Analytic Models of Trip Length Distributions

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This paper develops analytic models of trip length distributions. The models are derived from a destination choice model for a range of assumptions about the distributions of transportation level-of-service attributes and opportunities over the urban space. These models include all previously reported analytic trip length distributions. Their derivation from an explicit model of individual choice behavior illuminates their underlying assumptions about the urban space. It is shown how the parameters of the derived trip length distributions can be interpreted and estimated from available data that include estimated parameters of travel demand models and other readily available statistics on average speeds and fuel consumption. This makes these models useful for simplified analyses of various urban transportation policies, especially areawide pricing and travel time changes.

This paper derives a wide variety of analytic trip length distributions from underlying assumptions about travel behavior, transport system performance, and spatial distribution of travel opportunities. It develops the relations between parameters of trip length distributions and aggregate measures of transport level-of-service, land use, and socioeconomic variables. The resulting trip length distributions have policy-sensitive parameters and therefore can be used for simplified analyses of urban transportation policies and land use changes. The aggregate impacts of changes in the patterns of travel speeds and travel costs can be predicted by such a model with fewer input data and calculations than by a discrete destination choice model that distributes traffic among a number of origins and destinations.

A variety of models have been proposed and used to describe trip length patterns and urban densities (population, employment or joint populations, and employment densities). These include the exponential model [attributed to Clark (1) but first applied by Bleicher in 1892 (2) to analyze Frankfurt data], the square root exponential model (3), the gamma model (3,4), the normal model (5), the shifted normal model (6), the generalized normal model (7), the generalized gamma model (8), the power model (9-11), the beta-type model (12), and the combined exponential and gamma model (13). The normal model has also been used with a directionally dependent variance (7,14,15). Joint population and employment densities were modeled by the bivariate normal model (14) and the quadrivariate normal model (16-18). Theoretical justification is provided for several of the above models through the framework of spatial equilibrium of deterministic utility functions with or without a competitive housing market (17,19-23). Several models were derived by using the entropy maximizing approach (24-26) and from the gravity model (18). A few models have also been derived from random utility theory. Ben-Akiva and Watanatada (27) derived a truncated gamma-2 trip length distribution based on the continuous logit model and Goodwin (28) derived a gamma trip length distribution and Mogridge (29,30) a Weibull distribution based on a slightly different approach. Empirical validations and comparisons of alternative models exist in a number of sources [for example, Casetti (21), Genest (31), Pearce and others (32), McDonald and Bowman (33), Clickman and Oguri (34), and Horowitz (35)].

In this paper all the trip length distributions found in the published literature are derived as special cases of the continuous spatial choice logit model. Their derivation from the continuous logit model clarifies their underlying assumptions and offers ways for their improvement. It also offers a basis for comparison and selection among alternative models for specific applications. Furthermore, a few new, more general models are derived.

ASSUMPTIONS ABOUT URBAN SPACE UNDER CIRCULAR SYMMETRY

The derivation of the trip length distributions in this paper is based on the assumption of circular symmetry around the decisionmaker's origin. This is an approximation of the complex urban patterns that can be employed here for analytical convenience because of the nondirectional nature of the analysis. The results demonstrate that even this highly simplifying assumption leads to valid trip length distributions that are expressed as functions of a small number of parameters.

Assumptions About Generalized Transport Cost Surface

Assume a circularly symmetric (around the decisionmaker) generalized cost surface, $B(l,\phi|r_w,\theta_w)$, in units of generalized cost per unit distance, given by (see the coordinate system in Figure 1)

$$B(l,\phi|r_w,\theta_w) = [c/(l + c_o)] + b_1(l + b_o)^{\nu-1} \tag{1}$$

where c, c_o, b_o, ν are parameters that have specific values by mode and decisionmaker. Equation 1 says that the generalized cost surface depends only on distance and not on directionality. This approximation is most accurate in situations with no directional congestion and for trips that start or end at the city center. A detailed discussion on the derivation of this surface from transportation system performance and the interpretation of its parameters is presented later. For $\nu = 1$ this surface can be derived from the velocity field used by Blumenfeld and Weiss (36) and is also an approximation of a generalized cost surface based on the velocity field tested by Angel and Hyman (37). For this case b_1 can be interpreted as the generalized cost per unit distance at free flow (i.e., $l = \infty$) and c/c_o is the difference between the generalized cost at the most congested point ($l = 0$) and a free flow location.

From this circularly symmetric generalized cost surface, the value of the utility β for a trip from w to h can be derived as a function of the distance l, as follows (for $\nu>0$):

$$\begin{aligned}\beta(l) &= -\int_0^l B(\zeta,\phi|r_w,\theta_w)d\zeta \\ &= c\ln c_o + (b_1/\nu)b_o^\nu - c\ln(l + c_o) - (b_1/\nu)(l + b_o)^\nu\end{aligned} \tag{2}$$

