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Equilibrium Model for Carpools on an Urban Network

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Traffic equilibrium methods are presented in which the population of motorists consists of individuals who are minimizers of a linear combination of cost and travel time. The relative importance of travel time versus cost varies across the population, but fairly mild conditions for the existence and uniqueness of the equilibrium can nevertheless be identified. The paradigm is of particular interest for carpooling studies because the occupants of carpools can divide the cost among themselves but they cannot do the same with the travel time. Thus, vehicles that have different occupancy levels will have different relative values of travel time and cost. The model is specially well suited to the analysis of how vehicles that have different occupancies compete for segments of the roads that are crowded or have tolls. It is therefore very useful to predict the impacts of special carpooling lanes, lower tolls for high-occupancy vehicles, and other transportation-system-management strategies on the distribution of traffic over an urban network.

Current traffic-assignment practice takes two principal forms, which are applicable to congested and uncongested networks. Stochastic traffic-assignment models (1-5) ignore congestion but do not allocate all the traffic from an origin-destination (O-D) pair to the shortest route. Instead, they spread it over the network as if travel time was perceived with some random noise by a motorist population of travel-time minimizers.

Deterministic-equilibrium models assume that motorists are accurate minimizers of travel time but that travel time depends on the traffic flow because of congestion. Textbook-level treatments of deterministic equilibrium models can be found (6-9). The equilibrium condition for these models was stated by Wardrop (10). It can be paraphrased as follows: at equilibrium (a) routes that have flow are the shortest routes, or (b) no user can improve route travel time by unilaterally changing routes, or (c) links that have flow for a given destination are on a shortest path to the destination. Since a problem that is more closely related to deterministic-equilibrim models than to stochastic-assignment models will be addressed here, the discussion of the former is expanded below. A question that arises immediately is that of the existence and uniqueness of an equilibrium-flow pattern that satisfies all three equilibrium conditions.

Beckmann, McGuire, and Winsten (6); Netter (11); and Smith (12) have provided progressively more general existence results. It is currently known that if travel time on every link of the network is a continuously differentiable positive function of the link flows, Brouwer's fixed-point theorem guarantees the existence of the equilibrium flows.

Uniqueness was first studied for networks in which the travel time on a link depends only on its own flow (6). In this case and if travel time increases with flow for all links, the equilibrium exists and the resulting link-flow pattern is unique. This is because the equilibrium problem admits a formulation as the minimization of a strictly convex function subject to linear constraints. This formulation can be expressed in terms of link flows as follows:

$$(\text{MP}) \min \sum_i \int_0^{x_i} c_i(w)dw$$

subject to

$$\sum_{i \in I(r)} x_i^s - \sum_{i \in E(r)} x_i^s = q^{rs} \quad \forall r \neq s, \forall s$$

$$\sum_s x_i^s = x_i \quad \forall i$$

$$x_i^s \geq 0 \quad \forall i, s$$

In this program, the letters r and s represent nodes, and the letter i represents a link. I(r) represents the set of links that point to node r; E(r), the set of links that point out of node r; and $c_i(\cdot)$, the link-cost function that relates the flow on link x_i to the link travel time c_i. In addition, x_i^s is the total number of trips that have final destinations s and that use link i, and q^{rs} is the total number of trips that go from origin r to destination s.

In order to write equilibrium problems more succinctly, the set of feasible link-flow patterns is denoted by X; thus, program (MP) is written as follows:

$$(\text{MP}) \min_{x \in X} \sum_i \int_0^{x_i} c_i(w)dw$$

Link flows that are optimal for (MP) are equilibrium flows (and vice versa) because the Kuhn-Tucker conditions of (MP) are the mathematical expressions for Wardrop's principle as paraphrased under item (c) at the outset of this paper. This happens because the partial derivatives of the objective function are the link-cost functions, as follows:

$$\partial/(\partial x_j) \sum_i \int_0^{x_i} c_i(w)dw = c_j(x_j); \forall j \tag{1}$$

