13

It is clear from both of these figures that a sub-
stantial increase in longitudinal cracking can be
obtained by reducing the standard deviation (see the
examples in Figures 4 and 5 shown by a dotted line
for a depth ratio = 0.75). This indicates that rea-
sonable control of the quality of this construction
material (concrete), pavement thickness, and saw-cut
depth, all combined, can contribute to a reasonable
saw-cut depth.

CONCLUSIONS

The results of this study can be summarized as fol-
lows:

1. The development of longitudinal cracks in a
saw-cut groove can be explained by a model using the
concepts of variability in concrete strength and
thickness of pavement sections.
2. The model developed for this study is sensi-
tive to the construction quality of pavement. An im-
provement in construction quality can result in
reduction of saw-cut depths. The reliability of lon-
gitudinal cracks (being confined to saw-cut groove)
is also improved. This can save construction costs
as well as future maintenance and repair costs.
3. Figures 3-5 show that it is possible to in-
duce any desired amount of longitudinal cracking
along the saw-cut groove if an appropriate saw-cut
depth is provided.
4. The aggregates used in concrete affect the
development of longitudinal cracks along a saw-cut

groove. This finding is based on a study of two ag-
gregates (river gravel and limestone).

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Westergaard Solutions Reconsidered

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ABSTRACT

The pioneering analytical work of Harold Malcom Westergaard (1888-1950) has
been at the heart of slab-on-grade pavement design since the 1920s. Every code
of practice published since then makes reference to the "Westergaard solu-
tions." These solutions are only available for three particular loading condi-
tions (interior, edge, and corner) and assume a slab of infinite or semi-infi-
nite dimensions. Since their first appearance, beginning in the early 1920s,
Westergaard equations have often been misquoted or misapplied in subsequent
publications. To remedy this situation, a reexamination of these solutions
using the finite element method is described in this paper. A number of inter-
esting results are presented: (a) Several equations ascribed to Westergaard in
the literature are erroneous, usually as a result of a series of typographical
errors or misapplications, or both. The correct form of these equations and
their limitations have now been conclusively established. (b) Westergaard's
original equation for edge stress is incorrect. The long-ignored equation given
in his 1948 paper should be used instead. (c) Improved expressions for maximum
corner loading responses have been developed. (d) Slab size requirements for
the development of Westergaard responses have also been established.

The pioneering analytical work of Harald Malcom Westergaard (1888-1950) has been at the heart of slab-on-grade pavement design since the 1920s. Every code of practice published since then makes reference to the "Westergaard solutions." These solutions are only available for three particular loading conditions (interior, edge, and corner) and assume a slab of infinite or semi-infinite dimensions. In practice, the slab size required for the development of Westergaard responses is determined empirically. Several investigators, however, have noted repeatedly that although the Westergaard solution agreed fairly well with their observations for the interior loading condition, it failed to give even a close estimate of the response in the cases of edge and corner loading. The time-honored Westergaard solutions deserve a thorough reexamination using the tool of finite element analysis now available.

The highlights of an effort to reevaluate the Westergaard solutions (1) are presented. The form, theoretical background, limitations, and applicability of these equations have been examined, and what are considered to be the most accurate formulas are presented herein. Several empirical adjustments to the Westergaard solutions are also considered, and slab size requirements for the development of Westergaard responses are established.

The basic tool for this study is the ILLI-SLAB finite element computer program developed and extensively used at the University of Illinois (2). The ILLI-SLAB model is based on classical medium-thick plate theory, and employs the 4-noded, 12-degree-of-freedom plate-bending element, known in finite element literature as ACM or RPB12 (3). The Winkler-type subgrade assumed by Westergaard is modeled as a uniform, distributed subgrade through an equivalent mass formulation (4).

INTERIOR LOADING

As defined by Westergaard, this is the case of a wheel load at a "considerable distance from the edges," with pressure "assumed to be uniformly distributed over the area of a small circle with radius a" (5). After an extensive literature survey and comparisons with finite element results (6), the following interior loading equations are considered to be in their most general form.