Let

$$b \equiv b_1/\nu \tag{3a}$$

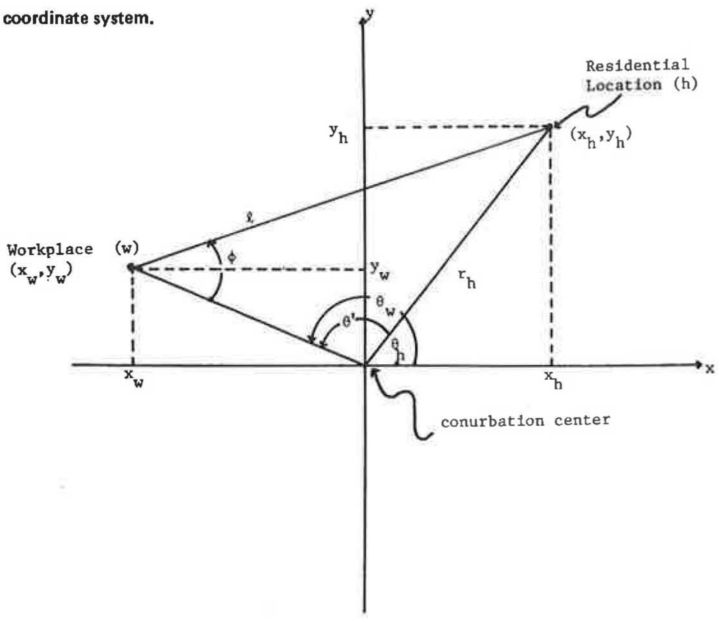
and

$$a_1 \equiv -c\ln c_o - b\,b_o^\nu \tag{3b}$$

to obtain,

$$\beta(l) = -a_1 - b(l + b_o)^\nu - c\ln(l + c_o) \tag{4}$$

Figure 1. Spatial coordinate system.



For the case $b_o = c_o = 0$ the β function can be defined as follows:

$$\beta(l) = -a_1 - bl^{\nu} - c\ln l \quad (5)$$

This form with $\nu = 1$ appears in the literature under the name Tanner function. Mogridge (29) has also suggested the form bl^{ν} with $0<\nu<1$ as a suitable approximation for the generalized cost.

The average utility of a trip from w to h is given by the above β function plus an additive constant that represents urban area, trip origin, mode, and decisionmaker-specific characteristics. Let a denote the sum of this constant with the above constant of integration a_1 to obtain the average utility as follows:

$$V(l) = -a - b(l + b_o)^{\nu} - c\ln(l + c_o) \quad (6)$$

Assumptions About Spatial Opportunity Density

Under circular symmetry, one of the most general assumptions for the opportunity density function that represents travel attractions is the form

$$\gamma(l) = \gamma_o(l + \lambda)^{\gamma-2}\exp[-\delta(l + \lambda)^{\xi}] \quad (7)$$

where γ_o, γ, δ, ξ, and λ are parameters that depend on the urban area and the decisionmaker. This form equals the kernel of the generalized gamma density function. We will show later that the generalized gamma density contains a rich set of density functions, including normal, hydrograph, Rayleigh, Maxwell, Weibull, chi-squared, and gamma [see, for example, Johnson and Kotz (38), a special case of interest primarily because most of the existing models can be derived by using some particular subcase of this form]. It is obtained by setting the translation parameter λ in Equation 7 equal to zero, to get

$$\gamma(l) = \gamma_o l^{\gamma-2}\exp(-\delta l^{\xi}) \quad (8)$$

CONTINUOUS LOGIT DESTINATION CHOICE MODEL WITH CIRCULARLY SYMMETRIC URBAN SPACE

The spatial choice logit density function for a circular attraction area is expressed in polar coordinates, as follows (27,39):

$$f(l,\phi) = \exp[V(l,\phi)]\,\gamma(l,\phi)/\int_0^{r^*}\int_0^{2\pi}\exp[V(l,\phi)]\,\gamma(l,\phi)\,l\,dl\,d\phi \quad (9)$$

where r* denotes the radius of the boundary of the attraction area. Substitute in this model the assumptions presented in the previous section (i.e., Equations 6 and 7) to obtain:

$$f(l,\phi) = \gamma_o(l + \lambda)^{\gamma-2}\exp[-\delta(l + \lambda)^{\xi} - a - b(l + b_o)^{\nu} - c\ln(l + c_o)]$$
$$\div \int_0^{r^*}\int_0^{2\pi}\gamma_o(l + \lambda)^{\gamma-2}\exp[-\delta(l + \lambda)^{\xi} - a - b(l + b_o)^{\nu} - c\ln(l + c_o)]\,l\,dl\,d\phi \quad (10)$$

The trip length distribution is defined as,

$$f(l) = \int_0^{2\pi} f(l,\phi)\,l\,d\phi = 2\pi l f(l,\phi) \quad (11)$$

Since under circular symmetry the density at a point can be obtained by dividing f(l) by $2\pi l$, the following analysis considers the derivation of trip length distributions only. Substitute Equation 10 in Equation 11 to obtain the following trip length distribution for a circularly symmetric logit model:

$$f(l) = \frac{(l + \lambda)^{\gamma-2}(l + c_o)^{-c}\,l\exp[-b(l + b_o)^{\nu} - \delta(l + \lambda)^{\xi}]}{\int_0^{r^*}(l + \lambda)^{\gamma-2}(l + c_o)^{-c}\,l\exp[-b(l + b_o)^{\nu} - \delta(l + \lambda)^{\xi}]\,dl} \quad (12)$$

Without the translation parameters ($\lambda = b_o = c_o = 0$) Equation 12 simplifies to