These results can be generalized for models in which link costs depend on the flows of other links. Dafermos (13) seems to have been the first to have studied this class of problems. She showed that the equilibrium problem admits an extremum formulation if the link-cost functions satisfy a condition similar to that in Equation 1. That is, if there is a function $C(x)$ whose partial derivatives are the link-cost functions $\partial C(x)/\partial x_i = c_i(x)$; $\forall i$, the equilibrium problem is as follows:

$$(\mathrm{MP}) \min_{x \in X} C(x)$$

To solve equilibrium problems, one does not have to find the function $C(x)$, since to solve (MP) only the derivatives of $C(x)$ are necessary. Furthermore, the existence of $C(x)$ can be verified from the (continuous) cross-derivatives of $c_i(x)$:

$$C(x) \quad \text{exists if } [\partial c_i(x)/\partial x_j] = [\partial c_j(x)/\partial x_i],\ \forall i, j$$

The uniqueness of the equilibrium link-flow pattern x^* can be established from the strict convexity of $C(x)$ or the positive definiteness of the Jacobian $J(x) = [\partial c(x)/\partial x]$. That is, if $J(x)$ is symmetric, there is an extremum formulation and if it is positive definite, the equilibrium solution is guaranteed to be unique. Recent research shows that this uniqueness condition holds even if $J(x)$ is not symmetric (12,14).

Another area of research that is closely connected is multimodal-equilibrium models. In these models, each vehicle type has a different impact on the overall congestion and imposes a different amount of delay on vehicles that share the road with it. In addition, vehicles of different types may exhibit different link travel times under the same link congestion. The most general formulation, short of letting vehicles on a link affect the travel times on another link (11), assumes that one has K vehicle types and that the travel time on link i for the kth vehicle class $c_i^{(k)}$ is as follows:

$$c_i^{(k)} = c_i^{(k)}[x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(K)}] \tag{2}$$

For example, if k = 1 represents automobiles and k = 2 represents trucks, one would expect $c_i^{(1)}$ to be smaller than $c_i^{(2)}$ for any combination of $x_i^{(k)}$'s, and one would also expect $x_i^{(2)}$ to influence $c_i^{(k)}$ more than $x_i^{(1)}$. Hypothetical curves could be as follows:

$$c_i^{(1)} = 100 + x_i^{(1)} + 5x_i^{(2)} \tag{3a}$$

$$c_i^{(2)} = 150 + 1.1[x_i^{(1)} + 5x_i^{(2)}] \tag{3b}$$

in which a truck is depicted as having the same effect on congestion as that of five passenger cars but also requiring more time units to travel the same distance.

Uniqueness results for multimodal networks can also be derived. Since one can visualize each traffic type as moving on its own transportation network, Equation 2 can be interpreted as an interaction among links of a network that consists of K copies of the original network instead of an intermodal interaction. With this mental picture, it is easy to see that multimodal networks are special cases of the single-mode network model that have general link-cost functions. Therefore, they share the same existence and uniqueness results (15). That is, the following equation guarantees existence of (MP) if the derivatives are continuous:

$$[\partial c_i^{(k)}/\partial x_j^{(\ell)}] = [\partial c_j^{(\ell)}/\partial x_i^{(k)}],\ \forall (i, j, k, \ell) \tag{4a}$$

$$J(x) = \partial[\ldots, c_i^{(k)}, \ldots]/\partial[\ldots, x_j^{(\ell)}, \ldots] \tag{4b}$$

Because $c_i^{(k)}$ depends only on the flows of link i, Equations 4 can be simplified as follows:

$$[\partial c_i^{(k)}/\partial x_i^{(\ell)}] = [\partial c_i^{(\ell)}/\partial x_i^{(k)}],\ \forall (i, k, \ell) \tag{5a}$$

$$J(x) = \begin{bmatrix} J_1(x) & 0 & \\ 0 & J_2(x) & \\ & & \ddots \end{bmatrix};\ J_i(x) = \partial[\ldots, c_i^{(k)}, \ldots] \div \partial[\ldots, x_i^{(\ell)}, \ldots] \tag{5b}$$