Maximum bending stress, σ_i

Ordinary theory

$$\text{BSIOT} = \left\{ [3P(1+\mu)]/2\pi h^2 \right\} \; [\ln (2\ell/a) \\ + 0.5 - \gamma] + \text{BSI2OT} \qquad (1a)$$

Special theory

$$\text{BSIST} = \left\{ [3P(1+\mu)]/2\pi h^2 \right\} \; [\ln (2\ell/b) \\ + 0.5 - \gamma] + \text{BSI2ST} \qquad (1b)$$

For square

$$\text{BSISQ} = \left\{ [3P(1+\mu)]/2\pi h^2 \right\} \; [\ln (2\ell/c') \\ + 0.5 - \gamma] + \text{BSI2SQ} \qquad (1c)$$

Supplementary, σ_2 (ordinary theory)

$$\text{BSI2OT} = \left\{ [3P(1+\mu)]/64h^2 \right\} \; [(a/\ell)^2] \qquad (1d)$$

Supplementary, σ_2 (special theory)

$$\text{BSI2ST} = \left\{ [3P(1+\mu)]/64h^2 \right\} \; [(b/\ell)^2] \qquad (1e)$$

Supplementary, σ_2 (for square)

$$\text{BSI2SQ} = \left\{ [3P(1+\mu)]/64h^2 \right\} \; [(c'/\ell)^2] \qquad (1f)$$

Maximum deflection, δ_i

Circle

$$\text{DEFIC} = (P/8k\ell^2) \left\{ 1 + (1/2\pi) \; [\ln (a/2\ell) \\ + \gamma - 5/4] \; (a/\ell)^2 \right\} \qquad (1g)$$

where

P = total applied load;
E = slab Young's modulus;
μ = slab Poisson's ratio;
h = slab thickness;
k = modulus of subgrade reaction;
a = radius of circular load;
c = side length of square load;
ℓ^4 = $\{Eh^3/[12(1 - \mu^2)k]\}$ which is radius of relative stiffness;
b = $[(1.6a^2 + h^2)^{1/2}] - 0.675h$
   if a < 1.724h
 ≡ a   if a > 1.724h;
c' = $(e^{\pi/4-1}/2^{1/2})c$; and
γ = Euler's constant (= 0.577 215 664 90).

These equations have been incorporated into WESTER, a computerized compendium of closed-form solutions for slabs on grade, developed in the course of this research (1).

Equation 1a follows from Equation 50 given in 1939, with the term $[\ln (2\ell/a) - \gamma]$ replacing the term $[\ln (2\ell/\gamma a)]$ used by Westergaard (7). Note that the symbol γ as used in Equation 1a is the Euler constant, whereas Westergaard uses this symbol to denote the antilog of the Euler constant.

Equation 1a also includes supplementary stress, σ_2, first derived by Westergaard in 1939. This is calculated according to Equation 1d, which is the same as Equation 6 in "Stresses in Concrete Runways of Airports" (7). This additional term was introduced to account for the effect of the finite size of the loaded area and is "satisfactorily applicable when a does not exceed ℓ" (7). Its contribution is usually small, but it is included because of its rigorous analytical nature. The effect of the size of the loaded area will be discussed further hereafter.

Equation 1b employs Westergaard's "special theory," first proposed in 1926, in which radius b replaces the true radius, a, of the loaded area. This was introduced to account for the effect of shear stresses in the vicinity of the load, which is neglected in the "ordinary theory" of medium-thick plates. As Westergaard stated, "the effect of the thickness of the slab is equivalent to a rounding off of the peak in the diagrams of moments" (5). To determine the relation among h, a, and b, Westergaard (5) performed "numerical computations . . . in accordance with an analysis which is due to A. Nadai." Results were fitted with a hyperbola, the equation of which may be written in the form presented earlier, "which is suitable for numerical calculations" (5). The validity of Westergaard's semiempirical adjustment and of the resulting "special theory" has been debated by various investigators [see, for example, Scott (8)], but a full discussion of this issue would be beyond the scope of this paper. The authors recommend, however, using "ordinary theory" when comparisons with finite element results are made.

