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Reliability of Soil Slopes

L. ALFARO AND M.E. HARR

Results of a study of the safety of soil slopes are reported in which the measure of safety used is "reliability" (or the "probability of failure"), an a priori quantitative estimate of the likelihood of the safety (or failure) of a slope. A closed-form solution to determine slope reliability is proposed in which a material with two resistance parameters (c and tan ϕ) is accommodated. Input to the model consists of a bivariate distribution of c and tan ϕ for the slope material and a line called the "critical boundary", which is independent of the operative strength parameters. This line is the locus of points in the c tan ϕ plane for which the slope in question is in a state of limiting equilibrium (factor of safety equal to unity). Beta distributions are assumed to model the variability of c and tan ϕ. The critical boundary is determined from two-dimensional and three-dimensional slope-stability analyses. For the former, the ordinary method of slices is adopted because of its simplicity (it requires no iterations) and because it is the only method that does not make the unrealistic assumption that the factor of safety takes the same value along the entire slip surface, thus permitting the analysis to yield some information regarding the failure process. For the three-dimensional analysis, Hovland's method is used. In concept, it is the three-dimensional equivalent of the ordinary method of slices. Output from the model is the probability of failure of the slope, which is information dependent and therefore can vary as new information is obtained. These probabilities can then be used to place the problem in the framework of decision theory.

Current procedures for evaluating the safety of slopes consist in determining a factor of safety (1-4) that is compared with allowable values found to be satisfactory on the basis of previous experience. The factor of safety suffers from the following:

1. Elements of uncertainty in analyses are not quantified when the factor of safety is used.
2. The scale of the factor of safety (F) is not known. For example, a structure with a factor of safety of 3.0 is not necessarily twice as safe as another with a factor of safety of 1.5.
3. Allowable values to be selected for the factor of safety are the result of experience. In dealing with new or different problems for which there is no previous experience, there is no allowable factor of safety.

To overcome these difficulties and permit the engineer to predict the performance of his or her designs, the concept of "reliability" or "probability of failure" is recommended (5-7).

Probability itself is a subjective interpretation. According to the definition of Tribus (8), "A probability assignment is a numerical encoding of a state of knowledge." A probability is understood to be an information-dependent quantity that may not be intrinsically related to the physical world. That is, the estimate of the reliability of a structure may change as new information regarding it is obtained, although the structure itself would remain unaltered.

This paper introduces a procedure to determine reliability that involves no approximations (from a probabilistic point of view) and can accommodate a material with the two customary operative strength parameters--i.e., "c" and "tan ϕ" = "t" (tan ϕ is designated t for simplicity).

FORMULATION OF PROBABILITY OF FAILURE

It is possible to locate points that represent combinations of c and t [on a (c - t) or [(c/γH) - t] plane] for which a slope is in a state of limiting equilibrium (factor of safety equal to unity). The locus of such points generally demonstrates a curve (AB in Figure 1), which will be called the "critical boundary". In concept, if the average values of the operative material parameters lie on or above curve AB, the slope will be safe. Consequently, the probability of failure is the likelihood that the point that represents the average values of the operative strength parameters lies below curve AB in the shaded region of Figure 1.

Since the strength parameters (c and t) are themselves random variables and not deterministic quantities, their description is given by their joint probability density function: f(c,t). In concept, the f(c,t) axis is normal to the plane of the paper in Figure 1. Random variables c and t will be assumed to be statically independent and to follow beta distributions; therefore, the joint density function of c and t is equal to the product of the marginal density functions of c and t (7,9). Since the beta distribution permits the selection of extremes of the variable (a minimum and a maximum value), in the most general case, $t_{min} \geq 0$, $c_{min} \geq 0$, $t_{max} < \infty$, and $c_{max} < \infty$ and the joint density function of (c,t) will appear as a rectangle on the c,t plane (Figure 1).

If the critical boundary is represented by t = g(c), then the following quantities are defined (see Figure 2):

$$Uc = g^{-1}(t_{min}) \tag{1}$$

$$Ut = g(c_{min}) \tag{2}$$

Given these values, the probability of failure can be quantified.

Figure 1. Interference between the probability density function of the material parameters and the critical boundary.