For uniqueness, it is sufficient thus that for all links, $J_i(x)$ be positive definite. Of course, the symmetry of all $J_i(x)$'s would also guarantee the existence of an extremal formulation. Unfortunately, these conditions are much too restrictive for multimodal networks because the off-diagonal terms of $J_i(x)$ can be large and asymmetric (14). For example, the hypothetical link defined by Equations 3 yields the following:

$$J_i(x) = \begin{bmatrix} 1 & 5 \\ 1.1 & 5.5 \end{bmatrix}$$

which violates the conditions because it is neither symmetric nor positive definite. Typically, vehicles of different sizes will result in asymmetric nondefinite Jacobians as in the example. In recognition of these problems, papers on multimodal public and private traffic-assignment problems tend to focus on computational schemes to finding equilibrium solutions but always recognize that multiple equilibria may exist [see papers by Florian (16) and Abdulaal and LeBlanc (17), for example].

It is shown next that there is a family of link-cost functions that have symmetric semidefinite Jacobians that well describe multimodal networks of similar-size vehicles and have application to carpooling problems.

The generalization of this family to vehicles of different sizes that is mentioned in the conclusion is in agreement with Jeevanantham's conjecture for general networks (18).

CARPOOLING MODEL

Assume that Equation 2 is of the following form:

$$c_i^{(k)} = c_{oi}^{(k)} + c_i(x_i);\ x_i = x_i^{(1)} + \ldots + x_i^{(K)} \tag{6}$$

where $c_i(x_i)$ is continuously differentiable and $c_{oi}^{(k)}$ can represent a constant that is independent of flow but can vary across traffic classes. Note that Equation 6 specifies that all traffic types have the same impact and are affected equally by congestion. Then

$$J_i(x) = [\partial c_i/\partial x_i] \begin{bmatrix} 1 & 1 \ldots 1 \\ 1 & 1 \ldots 1 \\ \cdots & \cdots \\ 1 & 1 \ldots 1 \end{bmatrix}$$

and if $c_i(x_i)$ is increasing, $J_i(x)$ is a positive semidefinite symmetric matrix. That $J_i(x)$ is positive semidefinite is seen by noting that for any vector, $\alpha = (\alpha_1 \ldots \alpha_K)$:

$$\alpha[J_i(x)]\alpha^T = [\partial c_i/\partial x_i] \left[\sum_{k=1}^{K} \alpha_k\right]^2 \geq 0$$

This implies that the equivalent minimization problem is a convex programming problem with a set of equilibrium solutions that is convex. Equilibrium in terms of modal flows $[\ldots, x_i^{(k)}, \ldots]$ is not necessarily unique (as it is not for the route flows for the single-mode problem) because C(x) is not strictly convex. Nevertheless, it is possible to show that all equilibrium-flow patterns must have the same link costs $c_{oi}^{(k)}$ and total link flows x_i (19).

In practical applications, the constants $c_{oi}^{(k)}$ may represent a number of things, including direct costs (expressed in travel-time units) that are independent of flow and may change across the motoring population. For example, the model could be applied to study a futuristic scenario in which a mixture of roadway powered vehicles (RPVs) and internal combustion engine vehicles share a transportation network. An RPV is an electrically powered vehicle that can draw its power from specially equipped links of the network. If we assume that these vehicles do not pay every time they use these special links (presumably they would be taxed differently from gasoline-powered vehicles), their routing incentive will tend to deviate from shortest routes within reason to take advantage of the lower operating costs on those links. The value of $c_{oi}^{(k)}$ on such links will be small for RPVs and relatively larger for gasoline-powered vehicles. On the standard links of the network, the values of $c_{oi}^{(k)}$ would be similar for both vehicle types.