To obtain the interior stress in the case of a square loaded area, radius a is replaced in Equation 1-c by a constant, c', related to the length of the side of the square, c, as follows:

$$c' = (e^{\pi/4-1}/2^{1/2}) \; c = 0.573804 \ldots c$$

The resulting expression is not stated explicitly by Westergaard, but follows directly from his theory (9,10). Timoshenko and Woinowsky-Krieger (11) provide a theoretical justification for this substitution by showing that, loaded by the same total load P, a square side c and a circle radius a give the same maximum interior stress.

In 1948 Westergaard presented an equation for the stress under an elliptical loaded area [Equation 3 in "New Formulas for Stresses in Concrete Pavements of Airfields" (10)]. Setting both axes of the ellipse to a, this equation can be compromised with Equation 1a provided that the following assumption is made.

$$\ln 2 + 1/2 - \gamma \approx 1/4 \ln 12 (1 - \mu^2)$$

For $\mu = 0.15$, this assumption gives

$$0.6159316 \approx 0.6155374$$

This indicates that the term 0.6159 in Equation 9 in "Stresses in Concrete Pavements Computed by Theoretical Analysis" (5) is a truncated form of the term involving Euler's constant, not slab Poisson's ratio, μ. Equation 1a is, therefore, more general than the 1948 equation.

Equation 1g follows from Equation 51 in "Stresses in Concrete Runways of Airports" (7), described previously, with the introduction of Euler's constant. This form is more general than the one obtained from Equation 5 in "New Formulas for Stresses in Concrete Pavements of Airfields" (10), which makes the approximation noted in the previous paragraph.

Westergaard also presented an equation for supplementary stress, σ_3, to account for "the effects of a plausible redistribution" of subgrade reactions (12,7). This was a semiempirical adjustment to reduce calculated stresses so that they agreed better with the 1932 Arlington tests (13). Bergstrom et al. (14) note that "it appears advisable to neglect σ_3 in design," because it is difficult to evaluate and causes considerable reduction in stress. Further discussions of this term are presented by Bradbury (15) and Kelley (16).

Slab Size Requirements for the Development of Interior Loading Westergaard Responses

As mentioned earlier, the closed-form Westergaard solutions assume a slab of infinite dimensions, although in practice empirical guidelines have been developed for the least slab dimension, L, required to achieve the Westergaard "infinite slab" condition. In this section, analyses will be presented to establish similar guidelines using the finite element method. A slab with a radius of relative stiffness, ℓ, of 23.16 in. was used with a mesh fineness ratio (2a/h) of 1.8. An earlier study (6) indicated that this ratio of element size, 2a, divided by slab thickness, h, must be about 0.8 for 98 percent accuracy.

ILLI-SLAB results from this investigation are shown in Figure 1. Both maximum deflection and bending stress converge to large slab values. The convergence of deflection is from above, indicating that a smaller slab settles more than a bigger one in a "punch-like" fashion. Bending stress converges from below, as expected. The rate of convergence, defined as the slab size at which the solution is essentially that for an infinite slab, is different for deflection (L/ℓ = 8.0) than for bending stress (L/ℓ = 3.5). Surprisingly, deflection appears to be much more sensitive to slab size changes for (L/ℓ) values of less than 3, because of the previously mentioned punch-like effect. The limit value approached by maximum deflection is the Westergaard solution (Equation 1g). The value to which bending stress converges when slab size is expanded is slightly lower than Westergaard's (Equation 1a) due to the coarseness of the mesh used.