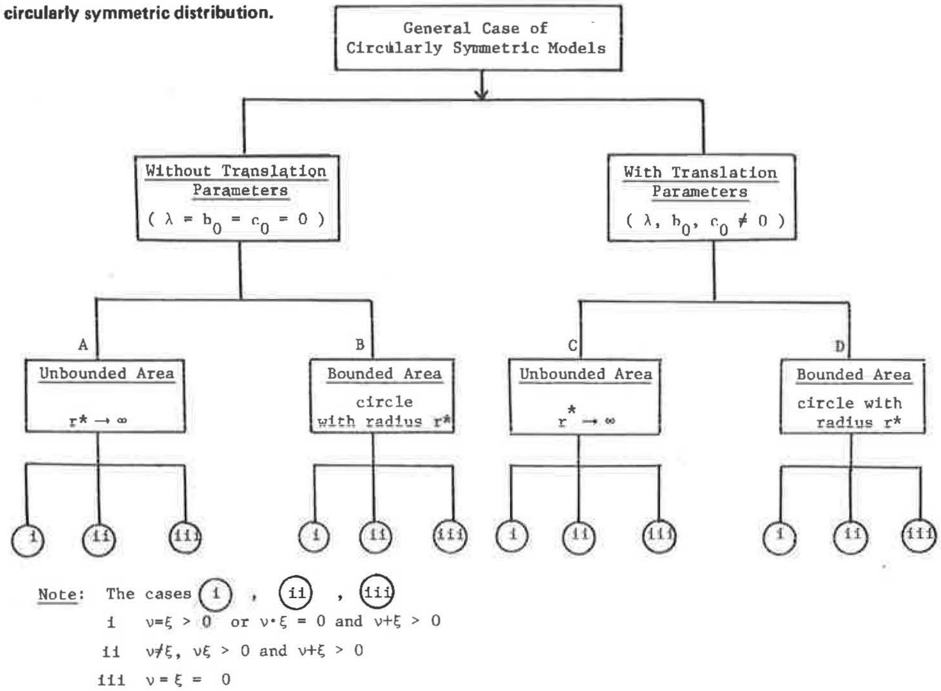
$$f(l) = l^{\gamma^*-1}\exp(-bl^{\nu} - \delta l^{\xi})/\int_0^{r^*} l^{\gamma^*-1}\exp(-bl^{\nu} - \delta l^{\xi})\,dl \quad (13)$$

where γ^* is identical to $\gamma - c$.

Equation 12 is the general form of the circularly symmetric continuous logit trip length distribution. The integral in the denominator cannot be evaluated analytically except for special cases. These analytic solutions are given in the following section for the classification of special cases shown in Figure 2. Each special case is defined as a combination of the following:

1. With or without translation parameters,
2. Finite or infinite radius r*, and
3. One of the following ranges of values for ν and ξ: (a) $\nu = \xi>0$ or $\nu\cdot\xi = 0$ and $\nu + \xi>0$, (b) $\nu \neq \xi$, $\nu\cdot\xi>0$ and $\nu + \xi>0$, or (c) $\nu = \xi = 0$.

Figure 2. Special cases of circularly symmetric distribution.



TRIP LENGTH DISTRIBUTIONS

The complete list of the analytic trip length density functions according to the classification of Figure 2 is given in Litinas and Ben-Akiva (39). This section summarizes the key results and discusses specific assumptions and the relations among special cases. Particular emphasis is placed on the special cases that correspond to the models that have appeared in the literature.

Models for Unbounded Urban Area Without Translation Parameters

In this case, the trip length density function is given by Equation 13, with $r^* = \infty$. Two major families of models are derived: the generalized gamma and the shifted generalized gamma. For the third case the distribution vanishes.

Generalized Gamma Models ($\nu = \xi = \nu^*$ or $\nu\xi = 0$ and $\nu + \xi > 0$)

The trip length density is given by

$$f(l) = \text{Generalized gamma}\,(\nu^*, \gamma^*, b^*) \tag{14}$$

where b^* is identical to $b + \delta$. Note that for the case $\xi = 0$ or $\delta = 0$ the model is

$$f(l) = \text{Generalized gamma}\,(\nu, \gamma^*, b) \tag{15}$$

and for the case $c = 0$ and $\xi = 0$ (or $\delta = 0$), it is

$$f(l) = \text{Generalized gamma}\,(\nu, \gamma, b) \tag{16}$$

Thus, the models in Equations 14-16, which are based on different assumptions, have the same form.

The generalized gamma model is equivalent to the model used by Blumenfeld and others (8) and labeled as the generalized Clark and Sherratt model. Depending on the particular values of ν^*, four types of distributions can be distinguished within the generalized gamma family: gamma, generalized Gauss, generalized Weibull, and other.

Gamma Models ($\nu^* = 1$)

For $\nu^* = 1$ or $[\nu = 1$ and $\delta = 0$ (or $\xi = 0)]$ or $[\nu = 1$ and $\delta = 0$ (or $\xi = 0)$ and $c = 0]$ the generalized gamma density becomes the gamma density:

$$f(l) = \text{Gamma}\,(\gamma^*, b^*) \tag{17}$$

This model was studied by Ajo (3), Aynvarg (4), and Blumenfeld (7) and has had numerous applications (20-22, 29,33,37,40-42).

