# Determining Composite Modulus of Subgrade Reaction by Use of the Finite-Element Technique

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**A finite-element computer program for rigid pavement analysis was used to develop curves for composite modulus of subgrade reaction versus subbase thickness. Such curves are useful in expediting rigid pavement thickness design when the elastic properties and the thickness of the subbase are known. These newly developed curves were compared with similar curves published by the Portland Cement Association. Little agreement was found. Results of concrete pavement analyses that used the finite-element technique were compared with the results of identical analyses that used Westergaard's slab theory. The two methods show similar trends, although the finite-element technique appears to be stiffer than Westergaard's theory. It was also found that the composite modulus of subgrade reaction depends on several variables, including slab thickness, modulus of elasticity of the subbase, and modulus of subgrade reaction.**

The strength and durability of a pavement system are based on the characteristics of its layers. Although the materials are affected by the applied loads to a depth of several feet, the layers placed close to the surface will have the greatest effect on pavement performance. The common approach has been to place the more rigid layers near the surface unless an inverted pavement is used.

Rigid pavements, which have a relatively strong resistance to bending under load, usually consist of three layers: (a) a portland cement concrete slab, (b) a subbase course (bituminous asphalt mix, gravel, etc.), and (c) a subgrade.

To ensure acceptable performance and long life for rigid pavements, the following elements are essential:

1. A thickness design in which slab stresses due to loads are kept within acceptable limits,
2. Reasonably uniform support,
3. Prevention of mud-pumping with a relatively thin subbase on projects where expected truck traffic will be heavy enough to cause this type of distress, and
4. Joint design.

Most thickness design methods for rigid pavements recognize the need for placing a subbase. In general, the subbase will be stiffer than the subgrade but not as stiff as the slab. With the addition of the subbase, the slab will be supported on a material with improved strength characteristics, which will result in a reduction in the slab thickness. These improved strength characteristics can be represented by a higher modulus of subgrade reaction (k) on the subbase. Including a subbase also (a) reduces potential joint deflection in the concrete pavement, (b) improves and maintains the effectiveness of the joint under repetitive loads for a longer period of time, and (c) provides an all-weather working platform for the paving contractor.

Several types of subbases are commonly used: untreated granular, asphalt-stabilized, and cement-treated. The choice of any of these will depend on the availability of material and on economic considerations. The Portland Cement Association (PCA) (1) has developed a set of curves representing a composite k-value on a cement stabilized subbase versus the thickness of the subbase, for different values of k on the subgrade.

The main purpose of this study was to develop a set of curves similar to the ones developed by the PCA for different values of the modulus of elasticity of the subbase material and then to compare the newly obtained curves with the ones given by the PCA. Curves developed in this fashion are useful in the thickness design of rigid pavements when the elastic properties and thickness of the subbase, as well as the modulus of subgrade reaction, are known. A finite-element computer program for the analysis of rigid slabs and Westergaard's model were used throughout the study.

## FINITE-ELEMENT METHOD IN ANALYSIS OF RIGID PAVEMENTS

Determining stresses and deflections due to wheel loads in concrete pavements has been a subject of major concern for a long time. In 1926, Westergaard (2), using the theory of elasticity and assuming that the subgrade soil behaved as a Winkler foundation (where the reactive pressure between the subgrade and the slab at any given point is proportional to the deflection at that point, with a constant of proportionality called modulus of subgrade reaction or k-value), developed mathematical expressions to determine critical stresses and deflections in rigid pavements resulting from three types of loading. These types of loading were (a) load applied near the corner of a large slab, (b) load applied near the edge of a large slab but far from any corner, and (c) load applied at the interior of a large slab at a large distance from the edges.

The method has been extensively used since then but, with the application of the finite-element technique to the analysis of rigid pavements, it is now possible to better understand slab behavior and to consider more complicated pavement profiles and types of loading. In the initial phase of the study, deflections and stresses calculated by using both Westergaard's model and the finite-element technique were compared to determine the accuracy of the finite-element technique.