The type of model implied by Equation 6 is particularly useful to study the effects of current transportation-system-management (TSM) strategies to encourage carpooling. In this case, the index k represents the number of people in an automobile and $c_{oi}^{(k)}$ represents the cost to any one of the occupants. If, reasonably, we assume that the k persons in the carpool divide all the costs (tolls and mileage, mainly) proportionately, $c_{oi}^{(k)}$ can be expressed as follows:

$$c_{oi}^{(k)} = \alpha^{(k)}\{[\tau_i^{(k)} + \beta^{(k)}d_i]/k\}, k = 1, 2, \ldots K \tag{7}$$

where $\tau_i^{(k)}$ represents the tolls (if any) on link i, d_i represents the distance of link i, and $\beta^{(k)}$ and $\alpha^{(k)}$ are factors that convert distance traveled into monetary units and monetary units into travel-time units, respectively.

The following TSM strategies can be studied:

1. Differential tolls,
2. Ramp metering,
3. Special lanes for high-occupancy vehicles, and
4. Parking privileges for carpools.

To model the effect of tolls that depend on vehicle occupancy, one defines $\tau_i^{(k)}$ accordingly. For example, if (βd_i) is negligible and no toll is levied for vehicles that have three or more occupants, we have the following:

$$c_{oi}^{(1)} = \alpha\tau, \quad c_{oi}^{(2)} = 1/2\,\alpha\tau, \quad c_{oi}^{(3)} = c_{oi}^{(4)} = \ldots = 0$$

To model differential treatment of vehicles that have different occupancy levels in a ramp-metering situation (e.g., vehicles that have more than three occupants may bypass the metering queue), one should represent the metered link as two parallel links--one that is not metered and is restricted to cars that have more than two passengers and a metered link for all other vehicles. To forbid the use of the unmetered link to motor vehicles of types 1 and 2, one simply sets a very high toll for these vehicle classes. For example, if the distance component is negligible and the original link is represented by metered link i and unmetered link i', one would have the following:

$$c_i^{(k)} = c_i(x_i), \quad c_{i'}^{(k)} = c_{oi'}^{(k)}$$

where $c_i(x_i)$ represents the delays encountered at the metering ramp when the metered flow is x_i and

$$c_{oi'}^{(k)} = 0 \quad \text{if } K \geq 3$$
$$= M\ (M \to \infty) \quad \text{if } k \leq 3$$

To model lanes for high-occupancy vehicles, one represents the special lane by a separate link and in the same way assigns it a very high differential toll that is applied only to vehicle classes that are forbidden to use it. Special parking privileges for carpools can be modeled similarly by assigning classes that are not allowed to park a very high fixed cost on links that go into the parking lot.

Example

Figure 1A is a graphic representation of the transportation problem from Marin County to San Francisco. It displays the central business district (SF) and a suburb (M) of a metropolitan area that are separated by a toll bridge. The central business district (CBD) can also be reached in 10 time units by using a ferry system. The ferry fleet is supposed to be large enough (or flexible enough) to guarantee this travel time independent of flow. The ferry fare is neglected, but the over-land taxicab fare (assessed jointly to the members of a carpool) is 10 monetary units. We assume that α equals 1. The toll bridge, on the other hand, is so short that its distance and free-flow travel time are negligible. However, congestion sets in very quickly and, for flow different from zero, the travel time is equal to the flow. Figure 1B summarizes this information. It also displays the parameters α and β and the O-D table for the morning rush hour: 10 vehicles per unit time that have one occupant and 10 more that have two occupants; all the traffic goes from M to the CBD.

The cost functions are as follows:

$$c_1^{(1)} = \tau + x_1 \quad c_2^{(1)} = 20$$

$$c_1^{(2)} = (\tau/2) + x_1 \quad c_2^{(2)} = 15$$

where τ is the toll on the bridge and $x_1 = x_1^{(1)} + x_1^{(2)}$. We will attempt to study the equilibrium flows on this problem as the toll τ is increased from zero. [For the simple network that is being studied, the equilibrium link costs are unique even though $c_2^{(1)}$ and $c_2^{(2)}$ are constant.]