Effect of Size of Loaded Area

In his attempt to develop equations for a loaded area of finite size, Westergaard used an approach in which a solution for a point load is first derived. Then, the loaded area is split into a number of small subareas, and each subarea is replaced by a statically equivalent point load acting at its center. A summation is performed over these subareas. In the limit of refinement, this summation tends to an exact integration (17). Westergaard suggested that his equations were valid for any size of loaded
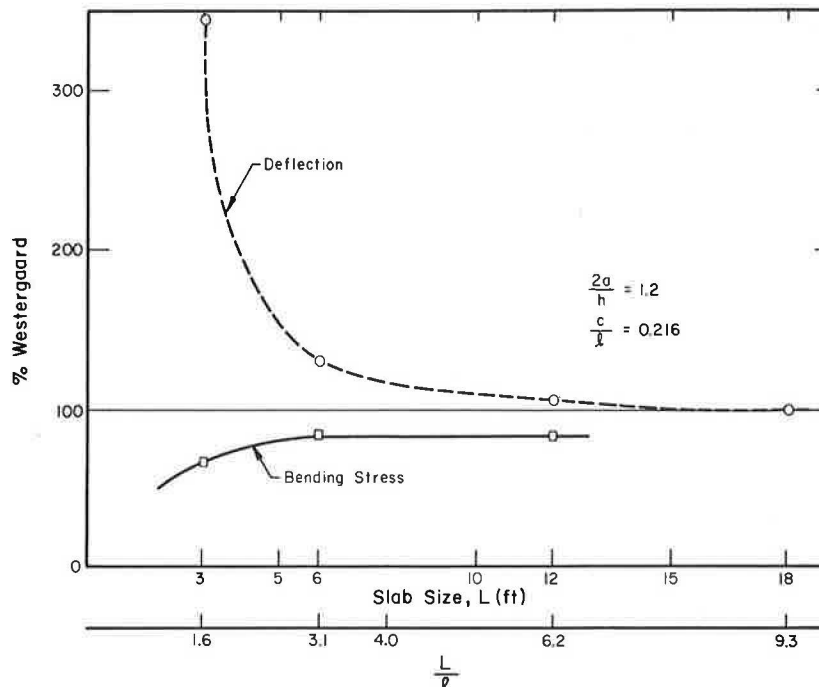

FIGURE 1 Effect of slab size on maximum interior loading responses.

area and that his "New Formulas" (10) assume that "the average width and length of the footprint of the tire is greater than the thickness of the slab in all significant cases."

Losberg (18) showed that the stress and deflection equations presented by Westergaard are only the first one or two terms of a rapidly converging infinite series. Westergaard's supplementary stress, σ_2, mentioned previously, for example, is an additional term of this series. The rate of convergence can be expected to vary depending on, among other things, the size of the loaded area.

Timoshenko and Woinowsky-Krieger (11) state that the equations apply only when the radius of the loaded area is "small in comparison with ℓ." Scott (8) attributes this restriction to the fact that "in the derivation of the equation a term of approximate value $0.1\ a^2/\ell^2$ was omitted." This cannot be the real cause of the restriction imposed by Timoshenko and Woinowsky-Krieger, because in most cases (even when the radius of the loaded area, a, is not "small in comparison with ℓ") this term is, indeed, negligible.

In this study the effect of the size of the loaded area was investigated using the finite element method. To eliminate slab size, mesh fineness, and element aspect ratio effects, a large (L/ℓ = 9.33) and fine (2a/h = 0.6) mesh, which consisted of square elements (aspect ratio = 1.0), was used. The results are plotted in Figure 2.

It is observed that Westergaard stress values (Equation la) agree with finite element results for a loaded area whose side length, c (if square), is about 0.2 times the radius of relative stiffness, ℓ; if the load is circular, its radius, a, must be about 0.1ℓ. As (c/ℓ) or (a/ℓ) increase, finite element stresses become progressively higher than Westergaard's. Therefore the consequences of Westergaard's truncation, mentioned previously, must be borne in mind when attempting such comparisons. The results in Figure 2 also suggest an effect related to the internal finite element discretization of the applied load. This is discussed in more detail elsewhere (1).

EDGE LOADING

Westergaard defined edge loading as the case in which "the wheel load is at the edge, but at a con-

siderable distance from any corner." The pressure is assumed to be "distributed uniformly over the area of a small semi-circle with the center at the edge" (5). Equations for a circular load at the edge were first presented in 1948 (10). The most general forms of the edge loading formulas follow.