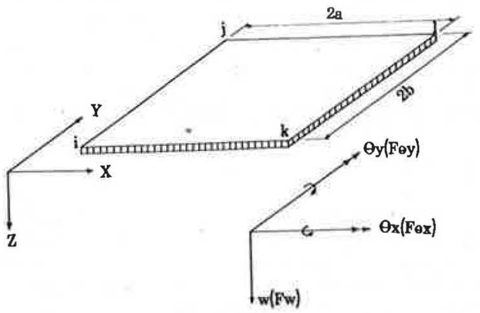
In the finite-element analysis of concrete pavements, the slab is idealized by a finite number of elements that are related to one another through common points or nodes. The engineer is responsible for choosing the coordinates of each node and numbering system that will affect the accuracy and speed of the computations. Additional data concerning material characterization (modulus of elasticity and Poisson's ratio) for each material type, locations of loads, and displacement boundary conditions are also required.

### Description of Method

The finite-element method used in the computer program is based on the classical theory of thin plates by assuming that the plane before bending remains plane after bending. In addition to the basic requirements that the slabs be homogeneous, isotropic, and elastic, it is further assumed that the subgrade acts as a Winkler foundation.

A complete description of the finite-element method as applied to the analysis of concrete pavements resting on a Winkler foundation is presented by Huang and Wang (3). The material presented here is based on their description.

Figure 1. Rectangular plate element.




Forces and Corresponding Displacements

Figure 1 shows a typical rectangular element with nodes i, j, k, and l. At each node there are three fictitious forces and three corresponding degrees of freedom (displacements). The three forces are a vertical force ($F_w$), a couple about the x-axis ($F_{\theta x}$), and a couple about the y-axis ($F_{\theta y}$). The three displacements are the deflection in the z direction (w), a rotation about the x-axis ($\theta_x$), and a rotation about the y-axis ($\theta_y$). These forces and displacements are related by the following expression:

$$\begin{Bmatrix} F_i \\ F_j \\ F_k \\ F_l \end{Bmatrix} = (K) \begin{Bmatrix} d_i \\ d_j \\ d_k \\ d_l \end{Bmatrix} + kab \begin{Bmatrix} d_i' \\ d_j' \\ d_k' \\ d_l' \end{Bmatrix} \quad (1)$$

where K is the stiffness matrix, the coefficients of which depend on dimensions a and b of the element and the Young's modulus and Poisson's ratio of the slab; k is the modulus of subgrade reaction; and, at a given node, i is

$$F_i = \begin{Bmatrix} F_{w_i} \\ F_{\Theta x_i} \\ F_{\Theta y_i} \end{Bmatrix}, d_i = \begin{Bmatrix} W_i \\ \Theta_{x_i} \\ \Theta_{y_i} \end{Bmatrix}, d_i' = \begin{Bmatrix} W_i \\ 0 \\ 0 \end{Bmatrix} \quad (2)$$

The stiffness matrix for this rectangular element was tabulated by Zienkiewics and Cheung (4) and was used in the study presented here. By superimposing the stiffness matrices over all elements and replacing the fictitious nodal forces with the statistical equivalent of the externally applied loads, a set of simultaneous equations was obtained for solving the nodal displacements. The nodal moments and stresses were then computed from the nodal displacements by using the stress matrix also tabulated by Zienkiewics and Cheung (4). Because the stresses at a given node computed by means of one element might be different from those of the neighboring elements, the stresses in all adjoining elements are computed and their average values are obtained.

## Computer Program

The computer program used in this study is called the Modified Kentucky Finite-Element Model. The program is a modified version of a computer program for finite-element analysis of rigid pavements based on the theory of plates on a Winkler foundation developed by Huang and Wang (3,5) at the University of Kentucky. Huang and Wang (3) compared Westergaard's exact solution for an infinite slab with a concentrated load on one edge far from any corner with the finite-element solution for a large slab. They concluded that the finite-element technique checked closely with Westergaard's results.

The finite-element model provides solutions for deflections and stresses due to loading or curling in concrete pavements consisting of one, two, or four slabs with load transfer at the joints. This study dealt with only one slab and did not consider the effect of transfer devices. The slab was assumed to be in full contact with the underlying soil layer and was divided into rectangular elements of various sizes. The elements and nodes were numbered consecutively from bottom to top along the y-axis and from left to right along the x-axis.

