tion. Mathematical models are suggested to evaluate signal coordination feasibility limits between a pair of intersections.

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Optimization Model for Isolated Signalized Traffic Intersections

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The existing methods for the optimization of isolated fixed-time signalized traffic intersections are applicable either to undersaturated stationary conditions or to oversaturated conditions. As far as is known, no model exists that is applicable to all conditions. A model is developed for the optimization of fixed-time signalized intersections that is applicable to undersaturated as well as to oversaturated conditions. In the model, the macroscopic approach to traffic flow is used. Although it is not so accurate as the microscopic approach, values are obtained for delay and number of stops that are accurate enough for practical purposes and that use much less computer time. Macroscopic simulation is then approximated by the geometric probability distribution. In this case also, values for delay and number of stops are obtained that are accurate enough for practical purposes and that use much less computer time. Consequently, the geometric probability distribution model is recommended for the optimization of fixed-time signalized traffic intersections.

The purpose of this paper is the development of a model for the optimization of fixed-time signalized intersections.

Most of the research in the field of signalized intersections has been done for undersaturated conditions. In this paper, however, we shall not refer to specific shortcomings, but as a result of these shortcomings, it has been decided to develop an accurate model for practical application to undersaturated and oversaturated conditions.

First, microscopic and macroscopic simulation are compared in the stationary zone with reference to average delay and number of stops. The difference is found to be negligible for practical purposes, and macroscopic simulation is used in the further development of the model because it uses much less computer time.

Second, average delay and number of stops are determined by macroscopic simulation in the nonstationary zone. Good agreement is found between the values obtained at the end of the nonstationary zone and those in the stationary zone. Macroscopic simulation in the nonstationary zone can therefore be deemed correct (see Figure 1).

Last, macroscopic simulation is approximated by the geometric probability distribution to further

reduce computer time. Good agreement is found for all practical purposes, and the geometric model is therefore recommended for the optimization of fixedtime signalized intersections.

COMPARISON BETWEEN MICROSCOPIC AND MACROSCOPIC SIMULATION

Macroscopic traffic flow at a signalized intersection is indicated in Figure 2, which shows average arrivals per unit time interval (q), overflow of vehicles at the end of the previous cycle (Q_B), overflow of vehicles at the end of the cycle (Q_E), cycle length (c) in seconds, effective green time (g) in seconds, effective red time (r) in seconds, and saturated flow (s) in vehicles per second. The total delay per cycle (D) is the area under the queue-length diagram:

$$D = [(2 \cdot Q_{B} + q \cdot r)r/2] + [(q \cdot r + Q_{B} + Q_{E})g/2]$$
(1)

The number of stops per cycle (N) is the number of vehicles that arrive while there is a queue plus the overflow at the start of the cycle (Q_B) :

$$N = c \cdot q + Q_B \tag{2}$$

Microscopic traffic flow is indicated in Figure 3.

In the macroscopic case, arrival of vehicles per cycle is obtained by generating random numbers. In the microscopic case, gaps between vehicles are obtained similarly.

By working from a zero origin, the times of arrival and departure are obtained; thus the delay is experienced. By summation of the delay for all vehicles, the total delay (D) is obtained. The average delay (d) is then the total delay divided by the sum of all the vehicles arriving during the period considered.

The number of stops is obtained as follows.

Figure 1. Transition from nonstationary to stationary zone as flow increases.







Figure 3. Arrivals and departures at signalized intersection.



If a vehicle is delayed, it is counted as a stop and if a vehicle does not depart during the cycle in which it arrives, an extra stop is counted. The average number of stops is obtained by dividing the total number of stops (N) by the total number of arrivals during the period considered.

The values for average delay and average number of stops as obtained for microscopic and macroscopic simulation are indicated in Table 1, from which it is clear that the difference between microscopic and macroscopic simulation is, for practical purposes, negligible. Because it uses less computer time, macroscopic simulation is used for further analysis.