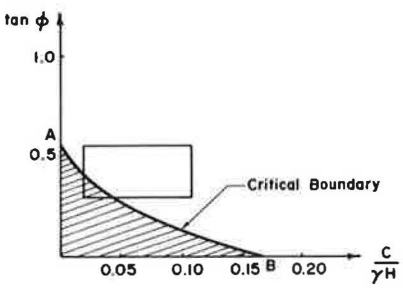
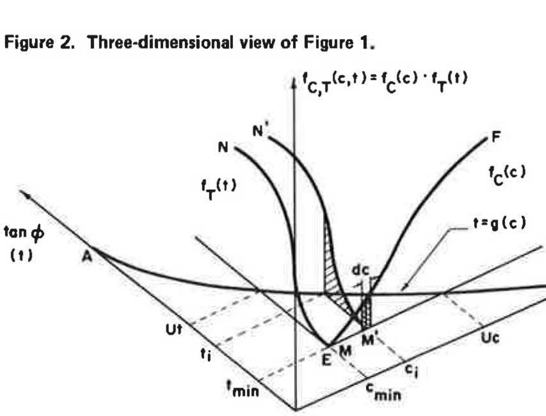


Figure 2. Three-dimensional view of Figure 1.



The probability of having the average value of c along the critical slip surface equal to c_i is

$$P(c_i < c < c_i + dc) = f(c_i)\,dc \tag{3}$$

The probability of having a value of t less than t_i, where $t_i = g(c_i)$, is

$$\int_{t_{min}}^{t_i} f(t)\,dt = F(t_i) \tag{4}$$

where $F(t_i)$ is by definition the cumulative distribution function of t. The probability of the joint occurrence of these two events is the product of their individual probabilities (the assumption of independence). Summing such products (since they are mutually exclusive) over the range c_{min} to Uc, the probability of failure is obtained as

$$P_f = \int_{c_{min}}^{U_c} f_c(c) \int_{t_{min}}^{t=g(c)} f_T(t)\,dt\,dc \tag{5}$$

or, alternatively,

$$P_f = \int_{c_{min}}^{U_c} f_c(c) F_T(t)\,dc \tag{6}$$

The procedure for determining the probability of failure just described presents several advantages over other existing methods (10-16):

1. It offers a closed-form solution that avoids the unknown errors in approximate methods such as error propagation (7,17).
2. The concept of limiting equilibrium is applied in the only state in which it is really valid, i.e., when the factor of safety is equal to unity.
3. A comparison is made of capacity and demand at the level of knowledge of the material parameters instead of at unknown stresses as is the case in common methods.
4. The procedure provides the means of visualizing the uncertainties associated with each factor in the analysis.

DISTRIBUTION OF THE DRAINED STRENGTH PARAMETERS

As stated earlier, it is assumed that c and t follow beta distributions. This assumption is supported by the results of laboratory tests (7,18) and by the following physical arguments:

1. Since the beta distribution may have finite extremes, its tails need not go to +∞ and/or -∞, a characteristic necessary in modeling real material parameters (7).
2. The beta distribution requires four parameters: a mean, a standard deviation, a minimum value, and a maximum value. Granted this information, it is very versatile and capable of assuming shapes that reflect the data themselves (7).

The mean value to be used in defining each beta density can be determined from experimental data. The extremes of each density (minimum and maximum values) reflect engineering judgment (19). The standard deviation of the distributions introduces some difficulties in the calculation of the probability of failure. One of the following procedures is recommended for its estimation:

1. Use typical values of the coefficient variation (the standard deviation divided by the mean value) reported in the literature (typical values

are approximately 50 percent for c and 10-15 percent for t) (20-23).

2. Assume an unbiased "prior" distribution for the standard deviation of each variable (8,24) and "update" it to obtain a "posterior" distribution as new information is obtained (new test results) through the use of Bayes' theorem (8,24). Then determine a weighted average of the probabilities of failure found with each discrete value of the standard deviation, weighting the probabilities of failure with respect to the likelihood of actually having such a value of the standard deviation. These likelihoods are directly obtained from the (prior or posterior) distribution of the standard deviation.

DETERMINATION OF THE CRITICAL BOUNDARY

The locus of the critical boundary is given here for both two-dimensional and three-dimensional slip surfaces. For the former, the "ordinary method of slices" (OMS) was used whereas, for the latter, Hovland's method (25) was selected. In each case, the minimum factor of safety was determined by investigating a number of potential slip surfaces. This provides the expression t = g(c) (for a factor of safety of unity).

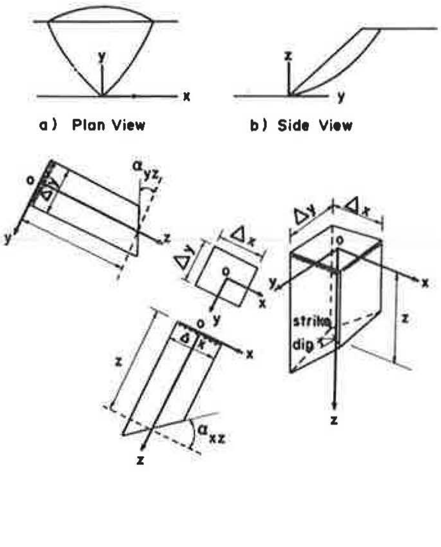
The OMS was selected over other limiting equilibrium methods for the following reasons:

1. It is simple to use and involves no iterations to determine a factor of safety.
2. It is the only limiting equilibrium method that does not make the unrealistic assumption that the local factors of safety are all equal and, in turn, are equal to the global factor of safety.