The function C(x) for our problem is (up to an additive constant) as follows:

$$C(x) = (x_1^2/2) + \tau x_1^{(1)} + [(\tau/2)\,x_1^{(2)}] + 20x_2^{(1)} + 15x_2^{(2)}$$

since $\partial C(x)/\partial x_i^{(k)} = c_i^{(k)}$. The flow conservation and nonnegativity constraints are as follows:

$$x_1^{(1)} + x_2^{(1)} = 10$$

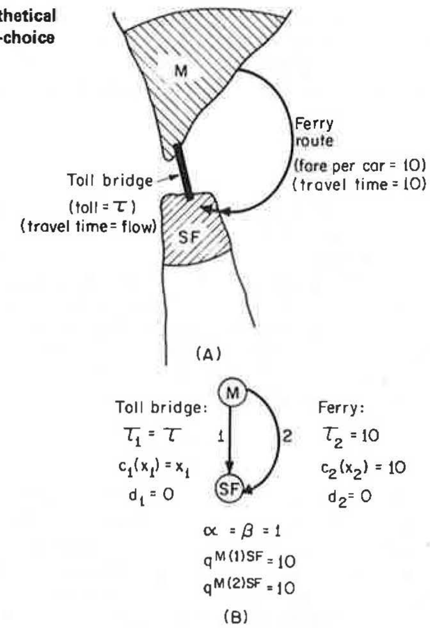


Figure 1. Hypothetical two-mode route-choice problem.

$$x_1^{(2)} + x_2^{(2)} = 10$$

$$x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_2^{(2)} \geq 0$$

Alternatively, we can solve the following:

$$\min C = [x_1^{(1)} + x_1^{(2)}]^2/[2 + \tau x_1^{(1)}] + [(\tau/2)\, x_1^{(2)}] + 20[10 - x_1^{(1)}] + 15[10 - x_1^{(2)}] \quad 0 \leq x_1^{(1)}, x_1^{(2)} \leq 10$$

Figure 2 plots the negative gradient field of C for different values of τ over the feasible range of $x_1^{(1)}$ and $x_1^{(2)}$:

$$-\nabla C^T = \begin{bmatrix} -x_1^{(1)} - x_1^{(2)} - \tau + 20 \\ -x_1^{(1)} - x_1^{(2)} - (\tau/2) + 15 \end{bmatrix}$$

The equilibrium solution is a point in the feasible region at which the gradient is perpendicular to the boundary.

As seen from Figure 2, there are two cases that result in different equilibrium solutions. If the toll is less than 10 monetary units, all the type-1 traffic takes the toll bridge and the type-2 traffic is split between the two routes so that the total cost to type-2 vehicles will be equal on both routes (type-2 vehicles are diverted from the bridge because they can split the taxicab fare). As the toll is increased more and more, type-2 vehicles are shifted to the long route in order to keep the costs on both routes equal for these vehicles. In the process, however, the relative attractiveness of route 1 for vehicles of type 1 is decreased (the toll affects these vehicles twice as heavily) until eventually route 2 becomes more attractive than route 1 to these vehicles. At that toll value, $\tau = 10$, the equilibrium solution is achieved by using 10 vehicles of any type on each route since they are then equally attractive to both classes. (This illustrates well the possible nonuniqueness of the modal link flows despite the uniqueness of the total flow.) A slight increase of τ beyond 10 shifts all type-2 vehicles to route 1 and all type-1 vehicles to route 2 because the toll is now sufficiently high to make the bridge unattractive to those who are not carpoolers. Increases beyond 10 will result in further decreases in bridge traffic as those who carpool find the bridge increasingly expensive.

