Statistical Evaluation of Random Versus Stratified Random Sampling for Pavement Test Sections

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In bituminous highway and runway pavement construction, specification procedures sometimes call for the selection of test specimens from a relatively small, rectangular section of pavement. These small sections are known as control strips, test strips, or test sections. The trial pavement sections are used to verify that the mix design, plant operations, and laydown procedures can meet specification requirements or to establish target values against which to evaluate actual production paving or both. An analysis and evaluation of two sampling schemes, random sampling and stratified random sampling, for use with pavement sections are presented. In both sampling schemes, the sample mean obtained is used as the estimator of the population parameter of interest, e.g., pavement density. Because of the small size of the test section it is assumed that the sample results are correlated and that the correlation between two sampling locations decreases exponentially with distance. Equations are developed to determine the theoretical variance of the sample mean by using the two sampling schemes. It is shown that as determined by the lower sampling variance, stratified sampling is preferred over random sampling when target characteristics of pavements are obtained.

In bituminous highway and runway pavement construction, specification procedures sometimes call for the selection of test specimens from a relatively small, rectangular section of pavement. These small sections are known as control strips, test strips, or test sections. There are typically two reasons for the use of these small trial pavement sections before actual pavement construction.

The first is to establish a target value against which to measure the results achieved during the actual pavement construction. This approach is referred to as the control-strip method and is employed by several state highway agencies (1) to establish a target value for the in-place pavement density. In this method, measurements are made on a small stretch of pavement, the control strip, to determine whether specification requirements for minimum in-place density have been achieved. The results of these measurements on the control strip establish a target density against which the densities achieved during paving operations can be evaluated.

The second use for a preliminary trial pavement section is to verify that the mix design, plant operations, and laydown procedures can satisfactorily meet the requirements and tolerances set forth in the specifications before the actual paving operations begin. In this application, the trial pavement section is usually referred to as a test strip or test section. The Federal Aviation Administration Eastern Region specification (2) for bituminous surface course is an example of a specification requiring a test section before approval of the commencement of paving. The reason for the test section is presented quite clearly in the note to the engineer that accompanies the specification (2): "The test section affords the Contractor and the Engineer an opportunity to determine the quality of the mixture in place, as well as performance of the plant and laydown equipment."

OBJECTIVES

Typical practice in deciding where to take measurements on the test section is to use a random-sampling procedure to determine the sampling location. This procedure differs from that usually employed for the actual pavement construction. The sampling procedure usually employed for the in-place pavement is to take a stratified random sample. In this approach the lot of the pavement to be evaluated is divided into a number of sublots, commonly four or five, and a sample is randomly selected from within each sublot. The reason for this sampling plan is to assure that all samples are not taken from a relatively small section of the total lot but are spread over the entire pavement length. It is assumed that stratified random sampling is not required on the test section because of its relatively small size and because the material for the test section is produced at the plant during a short time frame, thereby eliminating any chance of changes in the production process that may occur through the course of a day.

Nevertheless, the small size of the test sections assures that they will be located much closer together than those during the actual pavement construction. This proximity of testing locations yields a higher likelihood of correlation between the test results. It seems reasonable to assume that two sampling locations that are close together are more likely to yield similar test results than two locations that are far apart. One procedure to reduce the possibility of correlation effects is to use a stratified random sample on the test section similar to that employed on the actual production paving.

The objective of this paper is to evaluate the performance of two sampling plans, random and stratified random, in establishing a target value from a control strip or test section.

ANALYSIS PROCEDURE

The case to be considered in the analysis is the development of a target density as the average of a number of measurements taken on a control strip or test section. The test section requirements that will be used in the analysis are those from the FAA P-401 specification (section 401-3.3, Test Section) (2):

> Prior to full production, the Contractor shall prepare a quantity of bituminous mixture according to the job mix formula. The amount of mixture should be sufficient to construct a test section at least 100 feet (30.5 m) long and two spreader widths wide. . . . Four (4) samples of finished pavement, and four (4) samples that span the longitudinal joint, shall be randomly taken and tested to determine conformance to acceptance criteria.

In practice, in accordance with specification requirements, the sample mean based on a random sample is used to develop the target density. In this paper, the sample mean, well known to be an unbiased estimator for the population mean, is used as the estimator for the target density for stratified as well as random sampling. Expressions for the variance of the estimator for each of the two sampling plans are derived and compared.