The main modification introduced at the University of Illinois by Tabatabaie and Barenberg (6) has been the consideration of a stabilized base under the slab by assuming either perfect bond or no bond between the slab and the subbase. The thickness and modulus of elasticity of the slab and subbase layer as well as the subgrade reaction can be varied from node to node.

Because of the versatility of the computer program, many situations that could occur during pavement life can be represented during the design stage by the finite-element model. These situations appear in many cases to be fundamental in reducing pavement serviceability. With previous models, there had not been an appropriate representation of these key parameters of pavement performance.

## Program Input Variables

This research required the solution of two sets of examples: the first set analyzed the slab directly on top of the subgrade soil, whereas in the second set a subbase was placed between the slab and the subgrade. By using either maximum stress or maximum deflection at the surface of the slab as a parameter for comparison, an equivalent k-value on the subbase could be obtained.

The following input parameters are used in the calculations.

### Loading

A single tire load equal to 9,000 lb was applied on a square area at the edge of the concrete slab with a uniform tire pressure equal to 80 psi. The magnitude of this load corresponds to the load per tire of an 18-kip single-axle load, which is used as the equivalent load in most design methods. Load and tire pressure remained constant in all calculations.

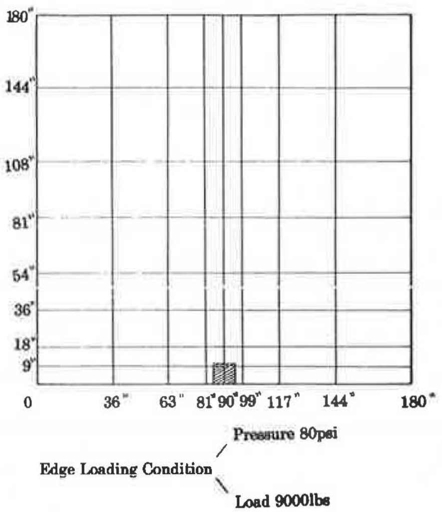
Edge loading was chosen because it is the one that gives the highest slab stress among edge, interior, and corner loading. In addition, this type of loading is quite common on pavements when vehicles travel close to the edge. The maximum stress is found directly under the load and compression at the top. Maximum surface deflection is also found at the same point.

### Slab Size and Mesh

Common two-lane highways are 30 ft wide with a longitudinal joint at the center. Statistical studies presented by Darter (7) show that plain concrete pavements crack in pieces 15 to 30 ft long when continuously built. This cracking is mainly caused by to the effects of curling (moisture-dependent) and warping (temperature-gradient-dependent). Thus, a 15 x 15-ft slab was believed to be appropriate for the complete calculations.

A standard finite-element mesh was used throughout the computations. The elements are smaller in

Figure 2. Standard finite-element mesh.




locations close to the loaded area because there is a greater variation of stresses and deflections at those points (see Figure 2).

Poisson's Ratio

The value of Poisson's ratio for concrete was assumed to be 0.15. For soils, it has not been well defined and it appears to depend on a number of factors, such as density, soil type, structure, water content, stress level, and stress path. Significant variations in values of Poisson's ratio are found even in uniformly prepared specimens and carefully controlled tests. However, its effect on the results is so small that it was not deemed necessary to vary it. Typical values for cement-treated materials are close to 0.1. Hence, this value was taken as representative for the subbase.

Modulus of Elasticity

The modulus of elasticity for concrete was assumed to be 4 million psi. For the treated subbase, three different values of modulus of elasticity were used: 100,000, 800,000 and 1.5 million psi.

Thickness

Slabs 4, 8, and 12 in. thick were studied. The subbase thickness varied as follows: 4, 6, 9, and 12 in.

Modulus of Subgrade Reaction

Westergaard's modulus of subgrade reaction (k) is expressed in pounds per square inch per inch or in pounds per cubic inch. The assumption that k is constant implies elasticity for the subgrade, but this assumption is only valid through a narrow range. The numerical value of k depends not only on factors that affect soil behavior, such as soil texture, density, and moisture, but also on structural and geometric factors such as slab rigidity and size of loaded area. Values of k are usually determined in the field by means of a plate-bearing test. Because test values vary with test conditions, a 30-in.-diameter plate is generally used. A study of the results of many tests indicates that the most representative value for k can be obtained using a pressure of 10 psi.

