Signal--II: Numerical Comparisons of Some Theoretical Expressions. Transportation Science, Vol. 6, 1972.
6. W.B. Cronjé. A Model for Use in the Optimization of Fixed-Time Signalized Intersections. Transactions of the Annual Transportation Convention, Vol. 4, Pretoria, Republic of South Africa, 1982.

*Publication of this paper sponsored by Committee on Traffic Control Devices.*

# Derivation of Equations for Queue Length, Stops, and Delay for Fixed-Time Traffic Signals

W.B. CRONJÉ

**The existing methods for the calculation of queue length, number of stops, and delay for isolated traffic intersections are applicable either to undersaturated stationary conditions or to oversaturated conditions. As far as is known, no model exists that is applicable to all conditions. Equations are derived for the calculation of queue length, number of stops, and delay for isolated fixed-time signalized intersections that are applicable to undersaturated as well as to oversaturated conditions. In the derivation the macroscopic approach to traffic flow is used. This approach has been shown to be sufficiently accurate for practical purposes. Traffic flow at a signalized intersection is considered a Markov process. Equations are derived for expected queue lengths, expected number of stops, and expected total delay. These equations can also be used for the optimization of isolated fixed-time signalized intersections.**

Traffic flow at a signalized intersection is a Markov process. The states being considered are the queue lengths at the beginning and the end of the signal cycle, and the time interval over which changes in these states take place is the length of the signal cycle.

The equation governing the states is as follows:

$$Q_E = Q_B + q \cdot c - s \cdot g \qquad Q_E \nless 0 \tag{1}$$

where

$Q_E$ = overflow of vehicles at end of cycle,
$Q_B$ = overflow of vehicles from previous cycle,
$q$ = average arrivals per unit time interval,
$c$ = cycle length (s),
$g$ = effective green time (s),
$s$ = saturated flow (vehicles/s),
$q \cdot c$ = number of arriving vehicles per cycle, and
$s \cdot g$ = maximum number of departing vehicles per cycle.

Equation 1 can be represented by the transition probability matrix shown in Figure 1, in which $P(Q_B,Q_E)$ is the probability of transition from state $Q_B$ to state $Q_E$, $P(s \cdot g)$ is the probability distribution of departing vehicles, and $P(q \cdot c)$ is the probability distribution of arriving vehicles.

Equation 1 is illustrated by Figure 2, in which r is effective red time in seconds.

## DERIVATION

Consider one approach to an intersection controlled by a fixed-time signal. Consider one cycle on the approach in which vehicles are expected to arrive according to a distribution $P(q \cdot c)$. The saturation-flow vehicles are distributed according to $P(s \cdot g)$.

Let $P(Q_B,Q_E)$ be the probability of an overflow of $Q_E$ vehicles at the end of the cycle, given an overflow $Q_B$ at the start of the cycle. The form of this probability is shown as a matrix in Figure 3.

The expected overflow at the end of the cycle is given by

$$E(Q_E) = \sum_{Q_E=0}^{\infty} Q_E \cdot P(Q_E)$$

$$= \sum_{s \cdot g} P(s \cdot g) \left[ \sum_{q \cdot c=0}^{\infty} P(q \cdot c) \sum_{Q_B=0}^{\infty} (Q_B + q \cdot c - s \cdot g) P(Q_B) - \sum_{q \cdot c=0}^{s \cdot g-1} P(q \cdot c) \sum_{Q_B=0}^{s \cdot g-q \cdot c-1} (Q_B + q \cdot c - s \cdot g) P(Q_B) \right]$$

$$= E(Q_B) + E(q \cdot c) - E(s \cdot g) - \sum_{s \cdot g} P(s \cdot g) \sum_{q \cdot c=0}^{s \cdot g-1} P(q \cdot c) \sum_{Q_B=0}^{s \cdot g-q \cdot c-1} (Q_B + q \cdot c - s \cdot g) P(Q_B) \tag{2}$$

A queue-length diagram with overflow at the end of the cycle is indicated in Figure 4.

The total number of stops per cycle is the number of vehicle arrivals while there is a queue. Overflow vehicles stop twice.

If there is overflow at the end of the cycle, the total number of stops per cycle is as follows:

$$N = Q_B + q \cdot c \tag{3}$$

A queue-length diagram without overflow at the end of the cycle is indicated in Figure 5.