In the derivation of the expressions for the variance of the estimated mean, it is assumed that a relationship exists between the distance separating two sampling points and the correlation between the measurements taken at those points. Intuitively, it is reasonable to assume that two points that are closer together will produce measurements that are more closely related. This is because the material is more likely to have come from the same truckload and to have been compacted under similar localized subgrade and temperature conditions. Specifically, the following relationship is assumed to apply between the correlation coefficient (ρ) and the distance between two points:

$$\rho(d_{ij}) = \exp(-\alpha d_{ij}^2) \quad (1)$$

where

α = some constant,
d_{ij} = distance between specimens i and j, and
$\rho(d_{ij})$ = correlation between specimens i and j.

Figure 1. Plots of correlation coefficient between samples as a function of distance between those samples for various α-values.

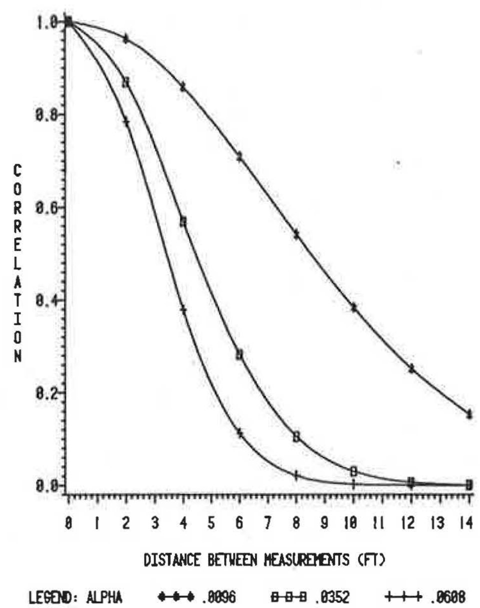


Figure 2. Random sampling of n points in n x a section of pavement.

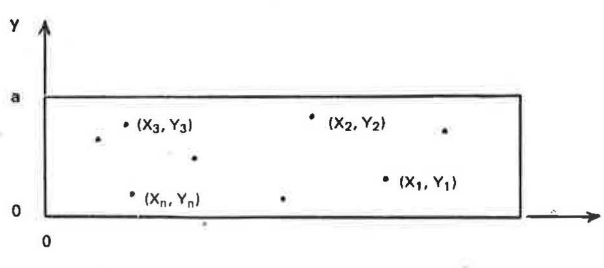
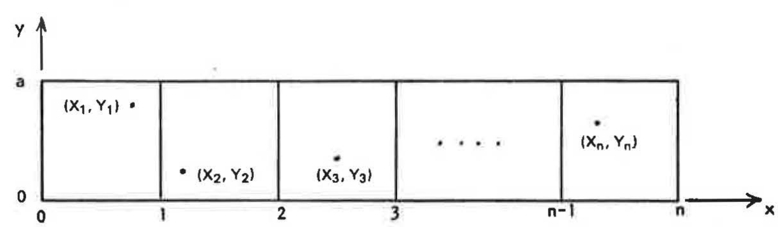


Figure 3. Stratified sampling of n points in n partitions of n x a section of pavement.



This relationship is selected to provide a derivation for the expected value of the correlation coefficient between two points that is analytically tractable. This relationship does, however, conform to what would be expected intuitively. Equation 1 yields high correlation values at close spacings between points that decrease exponentially with increasing distance between the points. As shown in Figure 1, the decrease in the correlation coefficient with distance is established by the selection of α.

Once expressions have been derived for the variance of the estimated mean for each sampling plan, these will be applied to several test section sampling situations that may develop under the FAA P-401 specification to determine which sampling scheme provides the mean estimate with the lower sampling variance.

THEORY

Random sampling is performed by choosing n measurement locations independently over the entire area of pavement. This means that the location of the ith measurement (X_i, Y_i) can be considered a random vector with X_i and Y_i having a joint uniform distribution over the range $0 < X_i < n$ and $0 < Y_i < a$ (see Figure 2). [Note: Without loss of generality the upper bound on X_i can be scaled to be n.]

Stratified random sampling is performed by choosing n measurement locations, one location in each of n 1 x a rectangles into which the strip of pavement has been partitioned. Here each pair of X_i and Y_i has a joint uniform distribution over the range $(i - 1) \leq X_i < i$ and $0 < Y_i < a$ (see Figure 3).

