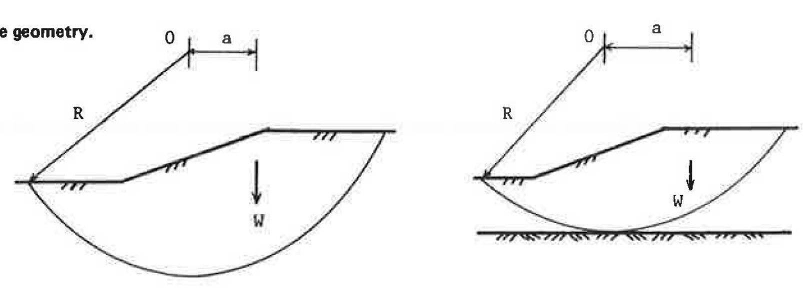
From Equation 1, it is seen that the total undrained shear strength of the slope is obtained by integrating c_u along the length L of the failure surface. If the soil medium is homogeneous and isotropic, then c_u is constant throughout the medium and the total resistance is equal to $c_u L$. In this case, the critical failure surface (i.e., the slip surface for which F_s becomes minimum) can be determined analytically. Thus, by expressing the equilibrium of moments around center O (Figure 1) as $aW = RL\tau$, or

$$aW = RL(N\gamma H) \tag{2}$$

where

τ = mean shear stress along the slip surface,
γ = unit weight of soil,
N = stability number (<u>1</u>), and
H = height of the slope,

Figure 1. Slope geometry.



(a) Infinite Depth Case (b) Finite Depth Case