Table 1 summarizes the results. Note also that although vehicle traffic on the bridge decreases smoothly with an increasing toll, there is a critical point when the composition of traffic changes drastically with an increase in the total number of bridge users. Figure 3 illustrates this.

The transportation cost to society (tolls are internal transfers) can be decreased by increasing tolls. This is logical because in this way the bridge is used only by cars that have high occupancy. The maximum revenue on the bridge is achieved when $\tau = 15$, but the maximum combined revenue (which also yields the minimum total travel time) is obtained for $\tau = 20$.

Methods

To solve problem (MP), one can use the Frank-Wolfe algorithm. Because the gradient of the objective function C(x) is the set of link costs for all vehicle classes, the linear subproblem is an all-or-nothing traffic-assignment problem. LeBlanc, Morlok, and Pierskalla (20) were the first to propose this algorithm for the one-vehicle traffic-assignment problem. The steps are as follows:

Step 0 (initialization): Set an arbitrary (nonnegative) cost vector $c = (\ldots, c_i^{(k)}, \ldots)$, assign the O-D table of each vehicle type to the corresponding shortest paths, and obtain a feasible link-flow pattern $x = (\ldots, x_i^{(k)}, \ldots)$.

Step 1 (cost updating): Recalculate c by using the new set of flows, $c = c(x)$.

Step 2 (assignment): Calculate the shortest paths and assign the O-D flows to them. Do this for all vehicle types. Label the flow pattern $y = (\ldots, y_i^{(k)}, \ldots)$.

Step 3 (interpolation): Find the value of ω, $\omega^* \in [0,1]$ that minimizes $f(\alpha) = C[x + (y - x)\omega]$ and let $x' = x + (y - x)\omega^*$ be the new flow pattern.

Step 4 (convergence check): If the new pattern is not substantially different from the old pattern, stop. Otherwise, repeat the process from step 1.

The easiest way of performing step 3 is to find the value of ω at which the derivative of $f(\omega)$ vanishes. If $f'(\omega)$ does not have a root in [0, 1], $\omega^* = 1$ because $f(\omega)$ is convex. In this way the objective function is never used, and one does not have to integrate the link-cost function:

$$f'(\omega) = \sum_{i,k} \left([y_i^{(k)} - x_i^{(k)}] \times c_i^{(k)} \left\{ x_i^{(k)} + \omega [y_i^{(k)} - x_i^{(k)}] \right\} \right)$$

Alternatively, one can use the method of successive averages (21-23) or, for sketch planning problems that have few links, some unconstrained methods (19,24).

Example

We do the example in Figure 1 by using $\tau = 0$ and start by using a cost vector that corresponds to an empty network:

$$c = [c_1^{(1)}, c_2^{(1)}, c_1^{(2)}, c_2^{(2)}] = (0, 20, 0, 15)$$

Step 0. The all-or-nothing flow pattern $x = [x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_2^{(2)}]$ is $x = (10, 0, 10, 0)$.

Step 1. The revised cost vector is $c = (20, 20, 20, 15)$.

Step 2. The all-or-nothing flow vector is $y = (0, 10, 0, 10)$.

Step 3. The $f'(\omega)$ function is as follows:

$$f'(\omega) = -10 \times [10(1-\omega) + 10(1-\omega)] + 10 \times 20$$
$$-10 \times [10(1-\omega) + 10(1-\omega)] + 10 \times 15$$
$$= -10[5 - 40\omega]$$

and $\omega^* = 0.125$.

The new flow vector is $x = (8.75, 1.25, 8.75, 1.25)$.

Another iteration yields $x = (9.3, 0.7, 4.9, 5.1)$, which is fairly close to the equilibrium solution $x^* = (10, 0, 5, 5)$. The Frank-Wolfe algorithm,

Figure 2. Negative gradient field and equilibrium solutions.