If there is no overflow at the end of the cycle, the total number of stops per cycle is as follows:

$$N = Q_B + r \cdot q + [(Q_B + r \cdot q)/(s - q)]\, q \tag{4}$$

**Figure 1. Transition probability matrix for Equation 1.**

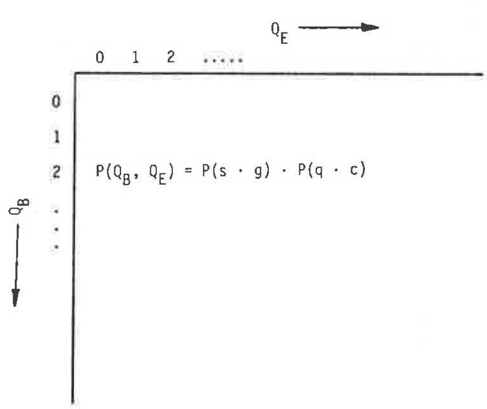


Figure 2. Queue-length diagram illustrating Equation 1.

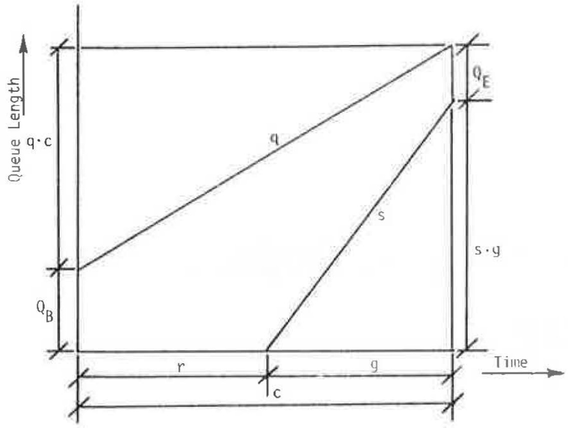
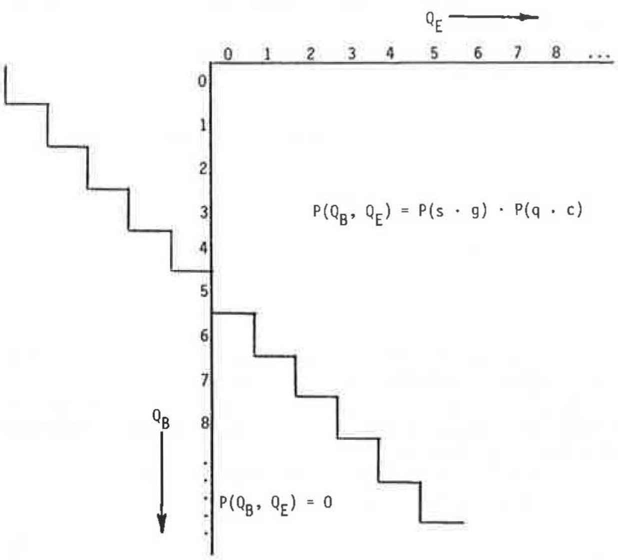



Figure 3. Transition probability matrix.




Replacing r in Equation 4 by (c − g) gives

$$N = Q_B + c{\cdot}q + [(q{\cdot}c)/c]\,\big((Q_B + q{\cdot}c - s{\cdot}g)/\{[(s{\cdot}g)/g] - [(q{\cdot}c)/c]\}\big) \tag{5}$$

Except for the third term of Equation 5, Equations 3 and 5 are identical. The third term of Equation 5 clearly covers the case of no overflow at the end of the cycle and is applicable to the zone to the left of the origin in Figure 3.

The expected number of stops per cycle is therefore given by

$$E(N) = \sum_0^\infty N{\cdot}P(N)$$

$$= \sum_{s\cdot g} P(s{\cdot}g) \left\{ \sum_{q\cdot c=0}^{\infty} P(q{\cdot}c) \sum_{Q_B=0}^{\infty} \Big[Q_B + q{\cdot}c + [(q{\cdot}c)/c] \big((Q_B + q{\cdot}c - s{\cdot}g)/\{[(s{\cdot}g)/g] - [(q{\cdot}c)/c]\}\big)\Big] \cdot P(Q_B) \right\}$$

$$= E(Q_B) + E(q{\cdot}c) + \sum_{s\cdot g} P(s{\cdot}g) \sum_{q\cdot c=0}^{s\cdot g-1} P(q{\cdot}c) \sum_{Q_B=0}^{s\cdot g-q\cdot c-1} P(Q_B) \cdot [(q{\cdot}c)/c]$$

$$\times\ (Q_B + q{\cdot}c - s{\cdot}g)/\{[(s{\cdot}g)/g] - [(q{\cdot}c)/c]\} \tag{6}$$

Figure 4. Queue-length diagram with overflow at end of cycle.