MACROSCOPIC SIMULATION MODEL

Analysis

The following equations are used:

 $Q_E = Q_B + q \cdot c - s \cdot g$ (3)

 $P(Q_E) = P(Q_B) \cdot P(q \cdot c) \cdot P(s \cdot g)$ (4)

Table 1. Average delay and average number of stops for microscopic and macroscopic simulation.

c		S	x	q	Microscopic Simulation		Macroscopic Simulation	
	g				d	n	d	n
40	12	0.5	0.83	450	25.92	1.23	25.67	1.27
	24	0.5	0.74	800	8.17	0.73	8.20	0.85
60	18	0.5	0.83	450	30.59	1.09	30.77	1.14
	36	0.5	0.74	800	10.74	0.71	10.62	0.81
80	24	0.5	0.83	450	36.23	1.04	36.15	1.07
	48	0.5	0.74	800	13.46	0.71	13.24	0.79
100	32	0.5	0.78	450	35.95	0.94	35.76	0.97
	64	0.5	0.69	800	12.94	0.63	12.68	0.69
120	38	0.5	0.79	450	42.30	0.94	42.24	0.97
	76	0.5	0.70	800	15.66	0.64	15.52	0.70

Notes: Average delay per vehicle in seconds: $d = D/(q \cdot c)$. Average number of stops per vehicle: $n = N/(q \cdot c)$. Ratio of average number of arrivals per cycle to the maximum number of departures per cycle: $x = (q \cdot c)/(s \cdot c)$. q = average number of arrivals per hour.

 $P(D) = P(Q_B) \cdot P(q \cdot c) \cdot P(s \cdot g)$ (5)

 $P(N) = P(Q_B) \cdot P(q \cdot c) \cdot P(s \cdot g)$ (6)

$$E(D) = \sum_{i} D_{i} \cdot P(D_{i})$$
⁽⁷⁾

$$E(N) = \sum N_i \cdot P(N_i)$$
(8)

where

q c = average number of arrivals per cycle,

s•g = number of departures per cycle,

 $P(Q_E) = probability of overflow Q_E$,

P(D) = probability of total delay, P(N) = probability of total number of stops,

E(D) = expected value of total delay, and

E(N) = expected value of total number of stops.

P(q.c) is obtained from the arrival distribution. P(s.g) is obtained from the departure rate, and P(QB) is obtained as follows.

Assume zero flow initially. With zero flow. there can be no overflow at the end of the cycle; thus $Q_E = 0$. But Q_E becomes Q_B for the next cycle; therefore, $Q_B = 0$ for the first cycle and it is clear that $P(Q_B = 0) = 1$ and that $P(Q_B = 1, 2, 3, ...) = 0$. The probability diagram is therefore as indicated in Figure 4.

By varying the number of arrivals per cycle (q.c) between zero and j so that

$$\sum_{i=0}^{j} P(q \cdot c = i) \ge 0.9999$$
(9)

and by substituting the possible values of $Q_{\rm B}$ and q c in Equation 3, different values for Q_E are obtained.

The procedure will be illustrated with an example. Assume that sog is an integer such that $P(s \cdot g) = 1.$

Example 1

If we use cycle length (c) of 40 s, saturated flow (s) of 0.5 vehicle/s, flow (q) of 800 vehicles/h, Poisson arrivals, and effective green time (g) of 16 s,

s•g = 0.5 • 16 = 8 vehicles departing per cycle,

 $m = q \cdot c = (800 \cdot 40)/3600 = 8.888 888 9$ arrivals per cycle.

It is found that j in Equation 9 equals 22. Therefore, q•c is varied between zero and 22. For the first cycle, $Q_{\rm B}$ = 0.

Put $Q_B = 0$ and s•g = 8 in Equation 3; then

 $Q_E = q \cdot c - 8 \tag{10}$

Figure 4. Probability distribution diagram for Q_B in first cycle.



Table 2. Probability of QE.

q.c	$Q_{\rm E}$	$P(Q_E)$	q·c	$Q_{\rm E}$	$P(Q_E)$
0	-8	0.000 137 9	12	4	0.070 054 9
1	-7	0.001 225 9	13	5	0.047 900 8
2	-6	0.005 448 4	14	6	0.030 413 2
3	-5	0.016 143 5	15	7	0.018 022 6
4	-4	0.035 874 4	16	8	0.010 012 6
5	-3	0.063 776 6	17	9	0.005 235 3
6	-2	0.094 483 9	18	10	0.002 585 4
7	-1	0.119 979 6	19	11	0.001 209 5
8	0	0.133 310 6	20	12	0.000 537 6
9	1	0.131 664 8	21	13	0.000 227 5
10	2	0.117 035 4	22	14	0.000 091 9
11	3	0.094 574 1			

Figure 5. Probability distribution diagram for Q_E.



Table 3. Average delay and number of stops by macroscopic simulation.