For cases in which a subbase was not included, the following values were given to k: 50, 100, 400, 700, 1,000, and 2,000 pci. This was done for each of the slab thicknesses corresponding to 4, 8, and 12 in. When a subbase was included in the analysis, the modulus of subgrade reaction was limited to 50, 100, and 200 pci, as shown in the PCA curves.

RESEARCH METHODOLOGY

To develop the proposed curves of composite k-value versus subbase thickness, 102 problems were solved by using the finite-element technique. Both maximum stress and maximum deflection criteria were used in the process. The steps given below were followed in developing the first set of curves (composite k-value versus subbase thickness) with the maximum edge stress criterion:

1. Obtain curves of maximum stress versus modulus of subgrade reaction for three different slab thicknesses, as shown in Figures 3a, 4a, and 5a. Maximum edge stresses calculated by using Westergaard's model are also included in these figures for the purpose of comparing the results of Westergaard's slab theory and the finite-element technique.
2. Keeping constant the slab thickness, the modulus of elasticity of the subbase, and the modulus of subgrade reaction, vary the subbase thickness and find the maximum stress for each subbase thickness.
3. Using the curve of maximum stress versus k-value obtained in step 1, for the same slab thickness determine for each maximum stress found in step 2 a composite k-value under the slab (on top of the subbase) that gives the same slab stress. A curve of k on the subbase (composite modulus of subgrade reaction) versus subbase thickness is then plotted jointly with the PCA curves for the modulus of subgrade reaction used in the analysis.
4. Vary the modulus of elasticity of the subbase and return to step 2 until the three values of the modulus of elasticity of the subbase have been considered for the given slab thickness and modulus of subgrade reaction. Then modify the modulus of subgrade reaction after resetting the modulus of elasticity of the subbase to its initial value and return to step 2. Repeat the process until all proposed values of the modulus of subgrade reaction have been used. Finally, reset both the modulus of elasticity of the subbase and the modulus of subgrade reaction to their initial values, vary the slab thickness, and repeat the process starting from step 2 until all of the proposed slab thicknesses have been considered.

Each computer program output from the previous cases also contains the maximum surface deflection. The process of determining curves of composite modulus of subgrade reaction versus subbase thickness based on a maximum deflection criterion is similar to the process described previously except that maximum surface deflection is used instead of maximum edge stress.

DISCUSSION OF RESULTS AND CONCLUSIONS

There is some agreement between the results obtained by the finite-element technique and by Westergaard's model for a given slab resting directly on top of the subgrade even though Westergaard's model is based on an infinite slab. For example, curves of maximum stress versus k on the subgrade as well as maximum deflection at the surface versus k on the subgrade (Figures 3 to 5) show the same trend even

Figure 3. Maximum stress and maximum deflection versus modulus of subgrade reaction for 4-in.-thick slab.

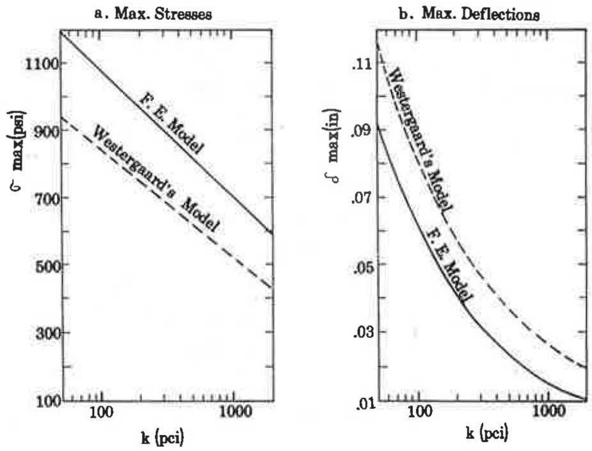



Figure 4. Maximum stress and maximum deflection versus modulus of subgrade reaction for 8-in.-thick slab.