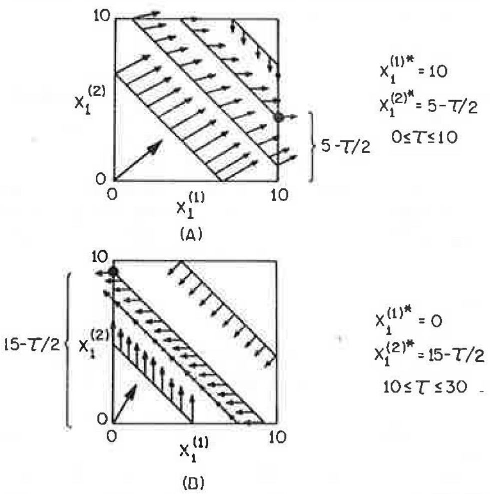
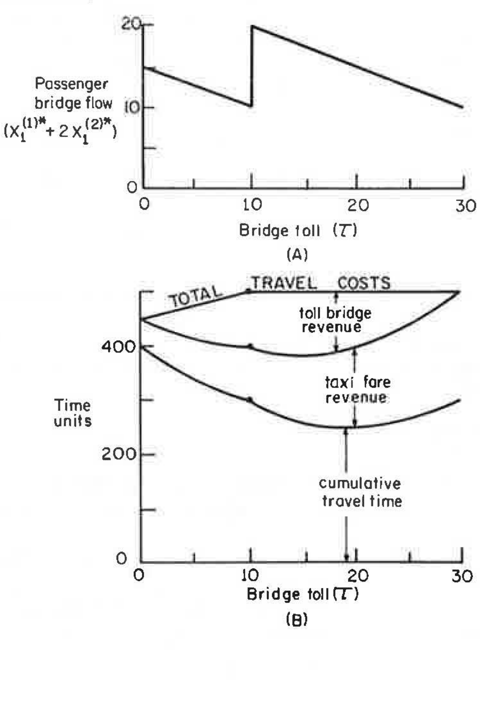


Table 1. Components of the equilibrium solution for different tolls.

τ	$x_1^{(1)*}$	$x_1^{(2)*}$	x_I	$c^{(1)*}$	$c^{(2)*}$
$\tau \leq 10$	10	$5-(\tau/2)$	$15-(\tau/2)$	$15+(\tau/2)$	15
$10 \leq \tau \leq 30$	0	$15-(\tau/2)$		20	

Figure 3. Effect of varying tolls on passenger bridge flow, revenues, and total travel time.



however, tends to slow down when the equilibrium is approached. This example is no exception, since, as the reader can verify, the next two flow vectors are $x = (9.47, 0.53, 6.13, 3.87)$ and $(9.57, 0.43, 4.97, 5.03)$.

CONCLUSION

This paper has demonstrated that many current TSM strategies related to carpooling can be investigated by using equilibrium theory. It was argued that multiple-vehicle-type network models that have link-cost functions of the following form are good descriptors of carpooling cost functions because the independent constant $c_{oi}^{(k)}$ can capture the difference in the fixed costs of a link to the different vehicle classes:

$$c_i^{(k)} = c_{oi}^{(k)} + c_i[x_i^{(1)} + x_i^{(2)} + \ldots + x_i^{(K)}] \tag{8}$$

It was mentioned that if the $c_i(\cdot)$'s were increasing functions, the total equilibrium flows $x_i = [x_i^{(1)} + \ldots + x_i^{(K)}]$ existed and were unique. Furthermore, because the Jacobian $\{\partial[\ldots, c_i^{(k)}, \ldots]/\partial[\ldots, x_i^{(k)}, \ldots]\}$ is continuous and symmetric, the equilibrium problem admits an extremum formulation that can be solved by using optimization procedures.