Special cases of the gamma model are obtained for different values of γ^*. The model for $\gamma^* = 1$ is

$$f(l) = \exp(b^*) \tag{18}$$

In this model, if $\delta = 0$ (or $\xi = 0$) the opportunities are assumed to decline with the reciprocal of distance. For $\gamma^* = 2$, the model is

$$f(l) = \text{Gamma}\,(2, b^*) \tag{19}$$

This model with $\delta = 0$ (or $\xi = 0$) and $c = 0$ represents an assumption of a featureless plane with uniformly distributed opportunities (27) and is equivalent to an exponential density at a point (11). It has had numerous applications (1,2,7,19, 21,23,27,32-34,43-48).

Generalized Gauss Models ($\nu^* = 2$)

For $\nu^* = 2$ or $[\nu = 2$ and $\xi = 0$ (or $\delta = 0)]$ or $[\nu = 2$ and $\delta = 0$ (or $\xi = 0)$ and $c = 0]$, the model is

$$f(l) = \text{Generalized Gauss}\,[\gamma^*, 0, (1/2b^*)] \tag{20}$$

It includes as special cases the following distributions: normal, Rayleigh (which is also called circular normal), and Maxwell (which is also sometimes called spherical normal). It is equivalent to the generalized Sherratt model used by Blumenfeld (7).

Special cases of the generalized Gauss model are obtained by specific values of γ^*. The model for $\gamma^* = 1$ is

$$f(l) = 2 \text{ Normal } [0, (1/2b^*)] \quad (21)$$

For $\delta = 0$ (or $\xi = 0$) and $c = 0$ it corresponds to declining opportunities with the reciprocal of distance and increasing travel costs with the square of distance. For $\gamma^* = 2$, the model is

$$f(l) = \text{Rayleigh } [0, (1/2b^*)] \quad (22)$$

which is the same as the Sherratt model for population density at a point. In rectangular coordinates the density at a point (x,y) is an independent bivariate normal with zero means and variances $1/2b^*$. For $\delta = 0$ (or $\xi = 0$) and $c = 0$ it represents a trip length distribution under a featureless plane assumption. The Rayleigh and the normal models have been used in numerous applications (5,7,20-22,32,33,49). The model for $\gamma^* = 3$ is

$$f(l) = \text{Maxwell } [0, (1/2b^*)] \quad (23)$$

Generalized Weibull Models ($\nu^* = n\gamma^*$)

The generalized Weibull distribution is obtained from the generalized gamma by letting $\nu^* = n\gamma^*$, where n is a positive integer.

$$f(l) = \text{Generalized Weibull } (\gamma^*, b^*, n) \quad (24)$$

For $n = 1$ it becomes the Weibull density,

$$f(l) = \text{Weibull } (\gamma^*, b^*) \quad (25)$$

This model was also derived by Mogridge (29). The model for $n = 2$ is

$$f(l) = \text{Modified Weibull } (\gamma^*, b^*) \quad (26)$$

Other Models

Other values of ν^* result in other types of distributions. A special case of interest is the model for $\nu^* = 1/2$,

$$f(l) = \text{Generalized square root } \exp(\gamma^*, b^*) \quad (27)$$

from which more specialized cases can be obtained for different values of γ^*. For example, for $\gamma^* = 1$ the model is

$$f(l) = \text{Square root } \exp(b^*) \quad (28)$$

and for $\gamma^* = 2$, it becomes

$$f(l) = \text{Generalized square root } \exp(2, b^*) \quad (29)$$

The last model was used by Ajo (1), Cassetti (21), and Papageorgiou (22).

Shifted Generalized Gamma Models ($\nu \neq \xi$ and $\nu\xi > 0$, $\nu + \xi > 0$)

The shifted generalized gamma distribution is defined by Equation 13 with $r^* = \infty$ and by restricting the exponents ν, ξ to unequal positive values. A closed form solution for the whole family of these models does not exist. However, solutions may be obtained for certain specific values of ν and ξ. The shifted generalized Gauss distribution is obtained when one exponent equals 2 and the other equals 1.

For $\nu = 2$ and $\xi = 1$ the model is

$$f(l) = \text{Shifted generalized Gauss } (\gamma^*, -\delta/2b, 1/2b) \quad (30)$$

The following special cases are of interest:

$$\text{For } \gamma^* = 1, f(l) = \text{Shifted normal } (-\delta/2b, 1/2b) \quad (31)$$

$$\text{For } \gamma^* = 2, f(l) = \text{Shifted Rayleigh } (-\delta/2b, 1/2b) \quad (32)$$

$$\text{For } \gamma^* = 3, f(l) = \text{Shifted Maxwell } (-\delta/2b, 1/2b) \quad (33)$$

For $\nu = 1$ and $\xi = 2$ the models are obtained by interchanging δ and b in the above distributions.

The shifted Rayleigh trip length distribution is equivalent to the shifted normal for population density at a point. It has numerous applications (6, 20,21,32-34).

Models for Bounded Urban Area Without Translation Parameters

For a bounded urban area without translation parameters three general families of models are derived from Equation 13:

1. Truncated generalized gamma density,
2. Truncated shifted generalized gamma density, and
3. Power density.

Distributions 1 and 2 have identical kernels to the distributions of the generalized gamma and the shifted generalized gamma. In many previous applications these models were applied in a form that contains the kernel of a distribution multiplied by a constant that was not derived explicitly. The power density is obtained from Equation 13 for $\nu = \xi = 0$,

$$f(l) = \text{Power density } (\gamma^*, r^*) \quad (34)$$

This distribution was proposed by Harwood (9) and Smeed (10,11), and applied by Pearce and others (32) and Pearce (50).