Hovland's method (25) is the three-dimensional equivalent of the OMS. Two specific slip surfaces--a spherical surface and a cylindrical surface [with the axis of the cylinder tilted in the x = 0 plane (see Figure 3)]--were investigated in this study. It is felt that such surfaces produce sliding masses that approximate reality better than the commonly used assumption (13) that the sliding mass is a cylinder with its axis in the x direction.

Hovland's method assumes that all movement leading to failure occurs along the y direction (Figure 3). A consequence of this assumption is that the forces that tend to produce failure are a function of the angle between the tangent to the slip surface and the horizontal direction in the yz plane (see

Figure 3. Plan and side view of a slide (top) and a three-dimensional slice (bottom).



α_{yz} in Figure 3). Those forces that oppose failure are a function of the dip angle of the tangent to the slip surface (DIP) (Figure 3). Therefore, the global factor of safety is given by

$$F = \sum_{i=1}^{n} [W_i \cos(DIP_i)] \tan\phi + \sum_{i=1}^{n} c A_i / \sum_{i=1}^{n} W_i \sin\alpha_{yz_i} \quad (7)$$

where

- n = number of slices (of the form shown in Figure 3),
- W_i = weight of slice i,
- DIP_i = dip (or maximum inclination) of the slice base,
- A_i = area of the base of slice i, and
- α_{yz_i} = angle in the yz plane between the horizontal and the tangent to the midpoint of the base of slice i.

Expressions for W, A, and DIP are given by Hovland (25). From Equation 7, it is possible to obtain a critical boundary as was done for the two-dimensional case.

The inclusion of in situ horizontal (K_o) forces acting in the x direction was also investigated. It was found (19) that they acted so as to reduce the probability of failure (considerably for high values of K_o). However, careful consideration must be given to these forces because soils with large values of K_o are generally overconsolidated and are likely to be fissured, thus rendering the proposed slip surfaces unrealistic, since the discontinuities in the soil mass would most probably control the true shape of the slip surface.

The critical boundary (shown in Figures 1 and 2) was approximated by a straight line. The curvature of the critical boundary results from the fact that, for every combination of c and t for which the factor of safety is equal to unity, a different slip surface will be critical. The straight-line critical boundary was obtained by using the critical slip surface obtained with the mean values of c and t for every combination of c and t in the plane. This approximation was made for convenience and can be justified by noting that the difference between the probabilities of failure, when the critical boundary is taken to be a straight line and when its true shape is considered, was very small. This was found to be the case in a number of problems we have solved (19). However, if the critical boundary is far from the mean values of the distributions of the strength parameters, the probability of failure itself will be very small. For this condition, the linearization of the critical boundary is not recommended.

There are four ways in which the critical boundary can intersect the rectangle that represents the distribution of material parameters shown in Figure 1. These are developed in great detail by Alfaro (19). Two examples are considered here to illustrate the procedure.

Example 1

As mentioned above, variations in the standard deviations (or coefficients of variation) of the strength parameters can greatly affect the probability of failure. The latter can change over several orders of magnitude, depending on values of the coefficients of variation for c and t, even though the factor of safety remains a single, constant value (7,26).

Example 1 is intended to illustrate how different probabilities of failure can be obtained for four slopes that have the same factor of safety and the same coefficients of variation for c and t.

Two slope angles, β = 30° and 60°, and two materials, A and B (see Table 1), were studied for the four combinations (see Table 2). The slope heights that produced a factor of safety of F = 1.3 were obtained by using the OMS. The slopes are drawn to scale in Figure 4. The resulting probabilities of failure are given in the sixth column of Table 2. As can be seen, the difference between the probabilities of failure is small for cases 1, 3, and 4, for which Janbu's dimensionless parameter λ (27),

$$\lambda = \gamma H \tan\phi / c \tag{8}$$

does not vary much. However, for case 2, λ changes considerably and so does the probability of failure. It is also interesting to note that these

Table 1. Statistics of strength parameters of soils A and B.