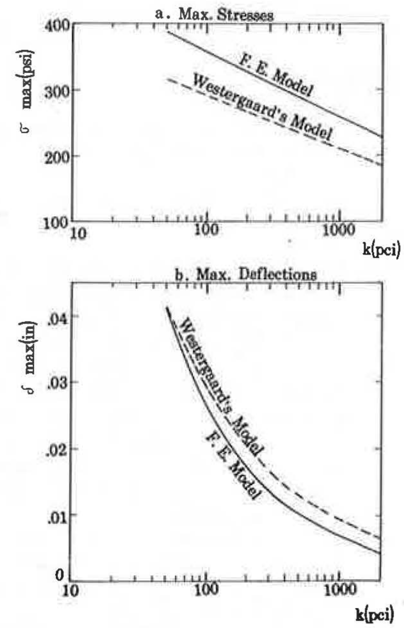
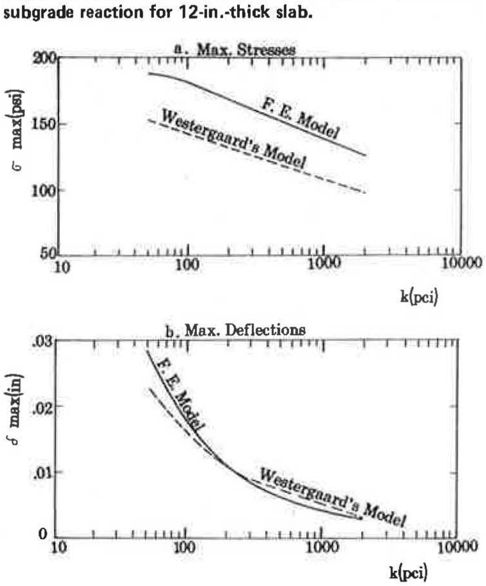



Figure 5. Maximum stress and maximum deflection versus modulus of subgrade reaction for 12-in.-thick slab.




for k-values greater than 500 pci, where Westergaard's model is considered not applicable.

The finite-element model appears to be stiffer than Westergaard's approach: stresses are higher in all cases and deflections are lower, the only exception being the case of a 12-in.-thick slab. In this case, the curves of deflection versus k on the subgrade merge at k = 200 pci (Figure 5b). The deflections found by using the finite-element method for k-values lower than the merging point were larger than Westergaard's and beyond this point they were smaller. There seems to be better agreement in the deflections calculated by both methods when the slab thickness increases.

There is very little agreement between the curves of composite k-value versus subbase thickness developed in this study and similar curves developed by the PCA, as shown in Figures 6 to 10. The disagreement is even more marked when the newly developed curves based on the deflection criterion are compared with the PCA curves than when the curves based on the stress criterion are compared. For the deflection criterion, a higher composite k-value resulted when a thinner slab and a higher modulus of elasticity of the subbase were used, whereas when a thick slab was used it did not make any difference in the composite k-value whether a subbase was included or not.

Curves developed based on the maximum stress criterion also show a higher composite k-value for thinner slabs and higher modulus of elasticity of the subbase, whereas the effect of placing a subbase will be smaller and almost imperceptible for thicker slabs and lower modulus elasticity of the subbase.

The curves of composite k-value versus subbase thickness found by means of the finite-element com-

Figure 6. Composite k-value versus subbase thickness for 4-in.-thick slab: deflection criterion.

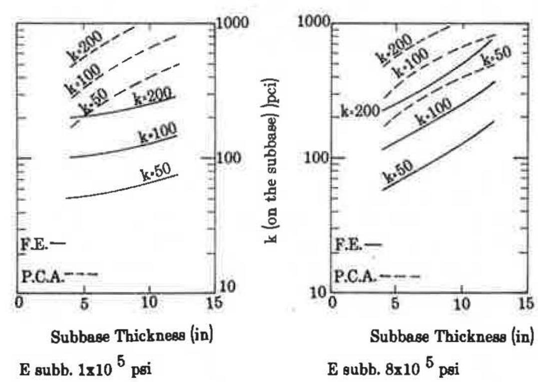



Figure 7. Composite k-value versus subbase thickness for 8-in.-thick slab: deflection criterion.