The following generalizations are of some merit:

1. Vehicles of different sizes: If $c_i^{(k)}$ can be expressed as follows:

$$c_i^{(k)} = c_{oi}^{(k)} + \delta^{(k)} c_i[\gamma_1 x_i^{(1)} + \gamma_2 x_i^{(2)} + \ldots + \gamma_K x_i^{(K)}]$$

where the γ_k's are nonnegative constants that reduce the flow of k-type vehicles to an equivalent flow of standard-type vehicles, the link costs and the standardized link traffic levels are given as follows:

$$x_i = \sum_{k=1}^{K} \gamma_k x_i^{(k)}$$

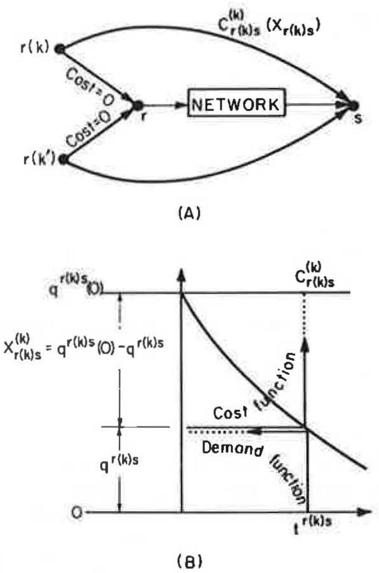
At equilibrium, these link costs and traffic levels exist and are unique, provided the $\delta^{(k)}$'s and γ_k's are positive constants and the $c_i(\cdot)$'s are nonnegative increasing continuously differentiable functions (19). [This is true even though the Jacobian $J(x)$ in this case is no longer positive-definite or symmetric.] This formulation is of interest in cases in which vehicles of different types are of different sizes and is of potential interest to study urban goods movement by trucks in an urban area.

2. Elastic demand: If the k-type O-D flow $q^{r(k)s}$ decreases with an increasing O-D cost for k-type vehicles $t^{r(k)s}$ and is also independent of other O-D costs, the O-D tables, link costs, and link-flow levels are unique at equilibrium. This is not difficult to see because, as with single-vehicle-type elastic-demand problems (24), multiple-vehicle-type problems admit an equivalent formulation that has fixed O-D tables. Figure 4A depicts the equivalent network, which includes additional nodes r(k) and r(k'). These nodes are only connected to r and s by dummy links. The k-type flow from r to s is now assumed to start at r(k) and to be equal to $q^{r(k)s}(0)$. Link r(k)r has zero cost and link r(k)s has a cost function $c_{r(k)s}^{(k)}(\cdot)$, which is defined from the demand function as shown in Figure 4B:

$$c_{r(k)s}^{(k)}(w) = q^{r(k)s^{-1}}[q^{r(k)s}(0) - w]$$

Similar definitions apply to flows of other types.

Figure 4. Fixed-demand network equivalent to elastic-demand network.



Since link r(k)s can carry k-type flow, it is clear that at equilibrium $c^{(k)}_{r(k)s}$ must equal $t^{r(k)s}$, and therefore the following holds:

$$q^{r(k)s^{-1}}\left[q^{r(k)s}(0) - x^{(k)}_{r(k)s}\right] = t^{r(k)s}$$

$$q^{r(k)s^{-1}}\left[q^{r(k)s}\right] = t^{r(k)s}$$

$$q^{r(k)s} = q^{r(k)s}\left[t^{r(k)s}\right]$$

which is the condition for elastic demand.

Elegant as it is, the elastic-demand formulation just explained is not realistic for carpooling problems because it fails to capture the important phenomenon of passenger jockeying among modes. That is, if the travel cost for carpools were to decrease substantially, one would expect to see an increase in carpooling but at the expense of noncarpooling traffic. Modal-split models that assume that the total O-D passenger flows remain unchanged are much better suited for this and other public transportation applications. In these cases, as passengers switch to high-occupancy vehicles, total O-D vehicular flows decrease. Unfortunately, equilibrium results have not been derived for these models.

Further research should concentrate on establishing uniqueness results for multiple-vehicle-type equilibrium flows in which there is passenger jockeying and on further exploration of optimal pricing strategies by using tolls. Some of these issues will be discussed in a forthcoming publication.

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