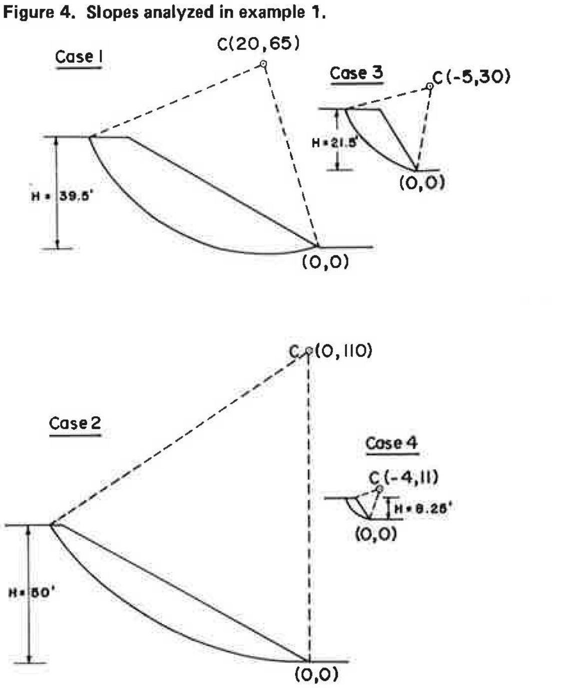
Parameter	Statistic	Soil A	Soil B
c	$\bar{c}$	400 lbf/ft^2	100 lbf/ft^2
	V_c	50 percent	50 percent
	c_{min}	0	0
	c_{max}	750 lbf/ft^2	200 lbf/ft^2
t	$\bar{t}$	0.268	0.577
	V_t	10 percent	10 percent
	t_{min}	0.19	0.4
	t_{max}	0.35	0.75

Table 2. Results of analysis in example 1.

Case	Slope Angle (°)	Soil	λ	F_2	p_{f2}	F_3	p_{f3}
1	30	A	2.91	1.30	0.234	1.40	0.214
2	30	B	31.75	1.30	0.021	1.30	0.029
3	60	A	1.58	1.30	0.306	1.38	0.249
4	60	B	5.24	1.30	0.241	1.43	0.210

Note: The subindex (2 or 3) refers to two- or three-dimensional analyses.

Figure 4. Slopes analyzed in example 1.



probabilities of failure are very high for what would normally be considered a tolerable factor of safety.

Example 2

Example 1 is repeated by using three-dimensional spherical slip surfaces. The same centers of the critical circles used in example 1 were used in this example. This was found to produce a negligible error in the determination of the probability of failure.

The last two columns in Table 1 compare the resulting probabilities of failure and the three-dimensional factors of safety with the corresponding values obtained from the two-dimensional analyses. The differences are seen to be small; however, it is noteworthy that only for case 2 is the probability of failure from a three-dimensional analysis greater than that from a two-dimensional analysis and this only slightly so.

The cylindrical sliding surface investigated (with the axis of the cylinder tilted in the x = 0 plane) was found to be less critical than the spherical slip surface for every case that was examined.

CONCLUSIONS

The following conclusions can be drawn from the work reported in this paper:

1. Different probabilities of failure can be obtained for slopes judged equally safe by conventional factors of safety.
2. For a given factor of safety, slopes with smaller values of Janbu's dimensionless parameter λ ($\lambda = \gamma H \tan \phi/c$) have higher probabilities of failure. This is because the "c parameter" of strength abounds in uncertainty.
3. From three-dimensional analyses, it was found that spherical slip surfaces have higher probabilities of failure than cylindrical ones (with the axis of the cylinder tilted in the same plane as the slope profile).
4. In materials that have low values of λ, circular two-dimensional slip surfaces yield slightly higher probabilities of failure than spherical three-dimensional slip surfaces; however, the latter is certainly more rational, since it approximates better real failed surfaces. Only minor differences were noted between two- and three-dimensional analyses in this study; however, the introduction of in situ lateral stresses (perpendicular to the slope profile) can decrease the probability of failure considerably.

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Risk Reduction Versus Risk Assessment: A Case for Preventive Geotechnical Engineering

THOM L. NEFF

The topic of risk analysis has become greatly sophisticated in recent years. Owners and regulatory agencies have the ultimate concern of cost-effective risk reduction. Uncertainty and risk do not lend themselves to precise quantification, a fact that has resulted in some risk analyses finding a less than enthusiastic response from clients. All facilities rest on geologic materials and thus have a degree of uncertainty that often expresses itself most strongly in geotechnical elements of the project. This "natural" problem, and consideration of synergy and entropy, logically leads one to emphasize prevention rather than precise prediction of event sequences. Other professions, notably medicine and dentistry, have recognized the importance of preventive efforts and have formulated formal preventive programs. The size, complexity, and cost of many modern facilities suggest that a prudent approach to continuing acceptable facility performance should include formal preventive efforts, even in the planning stages of the project. A conceptual outline of a preventive geotechnical engineering program for a constructed facility is presented.

The field of risk analysis has grown rapidly in recent years, incorporating sophisticated mathematics, theory of probability, and modeling techniques (1). The costs of failures remain so high