The distance (D_{ij}) between any two sampling locations i and j can be expressed as a function of the random variables U_{ij} and V_{ij}, defined as follows:

$$D_{ij} = [(X_i - X_j)^2 + (Y_i - Y_j)^2]^{1/2}$$
$$= (U_{ij}^2 + V_{ij}^2)^{1/2}$$

For random sampling U_{ij} and V_{ij} have, respectively, the following probability density functions:

$$f(u) = (u + n)/n^2 \quad -n \leq u \leq 0$$
$$= (n - u)/n^2 \quad 0 \leq u \leq n$$

and

$$f(v) = (v + a)/a^2 \quad -a \leq v \leq 0$$
$$= (a - v)/a^2 \quad 0 \leq v \leq a$$

For stratified random sampling for two points separated by p blocks, U_{ij} has the following probability density:

$$h(u) = u - p \quad p \leq u \leq p + 1$$
$$= p + 2 - u \quad p + 1 \leq u \leq p + 2$$

V_{ij} has probability density g(v) as defined above.

At each location (X_i, Y_i) some measurement of pavement quality (S_i) is taken. Assume that the S_i

are distributed with mean μ and variance σ^2. For both sampling plans the sample mean $(\overline{S})$ is to be used to estimate μ. Because $\overline{S}$ is an unbiased estimator of μ, the variance of $\overline{S}$ using random sampling $[\mathrm{Var}_r(\overline{S})]$ must be compared with the variance of $\overline{S}$ using stratified random sampling $[\mathrm{Var}_s(\overline{S})]$.

As discussed earlier, it is reasonable to assume that the S_i, $i = 1, \ldots, n$, are not independent. Furthermore, the correlation between S_i and S_j depends on the distance (D_{ij}) between (X_i, Y_i) and (X_j, Y_j). Let $\underline{D}$ represent the random vector of the $\binom{n}{2}$ distances between n locations. Thus

$$\begin{aligned}\mathrm{Var}(\overline{S}) &= E_{\underline{D}}\,[\mathrm{Var}(\overline{S}\mid \underline{D})] + \mathrm{Var}_{\underline{D}}\,[E(\overline{S}\mid \underline{D})]\\ &= E_{\underline{D}}\left\{(1/n^2)\left[\sum_{i=1}^{n}\mathrm{Var}(S_i|\underline{D}) + 2\sum_{j=2}^{n}\sum_{i=1}^{j-1}\mathrm{Cov}(S_i, S_j|\underline{D})\right]\right\} + 0\\ &= E_{\underline{D}}\left\{(1/n^2)\left[n\sigma^2 + 2\sum_{j=2}^{n}\sum_{i=1}^{j-1}\mathrm{Cov}(S_i, S_j|\underline{D})\right]\right\}\\ &= (1/n^2)\left[n\sigma^2 + 2\sum_{j=2}^{n}\sum_{i=1}^{j-1}E_{D_{ij}}\mathrm{Cov}(S_i, S_j|D_{ij})\right]\end{aligned} \quad (2)$$

because the covariance between S_i and S_j depends only on the distance between their sampling locations. In the random-sampling case the D_{ij} have the same distribution for all i and j, $i \neq j$. Denoting $\mathrm{Cov}(S_i, S_j|D_{ij}) = \sigma^2\rho(D)$, Equation 2 becomes

$$\mathrm{Var}_r(\overline{S}) = (\sigma^2/n)\left\{1 + (n-1)E[\rho(D)]\right\} \quad (3)$$

In the stratified case, let $\mathrm{Cov}(S_i, S_j|D_{ij}) = \sigma^2\rho(D_p)$, where $\rho(D_p)$ is the correlation between measurements in blocks that are separated by p other blocks. In this case, Equation 2 becomes

$$\mathrm{Var}_s(\overline{S}) = (\sigma^2/n)\left\{1 + (2/n)\sum_{i=0}^{n-2}(n-1-i)E[\rho(D_i)]\right\} \quad (4)$$

As discussed earlier (Equation 1), it is assumed that the correlation between measurement S_i and S_j is

$$\rho(d_{ij}) = \exp(-\alpha d_{ij}^2)$$

For a random distance D_{ij}

$$\begin{aligned}E[\rho(D_{ij})] &= E[\exp(-\alpha D_{ij}^2)]\\ &= E\left\{\exp[-\alpha(U_{ij}^2 + V_{ij}^2)]\right\}\\ &= E[\exp(-\alpha U_{ij}^2)\exp(-\alpha V_{ij}^2)]\end{aligned} \quad (5)$$