Models for Unbounded Urban Area with Translation Parameters

Three general families of models are derived from Equation 12 with $r^* = \infty$.

Combined Generalized Gamma Models ($\nu^* = \nu = \xi$)

It was not possible to obtain a closed-form solution for the general model except for the special cases with equal translation parameters, $\lambda = c_o = b_o = \lambda^*$. The model for $\nu^* = 1$ is independent of b_o and in general can be expressed as a sum of gamma densities.

For $\gamma^* = 2$ it reduces to gamma $(2,b^*)$. For $\gamma^* = 3$ it results in a convex combination of gamma $(2,b^*)$ and gamma $(3,b^*)$,

$$f(l) = [b^*\lambda^\#/(2 + b^*\lambda^\#)] \text{ gamma } (2, b^*) + [2/(2 + b^*\lambda^\#)] \text{ gamma } (3, b^*) \quad (35)$$

where $\lambda^\# = c_o = \lambda$.

This model divided by $2\pi l$ is equivalent to the model of population density at a point proposed by Reinhart (13) and further analyzed by Casetti (21) and Papageorgiou (22). Other values of γ^* result in different combinations of gamma densities.

Solutions can also be obtained for special cases of $\lambda \neq c_o$. For example, letting $c_o = 0$ results in a combination of a gamma 2 with an exponential density.

The general model for $\nu^* = 2$ is the combined generalized Gauss density. For $\gamma^* = 2$ it reduces to the shifted Rayleigh density $(\lambda^*, 1/2b^*)$ which has the same form as Equation 32 but with different interpretation of the parameters. The case $\gamma^* = 3$

results in a combination of a shifted Mawell with a shifted Rayleigh density.

Combined Shifted Generalized Gamma Models ($\nu \neq \xi$ and $\nu\xi > 0$, $\nu + \xi > 0$)

In the combined shifted generalized gamma models case even the special case for $\nu = 2$, $\xi = 1$ or $\nu = 1$, $\xi = 2$ could not be solved in closed form. However, for $\lambda = c_o = \lambda^{\#}$ and $\gamma^* = 2$ the shifted Rayleigh density is obtained and $\gamma^* = 3$ results in a combination of a shifted Maxwell with a shifted Rayleigh. These models have the same functional forms as those of the models for the combined generalized gamma models with $\nu^* = 2$ with different interpretations of the parameters.

Combined Generalized Beta Prime Models ($\nu = \xi = 0$)

The general form of the combined generalized beta prime models can be expressed as a combination of generalized beta prime densities. The model for $c_o > \lambda > 0$ is a combination of two generalized beta primes $[(\gamma, c - \gamma, \lambda, c_o)$ with $(\gamma - 1, c - \gamma + 1, \lambda, c_o)]$, which for $\lambda = 0$ reduces to

$$f(l) = \text{Generalized beta prime}\,(\gamma, c - \gamma, 0, c_o) \qquad c > \gamma > 0 \tag{36}$$

and further simplifies to

$$f(l) = \text{Beta prime}\,(\gamma, c - \gamma) \tag{37}$$

for $c_o = 1$. Similarly, the model for $\lambda > c_o > 0$ is another combination of two generalized beta primes $[(2 - c, c - \gamma, c_o, \lambda)$ with $(1 - c, c - \gamma + 1, c_o, \lambda)]$. For $c_o = 0$ this model reduces to

$$f(l) = \text{Generalized beta prime}\,(2 - c, c - \gamma, 0, \lambda) \qquad \gamma < c < 2 \tag{38}$$

and to

$$f(l) = \text{Beta prime}\,(2 - c, c - \gamma) \tag{39}$$

for $\lambda = 1$. For the third possibility of $\lambda = c_o = \lambda^{\#} > 0$, the model is

$$f(l) = \text{Generalized beta prime}\,(2, c - \gamma, 0, \lambda^{\#}) \qquad c > \gamma \tag{40}$$

which for $\lambda^{\#} = 1$ becomes

$$F(l) = \text{Beta prime}\,(2, c - \gamma) \tag{41}$$

The beta prime densities in Equation 37 with $\gamma = 2$, Equation 39 with $c = 0$, and Equation 41 correspond to the simple potential models proposed and used by Stewart and Warntz (51) and Warntz (52) for rural population densities. The generalized beta prime models in Equation 36 with $\gamma = 2$, Equation 38 with $c = 0$, and Equation 40 correspond to a modified simple potential model for population densities investigated by Papageorgiou (22). Casetti (21) used the simple potential model with good results in peripheral areas of a number of cities.

Models for Bounded Urban Area With Translation Parameters

For models for bounded urban area with translation parameters Equation 12 is used to derive the truncated counterparts of the distributions in the models for unbounded urban area with translation parameters.

In the combined generalized beta prime models for the following opportunity density function

$$\gamma(l) = \gamma_o^* (r^* - l)^{\gamma - 2} \tag{42}$$

and for $c_o = 0$, the model is

$$f(l) = \text{Generalized beta}\,(2 - c, \gamma - 1, 0, r^*) \tag{43}$$

This model with $c = 1$ (or $c = 0$ for the population density at a point) corresponds to a model proposed by Mills (12) and discussed by McDonald and Bowman (33).

ESTIMATION OF PARAMETERS

The purpose of this section is to relate the parameters of the models with the transportation system performance and the decisionmaker characteristics. For illustrative purposes, take the case of travel by automobile (A).