С	g	s			Stationary Condition		Nonstationary Condition	
			x	q	d	n	d	n
40	12	0.5	0.83	450	25.67	1.27	25.68	1.27
	24	0.5	0.74	800	8.20	0.85	8.21	0.86
60	18	0.5	0.83	450	30.77	1.14	30.82	1.14
	36	0.5	0.74	800	10.62	0.81	10.64	0.81
80	24	0.5	0.83	450	36.15	1.07	36.27	1.08
	48	0.5	0.74	800	13.24	0.79	13.23	0.79
100	32	0.5	0.78	450	35.76	0.97	35.79	0.97
	64	0.5	0.69	800	12.68	0.69	12.67	0.69
120	38	0.5	0.79	450	42.24	0.97	42.25	0.97
	76	0.5	0.70	800	15.52	0.70	15.51	0.70

By substituting $q \cdot c$ in Equation 10 and $P(Q_B = 0) = 1$, $P(s \cdot g) = 1$, and $P(q \cdot c)$ from the Poisson model in Equation 4, the values of $P(Q_E)$ in Table 2 are obtained.

Q_E cannot be negative; thus

$$P(Q_E = 0) = \sum_{i=-8}^{0} P(Q_E = i) = 0.4703808$$

By adjusting the values in Table 2 so that Σ $P\left(Q_{\rm E}\right)$ = 1, the probability diagram in Figure 5 is obtained.

For the next cycle, $Q_{\rm E}$ becomes $Q_{\rm B}.$ Values of $Q_{\rm B}$ from zero to 14 are therefore available. For each value of $Q_{\rm B}$, the series of values of q·c from zero to 22 is substituted in Equation 3 and a probability distribution diagram for $Q_{\rm E}$ for each value of $Q_{\rm B}$ is obtained.

The probabilities of $Q_{\rm E}$ on these diagrams are then summed over all the diagrams, that is, over $Q_{\rm B},$ according to the following equation:

$$P(Q_E) = \sum_{Q_B} P(Q_B) \cdot P(q \cdot c) \cdot P(s \cdot g)$$
(11)

The probabilities obtained are again adjusted to sum to 1, $Q_{\rm E}$ again becomes $Q_{\rm B},$ and the probability distribution for the next cycle is determined.

This procedure is repeated until average delay becomes constant in the unsaturated case or until the increase in average delay from cycle to cycle becomes constant in the oversaturated case.

If this constant average delay in the nonstationary undersaturated case is equal to the average delay as obtained for stationary conditions, then the method whereby average delay is obtained for nonstationary conditions by macroscopic simulation can be deemed to be correct.

The values obtained for average delay and average number of stops for stationary and nonstationary conditions are indicated in Table 3.

From Table 3 it is clear that the differences are negligibly small. Thus the nonstationary analysis can be deemed correct.

APPROXIMATING MACROSCOPIC SIMULATION BY GEOMETRIC PROBABILITY DISTRIBUTION MODEL

The following form of the geometric distribution is suggested as an approximation model:

$$P(K) = (1 - f)f^{K}$$
(12)

where P(K) is the probability of a queue of length K at the start of the cycle and f is the probability of a queue at the start of the cycle. Let

$$f = E(Q_B)/[1 + E(Q_B)]$$
 (13)

In another paper in this Record, I have shown that for the geometric approximation model the expected overflow at the end of the cycle is

$$E(Q_E) = E(Q_B) + E(q \cdot c) - E(s \cdot g)$$

-
$$\sum_{s \cdot g} P(s \cdot g) \sum_{q \cdot c = 0}^{s \cdot g - 1} P(g \cdot c) [E(Q_B)(1 - f^{s \cdot g - q \cdot c}) + q \cdot c - s \cdot g]$$
(14)

The expected number of stops is

$$E(N) = E(Q_B) + E(q \cdot c) + \sum_{s \cdot g} P(s \cdot g) \sum_{q \cdot c^{=} 0}^{s \cdot g^{-} 1} P(q \cdot c) \left((q \cdot c)/c / \left\{ [(s \cdot g)/g] - [(q \cdot c)/c] \right\} \right) [E(Q_B)(1 - f^{s \cdot g - q \cdot c}) + q \cdot c - s \cdot g]$$
(15)



Table 4. Macroscopic simulation and geometric model at undersaturation.