For random sampling Equation 5 becomes

$$\begin{aligned}E[\rho(D)] &= \int_{-a}^{a}\int_{-n}^{n}\exp(-\alpha u^2)\exp(-\alpha v^2)f(u)g(v)\,du\,dv\\ &= (4/a^2n^2)\left\{n(\pi/\alpha)^{1/2}\,\Phi[n(2\alpha)^{1/2}] + (1/2\alpha)[\exp(-\alpha n^2) - 1]\right\}\\ &\quad \times\left\{a(\pi/\alpha)^{1/2}\Phi[a(2\alpha)^{1/2}] + (1/2\alpha)[\exp(-\alpha a^2) - 1]\right\}\end{aligned} \quad (6)$$

where

$$\Phi(x) = \int_0^x [1/(2\pi)^{1/2}]\exp[-(t^2/2)]\,dt$$

For the stratified case Equation 5 becomes

$$\begin{aligned}E[\rho(D_p)] &= \int_{-a}^{a}\int_{p}^{p+2}\exp(-\alpha u^2)\exp(-\alpha v^2)h(u)g(v)\,du\,dv\\ &= (2/a^2)\left\{a(\pi/\alpha)^{1/2}\,\Phi[a(2\alpha)^{1/2}] + (1/2\alpha)[\exp(-\alpha a^2) - 1]\right\}\\ &\quad \times\left\{(1/2\alpha)\exp[-\alpha(p+2)^2] - (1/\alpha)\exp[-\alpha(p+1)^2]\right.\\ &\quad + (1/2\alpha)\exp(-\alpha p^2) + (p+2)(\pi/\alpha)^{1/2}\,\Phi[(p+2)(2\alpha)^{1/2}]\\ &\quad \left. - (2p+2)(\pi/\alpha)^{1/2}\,\Phi[(p+1)(2\alpha)^{1/2}] + p(\pi/\alpha)^{1/2}\,\Phi[p(2\alpha)^{1/2}]\right\}\end{aligned} \quad (7)$$

If the limit is taken as $a \to 0$ in Equations 6 and 7, the expected correlation between measurements is obtained when sampling is done along a straight line. Because

$$\lim_{a\to 0}(2/a^2)\left\{a(\pi/\alpha)^{1/2}\,\Phi[a(2\alpha)^{1/2}] + (1/2\alpha)[\exp(-\alpha a^2) - 1]\right\} = 1$$

the expected correlations for random and stratified sampling along a straight line are, respectively,

$$E[\rho(D)] = (2/n^2)\left\{n(\pi/\alpha)^{1/2}\,\Phi[n(2\alpha)^{1/2}] + (1/2\alpha)[\exp(-\alpha n^2) - 1]\right\} \quad (8)$$

and

$$\begin{aligned}E[\rho(D_p)] &= (1/2\alpha)\exp[-\alpha(p+2)^2] - (1/\alpha)\exp[-\alpha(p+1)^2]\\ &\quad + (1/2\alpha)\exp(-\alpha p^2) + (p+2)(\pi/\alpha)^{1/2}\,\Phi[(p+2)(2\alpha)^{1/2}]\\ &\quad - (2p+2)(\pi/\alpha)^{1/2}\,\Phi[(p+1)(2\alpha)^{1/2}] + p(\pi/\alpha)^{1/2}\,\Phi[p(2\alpha)^{1/2}]\end{aligned} \quad (9)$$

Using Equations 6, 7, 8, and 9 appropriately in Equations 3 and 4 yields the necessary variances for comparing the effectiveness of the sample mean as an estimator for stratified sampling as opposed to random sampling. Such comparisons are discussed by example in the remaining sections.

APPLICATION TO TYPICAL TEST SECTIONS

Equations 3 and 4 are general in nature. In this section, these equations are applied to several test sections that are typical of those that may be encountered under the FAA specification. Two test sections, one 25 ft wide and 100 ft long (section 1) and one 25 ft wide and 200 ft long (section 2), are considered. These correspond to ratios of n to a of 4:1 and 8:1, respectively. In addition, the joint-sampling requirement for the two test sections is also considered by taking the case where a goes to zero (Equation 8). This provides the solution for a line that corresponds to the joint in the test section. A discussion follows of the results for each case considered.