First consider performance of the transportation system. Continuous surfaces are used to describe the travel time and travel cost per unit distance at each point of the urban space. Consider the following circularly symmetric travel time surface centered at the traveler's origin (see Figure 1 for the coordinate system):

$$t_A(l, \phi \mid r_w, \theta_w) = (c_A'/l + c_{oA}) + b_{lA}'(l + b_{oA})^{\nu_A - 1} \tag{44}$$

where $t_A(l, \phi \mid r_w, \theta_w)$ is the travel time [min/unit distance at the point (l, ϕ)] and c_A', c_{oA}, b_{lA}', b_{oA}, ν_A are parameters that can be estimated from observation and can be influenced by policies.

The first term of Equation 44 decreases with increasing l, and the behavior of the second term depends on the value of ν_A. For $0 < \nu_A < 1$ it also decreases with l but less rapidly than does the first term. For $\nu_A = 1$ it is constant, and for $\nu_A > 1$ it increases with l. It is reasonable to assume that $\nu_A \geq 1$. Then, the effect of the first term is more important in locations close to the travelers' origin and the weight of the second term is greater for distant locations. This functional form has enough flexibility to allow the representation of a wide range of travel time fields observed in urban areas and used by Blumenfeld and Weiss (36) and Angel and Hyman (37).

Consider the following travel cost surface:

$$C_A(l, \phi \mid r_w, \theta_w) = AVMMC_A \cdot t_A(l, \phi \mid r_w, \theta_w) \tag{45}$$

where $C_A(l, \phi \mid r_w, \theta_w)$ is travel cost [¢/unit distance (l, ϕ)] and $AVMMC_A$ is the average monetary cost of travel by automobile (¢/min).

Equation 45 implies that the travel cost per unit distance increases as the travel speed decreases. The average travel cost per minute ($AVMMC_A$) can be approximated from an average travel cost per mile ($AVMC_A$) as follows:

$$AVMMC_A = AVMC_A \cdot \bar{V}_A/60 \tag{46}$$

where $\bar{V}_A$ is average travel speed (miles/h) and $AVMC_A$ is average monetary cost of travel by automobile (¢/mile).

The average cost per mile can be related to gasoline price, fuel efficiency, and other costs such as maintenance costs as follows:

$$AVMC_A = (GPRICE/MPG) + MC \tag{47}$$

where

GPRICE = gasoline price (¢/gal),
MPG = miles per gallon of gasoline, and
MC = maintenance costs (¢/mile).

A different travel cost surface can be derived from the following relation of gasoline consumption to specific automobile characteristics:

$$\varphi_A(l,\phi|r_w,\theta_w) = K_{1A} + (K_{2A}/60)t_A(l,\phi|r_w,\theta_w) \quad (48)$$

where

$\varphi_A(l,\phi|r_w,\theta_w)$ = gasoline consumption [gal/unit distance at (l,ϕ)],
K_{1A} = gasoline consumed to overcome the rolling resistance (gal/unit distance), and
K_{2A} = gasoline consumed to overcome mechanical losses (gal/h).

The above equation has been investigated by several researchers in numerous experiments. This equation was found to adequately explain fuel consumption for different drivers who are driving normally in urban traffic and for speed $\lesssim$70 km/h [see for example, Evans and others (53), Evans and Herman (54,55), Chang and others (56), and Chang and Herman (57)]. The use of this equation is associated with a simplified fuel-consumption model theoretically derived by Amann and others (58). The parameters K_{1A} and K_{2A} can be inferred from the weight of the car and the idle fuel flow rate as follows:

$$K_{1A} = k_{1A}W_A \quad (49a)$$
$$K_{2A} = k_{2A}I_A \quad (49b)$$

where

W_A = weight of the car (lb),
I_A = idle fuel flow rate (gal/h), and
k_{1A}, k_{2A} = constants.

Evans and Herman (55) provide values of k_{1A}, k_{2A}, W_A, and I_A for various cars. Based on Equation 48 the travel cost surface is derived as follows:

$$C_A(l,\phi|r_w,\theta_w) = MC + GPRICE\,\varphi(l,\phi|r_w,\theta_w) \quad (50)$$

This cost surface allows a more elaborate analysis of automobile-related policies (for example, the effect of smaller-size cars on gasoline consumption).

The generalized cost surface $[B_{tA}(l,\phi|r_w,\theta_w)]$, which expresses the disutility per unit distance at the point (l,θ) perceived by traveler t, is derived from the above travel time and travel cost surfaces as follows. Assume that the disutility of travel is a linear combination of travel times and travel costs. Then, $B_{tA}(l,\phi|r_w,\theta_w)$ can be written as,

$$B_{tA}(l,\phi|r_w,\theta_w) = MUTT_t\, t_A(l,\phi|r_w,\theta_w) + (MUTT_t/VT_t)C_A(l,\phi|r_w,\theta_w) \quad (51)$$

where $MUTT_t$ is the marginal utility of travel time for decisionmaker t, which can be inferred from existing estimated discrete logit models, and VT_t is the value of time for decisionmaker t (¢/min).

The value of time (VT_t) is often estimated as a percentage of the wage rate as follows:

$$VT_t = (PWRVT_t)(INC_t)/1200 \quad (52)$$

where

INC_t = annual income for decisionmaker t ($),
$PWRVT_t$ = percentage of wage rate for the value of time, and
1200 = factor that converts annual income to wage rate, assuming 250 working days/year, (¢/min).