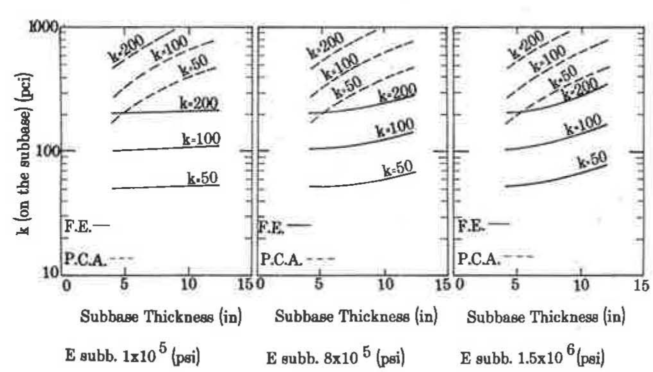


Figure 8. Composite k-value versus subbase thickness for 4-in.-thick slab: stress criterion.

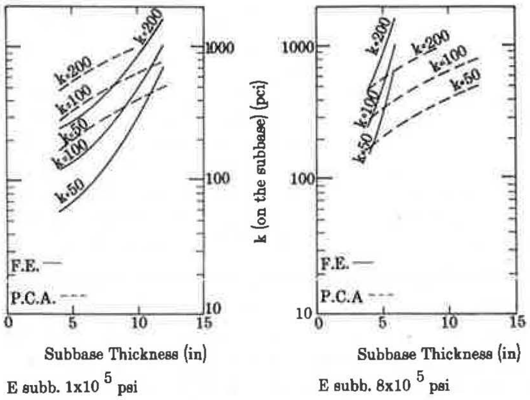



Figure 9. Composite k-value subbase thickness for 8-in.-thick slab: stress criterion.

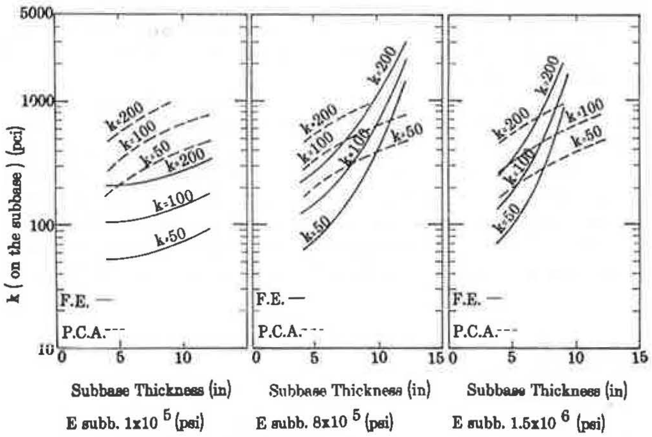
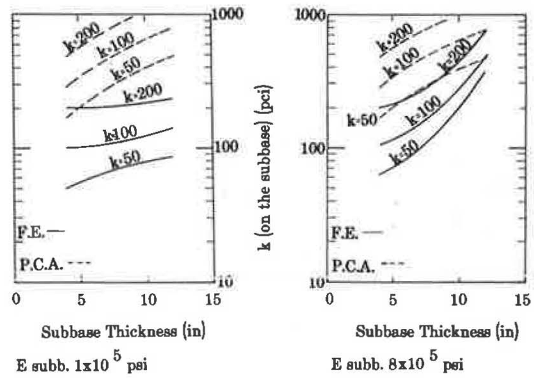



Figure 10. Composite k-value versus subbase thickness for 12-in.-thick slab: stress criterion.




puter program have an opposite curvature and lie below those developed by the PCA.

For the edge type of loading, at least, it can be concluded that there is not a definite generalization of the composite k-value with respect to subbase thickness for a given modulus of subgrade reaction. This study has shown that the composite k-value depends on at least the following variables: slab thickness, subbase thickness, modulus of elasticity of the subbase, and modulus of subgrade reaction.

Future research should be done to determine whether the same behavior is observed in both corner and interior loading for rigid pavements.

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