Figure 4 shows the geometry of test section 1. For the specification sample size of 4, the test section is divided into four 25 x 25-ft segments for stratified sampling (Figure 5). The results of the analysis are presented in Table 1 for α-values of 0.0096, 0.0352, and 0.0608. Without loss of generality, assuming $\sigma^2 = 1$, the data in Table 1 show that the variance of the sample mean $(\sigma_{\overline{X}}^2)$ is always smaller for stratified sampling than for random sampling. Thus, statistically, stratified sampling yields a better estimator than random sampling.

Table 1 also includes the results of the analysis on test section 2. Section 2 is 25 x 200 ft (see Figure 6) and is subdivided into four 50-ft segments for stratified sampling (Figure 7). As with test

Figure 4. Geometry of test section 1.

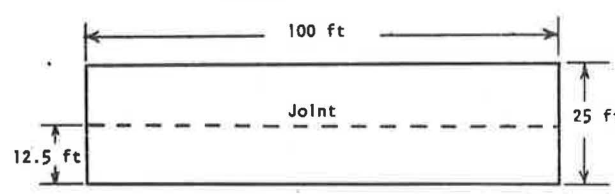


Figure 5. Stratified sampling subsections for test section 1.

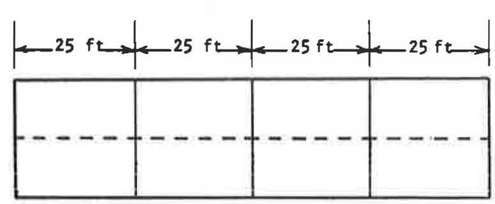


Table 1. Variance of sample mean for selected pavement sections and longitudinal joints.

	Test Sections		Longitudinal Joints	
α	Random Sampling	Stratified Sampling	Random Sampling	Stratified Sampling
Test Section 1				
0.0096	0.3212	0.2674	0.3779	0.2812
0.0352	0.2728	0.2528	0.3187	0.2585
0.0608	0.2638	0.2519	0.3027	0.2549
Test Section 2				
0.0096	0.2867	0.2544	0.3159	0.2578
0.0352	0.2616	0.2507	0.2849	0.2521
0.0608	0.2570	0.2503	0.2766	0.2512
Roadway Section				
0.0096	0.252512	0.250019		
0.0352	0.250784	0.250003		
0.0608	0.250469	0.250001		

Figure 6. Geometry of test section 2.

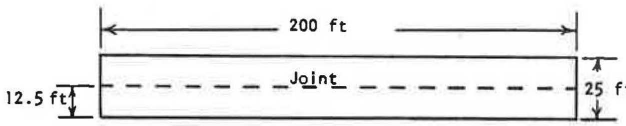
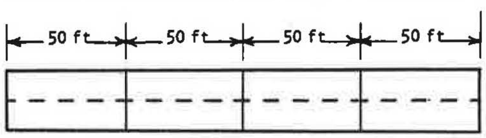


Figure 7. Stratified sampling subsections for test section 2.



section 1, stratified sampling produces a smaller variance of the sample mean for all three α-values considered.

The joint-sampling results (a = 0) for both test sections are also given in Table 1. Once again, in all instances stratified sampling yielded a lower variance of the sample mean than did random sampling.

APPLICATION TO PRODUCTION PAVING

Equations 3 and 4 can also be used to evaluate the mean estimate obtained from stratified random-sampling measurements on the in-place pavement against that obtained from a simple random-sampling plan. Because the width of the pavement, usually up to 24 ft, is much less than the length of paving in the lot, often in the thousands of feet, the production paving case approaches the case where a equals zero that was presented previously.

To evaluate the mean estimate, the variance of the estimate using random and stratified random sampling on a pavement section 25 ft wide and 3,000 ft long will be considered. The results of this analysis are given in Table 1. For this case, the values are much closer together, but stratified sampling still produces a lower sampling variance for all three α-values. The stratified sampling in this case produces sampling variances that closely approach the value for $\sigma_{\bar{x}}^2/\sigma^2$ of 2.25 that is the value when there is no correlation between the sample results. [Note: $\sigma_{\bar{x}}^2 = \sigma^2/n$; therefore $\sigma_{\bar{x}}^2/\sigma^2 = 0.25$ when n = 4.]

CONCLUSION

It has been shown by using an intuitively appealing relationship between correlation and distance that the sample mean is statistically a better estimator when stratified sampling is used than when random sampling is used. It is reasonable that this would be true for any such relationship between correlation and distance that damps exponentially. Practically, because stratified sampling does not result in extra cost, it is preferred over random sampling when target characteristics of pavement are obtained.

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