Substitution of the above travel time and travel cost surfaces in equation 51 yields the form of Equation 1 as follows:

$$B_{tA}(l,\phi|r_w,\theta_w) = [c_{tA}/(l + c_{oA})] + b_{ltA}(l + b_{oA})^{\nu_A - 1} \quad (53)$$

where

$c = c_{tA}$,
$b_1 = b_{1tA}$,
$\nu = \nu_A$,
$c_o = c_{oA}$, and
$b_o = b_{oA}$.

For the cost surface assumption of Equation 45 the parameters are evaluated as follows:

$$c_{tA} = MUTT_t\, c'_A\Big(\big\{[(GPRICE/MPG) + MC]\,20\bar{V}_A/(PWRVT_t) \times (INC_t)\big\} + 1\Big) \quad (54a)$$

$$b_{ltA} = MUTT_t\, b'_{lA}\Big(\big\{[(GPRICE/MPG) + MC]\,20\bar{V}_A/(PWRVT_t) \times (INC_t)\big\} + 1\Big) \quad (54b)$$

For the cost assumption of Equation 50 and for the case of $\nu_A = 1$ the following expressions for the parameters are obtained:

$$c_{tA} = MUTT_t\, c'_A\,\big\{1 + [20k_{2A}I_A GPRICE/(PWRVT_t)(INC_t)]\big\} \quad (55a)$$

$$b_{tA} = MUTT_t\big\{b'_{lA} + [20k_{2A}I_A b'_{lA} GPRICE/(PWRVT_t)(INC_t)] + [1200/(PWRVT_t)(INC_t)]\,(MC + k_{1w}W_A GPRICE)\big\} \quad (55b)$$

Note that in this case $b_{tA} = b_{1tA}$.

Thus, all the parameters of the generalized cost surface of Equation 1 have been related to the transportation system performance and the decisionmaker characteristics. The β function derived from this surface is given in Equation 2. The average utility function (V) given in Equation 6 equals the sum of this β function with the trip origin, mode, and traveler-specific constants. However, the additional parameter (a) of Equation 6 (denoted here as a_{tAw}) does not enter the expressions for the automobile trip length distributions and therefore it will not be evaluated here in terms of other variables.

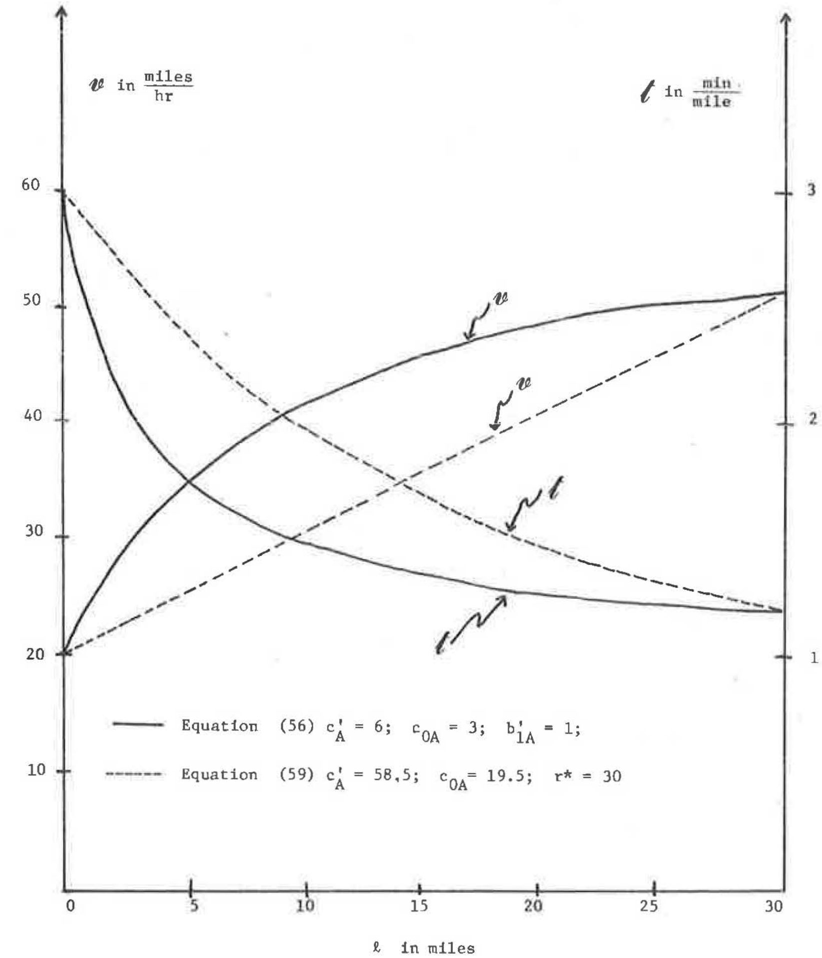
The above relations give a behavioral interpretation and a method of calculation for the parameters of all the derived trip length distributions and permits the use of these models for simplified policy analysis.

Below, two special cases of the above results are presented in more detail. For these cases a slightly different interpretation of the parameters is also possible, such as the case $\nu_A = 1$. This case covers a broad range of the derived trip length models. The travel time surface of Equation 44 for $\nu_A = 1$ is

$$t_A(l,\phi|r_w,\theta_w) = [c'_A/(l + c_{oA})] + b'_{lA} \quad (56)$$

This surface has been used by Blumenfeld and Weiss

Figure 3. Simplified velocity and travel time surface.