с					Simulation		Geometric Mode	
	g	S	q	x	d	n	d	n
40	12	0.5	450	0.83	25.68	1.27	22.26	1.19
	24	0.5	800	0.74	8.21	0.86	7.88	0.85
60	18	0.5	450	0.83	30.82	1.14	27.93	1.09
	36	0.5	800	0.74	10.64	0.81	10.45	0.81
80	24	0.5	450	0.83	36.27	1.08	33.82	1.05
	48	0.5	800	0.74	13.23	0.79	13.12	0.79
100	32	0.5	450	0,78	35.79	0.97	35.02	0.97
	64	0.5	800	0.69	12.67	0.69	12.66	0.69
120	38	0.5	450	0.79	42.25	0.97	41.53	0.96
	76	0.5	800	0.70	15.51	0.70	15.50	0.70

and the expected total delay is

 $E(D) = E(Q_B) \cdot c + 0.5[E(q \cdot c) \cdot c - E(s \cdot g)g] + \sum_{s \cdot g} P(s \cdot g)$

 $x \sum_{q\cdot c=0}^{s\cdot g-1} P(q\cdot c) \left(\frac{1}{2} \left\{ [(s\cdot g)/g] - [(q\cdot c)/c] \right\} \right)$ $x \left\{ E(Q_B) \left[2 - f^{s\cdot g-q\cdot c} (1+f) \right] + (q\cdot c - s\cdot g)^2 - (q\cdot c - s\cdot g - 1)^2 f_{l} \right\}$ $x \left[1/(1-f) \right]$ (16)

By applying Equations 12 through 16 to example 1, the probability distribution below is obtained:

QB	$P(Q_B)$			QB	$P(Q_B)$			
0	0.376	540	9	8	0.008	595	6	
1	0.234	757	9	9	0.005	359	0	
2	0.146	361	9	10	0.003	341	1	
3	0.091	250	7	11	0.002	083	1	
4	0,056	891	1	12	0.001	298	7	
5	0.035	469	2	13	0.000	809	7	
6	0.022	113	6	14	0.000	504	8	
7	0.013	786	9					

The probability distributions in Table 2 and the tabulation above are indicated in Figure 6.

In Table 4, macroscopic simulation is compared with the geometric model at undersaturation and in Table 5 at oversaturation.

From Figure 6 and Tables 4 and 5, it is clear that there is close agreement between macroscopic simulation and the geometric model.

An application to a two-phase intersection is illustrated in example 2. The intersection data are given below $(T_1, T_2, \text{ and } T_3 \text{ are consecutive time periods in seconds}):$

Phase	s	g	q 1	Tl	g2	T ₂	q 3	T ₃
1	0.5	16	400	1800	600	2400	800	1200
2	0.5	18	500	1800	750	2400	1000	1200

Table 5. Macroscopic simulation and geometric model at oversaturation.

с			x	q	Simulation		Geometric Model	
	g	S			đ	n	d	n
40	24 24	0.5 0.5	1.00 1.10	1080 1188	36.46 69.38	1.72 2.60	34.91 70.68	1.68 2.63
60	36 36	0.5 0.5	1,00 1,10	$\begin{array}{c} 1080 \\ 1188 \end{array}$	46.86 99.03	1.58 2.51	44.70 100.74	1.55 2.54
80	48 48	0.5 0.5	1.00 1.10	$1080 \\ 1188$	56.25 128.54	1.50 2.47	53.58 130.54	1.47 2.50
100	64 64	0.5 0.5	1.00 1.10	1152 1267	61.59 155.38	1.43 2.44	58.52 157.47	1.40 2.46
120	76 76	0.5 0.5	1.00 1.10	1140 1254	69.99 185.20	1.40 2.42	66.51 187.41	1.37 2.44

Rates of $3.1 \cdot 10^{-2}$ rand/stop and $1.74 \cdot 10^{-4}$ rand/s for total delay, as obtained from research by the National Institute for Transportation and Road Research in Pretoria, Republic of South Africa, are used. The rand is the unit of currency in the Republic of South Africa [1 rand = \$0.78 (1983 U.S.)].

The results obtained for a range of cycle lengths are indicated below, from which it is seen that the minimum cost occurs at the same underlined cycle length:

	Cost (rands)	
C	Simulation Model	Geometric Model
40	122.54	115.24
50	95.13	91.24
60	82.56	79.77
70	74.60	72.29
80	70.90	68.82
90	65.70	63.95
100	65.45	63.83
110	59.08	57.65
120	62.79	61.34

CONCLUSIONS

It is clear from the results indicated in the graphs and tables that the differences between macroscopic simulation and the geometric model are for all practical purposes negligibly small. The fact is substantiated by the results obtained for example 2 and indicated at the end of the previous section. The geometric approximation model for the cost optimization of fixed-time signalized traffic intersections is therefore recommended because it uses much less computer time.

Only the Poisson probability distribution model was used in the research. I have shown $(\underline{1})$ that, irrespective of which probability model of the Poisson, binomial, and negative binomial is used for the arrival of vehicles at a signalized intersection, the minimum cost occurs at the same cycle length. The Poisson distribution, being simpler, was therefore used in this research.

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