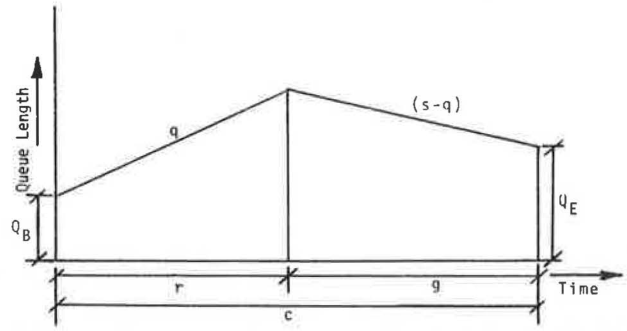
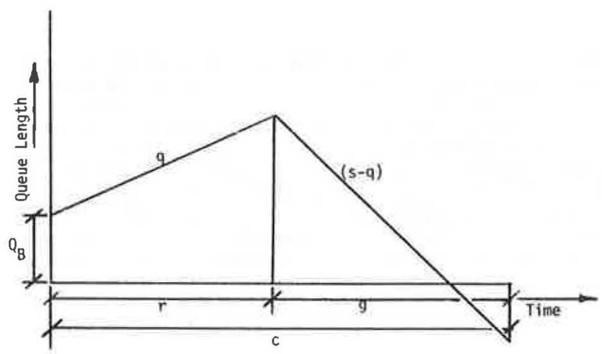



Figure 5. Queue-length diagram without overflow at end of cycle.




A queue-length diagram with overflow at the end of the cycle can also be represented by Figure 2.

Total delay is the area under the queue-length curve.

If there is overflow at the end of the cycle, total delay is given by (see Figure 2)

$$D = Q_B{\cdot}c + 0.5(q{\cdot}c{\cdot}c - s{\cdot}g{\cdot}g) \tag{7}$$

If there is no overflow at the end of the cycle, total delay is given by (see Figure 5)

$$D = Q_B \cdot r + q{\cdot}r \cdot (r/2) + [(Q_B + q{\cdot}r)/(s - q)] \cdot [(Q_B + q{\cdot}r)/2] \tag{8}$$

Replacing r in Equation 8 by (c − g) gives

$$D = Q_B \cdot c + 0.5(q{\cdot}c{\cdot}c - s{\cdot}g{\cdot}g) + 0.5\,\big((Q_B + q{\cdot}c - s{\cdot}g)$$

$$\div \{[(s{\cdot}g)/g] - [(q{\cdot}c)/c]\}\big) \tag{9}$$

If the same reasoning is applied to Equations 7 and 9 as in the case of the number of stops, the expected total delay is given by

$$E(D) = c \cdot E(Q_B) + 0.5[c \cdot E(q{\cdot}c) - g \cdot E(s{\cdot}g)] + \sum_{s\cdot g} P(s{\cdot}g) \sum_{q\cdot c=0}^{s\cdot g-1} P(q{\cdot}c)$$

$$\times \sum_{Q_B=0}^{s\cdot g-q\cdot c-1} P(Q_B)\big((Q_B + q{\cdot}c - s{\cdot}g)^2/\{[(s{\cdot}g)/g] - [(q{\cdot}c)/c]\}\big) \tag{10}$$

Assume that the overflow at the start of the cycle is distributed according to the following geometric distribution:

$$P(Q_B) = (1 - f)\,f^{Q_B} \tag{11}$$

with

$$f = E(Q_B)/[1 + E(Q_B)] \tag{12}$$

Some properties of the geometric probability distribution are

$$\sum_{i=0}^{n} (1 - f) f^{i} = 1 - f^{n+1} \tag{13}$$

$$\sum_{i=0}^{n} (i - n)(1 - f) f^{i} = [f/(1 - f)](1 - f^{n}) - n \tag{14}$$

$$\sum_{i=0}^{n} (i - n)^{2}(1 - f) f^{i} = [1/(1 - f)]\{[f/(1 - f)][2 - f^{n}(1 + f)] + n^{2} - f(n + 1)^{2}\} \tag{15}$$