(36) and its shape is shown in Figure 3. Note that the following relation exists between its parameters:

$$c'_A/c_{oA} = tt^o_A - tt^*_A \qquad (57)$$

where tt^*_A is the free-flow travel time [min/unit distance (i.e., $l = \infty$)], which is equal to b'_{1A} and tt^o_A is the travel time [min/unit distance at (r_w, θ_w)].

Apply the previously described assumptions to derive the average utility function as follows:

$$V_{tA}(l) = -a_{tAw} - c_{tA} \ln(l + c_{oA}) - b_{tA}\, l \qquad (58)$$

To get the parameter values of Equation 58, first estimate c'_A, c_{0A}, and b'_{1A} from travel time field observations. Assume the travel cost surface of Equation 45 and obtain estimates of GPRICE, MPG, MC, $\bar{V}_A$, $PWRVT_t$, INC_t, and $MUTT_t$. Substitute these values in Equation 54 to compute c_{tA}, b_{1tA}. For this case $b_{tA} = b_{1tA}$.

For the case of a bounded urban area another interpretation of Equation 58 is possible. Assume that the travel speed is linearly increasing from the traveler's origin to the city boundary. The resulting travel time surface (see Figure 3) is as follows:

$$t_A(l, \phi | r_w, \theta_w) = c'_A/(l + c_{oA}) \qquad (59)$$

where

$$c_{oA} = r^* V^o_A/(V^*_A - V^o_A), \qquad (60a)$$

$$c'_A = 60\, r^*/(V^*_A - V^o_A), \qquad (60b)$$

V^o_A = travel speed (mph at traveler's origin),
V^*_A = travel speed (mph at r^*), and
r^* = the radius of city boundary from traveler's origin (miles).

Assume that the travel cost surface is given by Equation 50. Then substitute Equation 60 in Equation 55 for $b'_{1A} = 0$ to obtain the values of c_{tA} and b_{tA}.

For the case of $c'_A = 0$ it is also required that $b_{oA} = 0$. The travel time surface is then

$$t_A(l, \phi | r_w, \theta_w) = b'_{lA}\, l^{\nu_A} \qquad (61)$$

Assume that the travel cost surface of Equation 45 is applicable. Then, these assumptions result in the following utility function:

$$V_{tA}(l) = -a_{tAw} - b_{tA}\, l^{\nu_{tA}} \qquad (62b)$$

where b_{tA} is given by Equation 54b. Note that this utility function covers all the trip length distributions of categories A and B (i.e., without translation parameters) by substituting $\gamma^* = \gamma$.

Now consider the following alternative behavioral assumption that leads to the same functional form of the utility function. The decisionmaker perceives the disutility of travel as a generalized cost to the ν_t power. Then, the following average utility is derived

$$V_{tA}(l) = -a_{tAw} - b_{tA}\, l^{\nu_A} \tag{62a}$$

where

$$\nu_{tA} = \nu_t \nu_A \tag{63a}$$

and

$$b_{tA} = \mathrm{MUTT}_t \left[(b'_{iA}/\nu_A) \left(\left\{ [(\mathrm{GPRICE}/\mathrm{MPG}) + \mathrm{MC}]\, 20\bar{V}_A \div (\mathrm{PWRVT}_t)(\mathrm{INC}_t) \right\} + 1 \right) \right]^{\nu_t} \tag{63b}$$

This interpretation allows the use of models that have an exponent $\nu_{tA} \neq 1$ under the assumption of a constant travel time surface (i.,e., $\nu_A = 1$). For this case, the more elaborate travel cost surface of Equation 50 may also be used.

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Publication of this paper sponsored by Committee on Passenger Travel Demand Forecasting.

Optimal Bus Scheduling on a Single Route

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This paper develops a method for scheduling bus runs on a single route so as to minimize the total waiting time for patrons. The demand for bus travel is assumed to be time-dependent with a given origin-destination pattern. The problem is formulated as a mathematical program subject to bus capacity and possible other (e.g., service standards) constraints. A dynamic programming procedure is suggested for the solution of this program. Finally, some issues associated with optimization of the schedule under stochastic demand are explored as well.

The conventional wisdom in setting up a schedule for a bus route typically involves supplying enough capacity at the maximum load point, subject to some service standards. This approach is useful when the period for which the schedule (timetable) is set is one in which the arrival rate of patrons is constant, and when this period is long in comparison with the bus roundtrip time on the route. When these conditions are not met (for example, the maximum load point may not be stationary or the peak period may be shorter than a bus trip), the schedule may not be optimal in the sense that unnecessary crowding may exist on some buses and slack capacity may exist on others. Better scheduling may bring about a higher level of service and increased productivity.

The problem referred to in this paper is that of preparing a desirable schedule for a given route, not of scheduling the actual buses to the runs. In other words, bus availability and interlining considerations are not taken into account. The inputs to the schedule preparation problem discussed here are the route geometry (including stops and speeds between stops), the total number of bus runs to be operated, and the desired trip rates (i.e., the demand pattern). The output is the route's schedule.

The objective of the schedule preparation is to find the timetable that would give passengers the maximum level of service for a given level of resources. The level of service is expressed in terms