The last part of Equation 2 is as follows:

$$\sum_{Q_B=0}^{s \cdot g - q \cdot c - 1} (Q_B + q \cdot c - s \cdot g)P(Q_B) \tag{16}$$

In expression 16, put n = (s•g - q•c) and substitute $P(Q_B)$ from Equation 11. Then expression 16 becomes

$$\sum_{Q_B=0}^{n-1} (Q_B - n)(1 - f) f^{Q_B} = \sum_{Q_B=0}^{n} (Q_B - n)(1 - f) f^{Q_B} - (n - n)(1 - f) f^{n} = \sum_{Q_B=0}^{n} (Q_B - n)(1 - f) f^{Q_B} \tag{17}$$

If Equation 14 is applied, Equation 17 becomes

$$[f/(1 - f)](1 - f^{n}) - n = \{E(Q_B)/[1 + E(Q_B)]\}/(1 - \{E(Q_B) \div [1 + E(Q_B)]\})(1 - f^{n}) - n = E(Q_B)(1 - f^{n}) - n = E(Q_B)(1 - f^{s \cdot q - q \cdot c}) - s \cdot g + q \cdot c$$

Equation 2 therefore becomes

$$E(Q_E) = E(Q_B) + E(q \cdot c) - E(s \cdot g) - \sum_{s \cdot g} P(s \cdot g) \sum_{q \cdot c = 0}^{s \cdot g - 1} P(q \cdot c) \times [E(Q_B)(1 - f^{s \cdot g - q \cdot c}) + q \cdot c - s \cdot g] \tag{18}$$

The transformation of Equation 6 is identical, which gives

$$E(N) = E(Q_B) + E(q \cdot c) + \sum_{s \cdot g} P(s \cdot g) \sum_{q \cdot c = 0}^{s \cdot g - 1} P(q \cdot c) \left( [(q \cdot c)/c]/\{[(s \cdot g)/g] - [(q \cdot c)/c]\} \right) [E(Q_B)(1 - f^{s \cdot g - q \cdot c}) + q \cdot c - s \cdot g] \tag{19}$$

The numerator of the last part of Equation 10 is

$$\sum_{Q_B=0}^{s \cdot g - q \cdot c - 1} P(Q_B)(Q_B + q \cdot c - s \cdot g)^{2} \tag{20}$$

If the same transformation is applied, expression 20 becomes

$$\sum_{Q_B=0}^{n} (Q_B - n)^{2}(1 - f) f^{Q_B} - (n - n)(1 - f) f^{n} = \sum_{Q_B=0}^{n} (Q_B - n)^{2} \times (1 - f) f^{Q_B} \tag{21}$$

If Equation 15 is applied, Equation 21 becomes

$$[1/(1 - f)]\{f/(1 - f)[2 - f^{n}(1 + f)] + n^{2} - (n + 1)^{2} f\} = [1/(1 - f)]\{E(Q_B)[2 - f^{s \cdot g - q \cdot c}(1 + f)] + (s \cdot g - q \cdot c)^{2} - (s \cdot g - q \cdot c + 1)^{2} f\}$$

Equation 10 therefore becomes

$$E(D) = c \cdot E(Q_B) + 0.5[c \cdot E(q \cdot c) - g \cdot E(s \cdot g)] + \sum_{s \cdot g} P(s \cdot g) \sum_{q \cdot c = 0}^{s \cdot g - 1} P(q \cdot c) \times \left( 1/2\{[(s \cdot g)/g] - [(q \cdot c)/c]\} \right)\{E(Q_B)[2 - f^{s \cdot g - q \cdot c}(1 + f)] + (q \cdot c - s \cdot g)^{2} - (s \cdot g - q \cdot c + 1)^{2} f\}[1/(1 - f)] \tag{22}$$

where Equation 18 gives $E(Q_E)$, the expected overflow at the end of the cycle; Equation 19 gives E(N), the expected number of stops per cycle; and Equation 22 gives E(D), the expected total delay per cycle.

## CONCLUSIONS AND RECOMMENDATIONS

I have shown in another paper in this Record that the equations developed in this paper for expected queue length, expected number of stops, and expected total delay show very close agreement with simulation.

They are also applicable to undersaturated as well as to oversaturated conditions. By assigning monetary rates to number of stops and delay as calculated by Equations 19 and 22 and varying the green times for the various phases, the optimum cycle length at the minimum cost can be obtained.

*Publication of this paper sponsored by Committee on Traffic Control Devices.*