Maximum bending stress, σ_e

Ordinary theory (semicircle)	$BSEWOT = 0.529\,(1+0.54\mu)\,(P/h^2)\,[\log_{10}\,(Eh^3 \div ka_2^4) - 0.71]$	(2a)
Special theory (semicircle)	$BSEWST = 0.529\,(1+0.54\mu)\,(P/h^2)\,[\log_{10}\,(Eh^3 \div kb_2^4) - 0.71]$	(2b)
"New" formula (circle)	$BSEIC = [3(1+\mu)P/\pi(3+\mu)h^2]\,\{\ell n\,(Eh^3/100ka^4) + 1.84 - 4\mu/3 + [(1-\mu)/2] + 1.18\,(1+2\mu)\,(a/\ell)\}$	(2c)
"New" formula (semicircle)	$BSEIS = [3(1+\mu)P/\pi(3+\mu)h^2]\,[\ell n\,(Eh^3/100ka_2^4) + 3.84 - 4\mu/3 + 0.5\,(1+2\mu)\,(a_2/\ell)]$	(2d)
Simplified "new" formula (semicircle)	$BSELS = (-6P/h^2)\,(1+0.5\mu)\,[0.489\,\log_{10}\,(a_2/\ell) - 0.091 - 0.027\,(a_2/\ell)]$	(2e)
Simplified "new" formula (circle)	$BSELC = (-6P/h^2)\,(1+0.5\mu)\,[0.489\,\log_{10}\,(a/\ell) - 0.012 - 0.063\,(a/\ell)]$	(2f)

Maximum deflection, δ_e

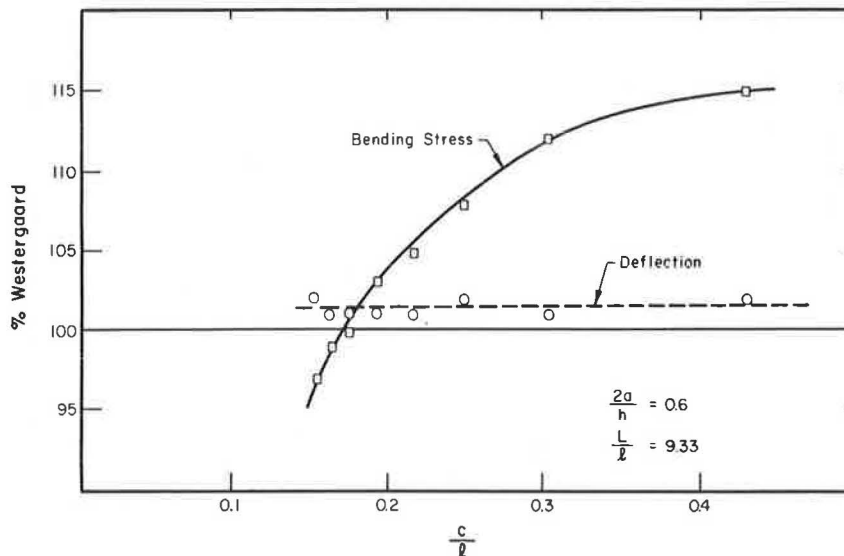| Original formula | $DEFEW = (1/6^{1/2})\,(1+0.4\mu)\,(P/k\ell^2)$ | (2g) |
"New" formula (circle)	$DEFEIC = \left(\{P[(2+1.2\mu)^{1/2}]\}/[(Eh^3k)^{1/2}]\right)\,[1 - (0.76+0.4\mu)\,(a/\ell)]$	(2h)
"New" formula (semicircle)	$DEFEIS = \left(\{P[(2+1.2\mu)^{1/2}]\}/[(Eh^3k)^{1/2}]\right)\,[1 - (0.323+0.17\mu)\,(a_2/\ell)]$	(2i)
Simplified "new" formula (semicircle)	$DEFELS = (1/6^{1/2})\,(1+0.4\mu)\,(P/k\ell^2)\,[1 - 0.323\,(1+0.5\mu)\,(a_2/\ell)]$	(2j)
Simplified "new" formula (circle)	$DEFELC = (1/6^{1/2})\,(1+0.4\mu)\,(P/k\ell^2)\,[1 - 0.760\,(1+0.5\mu)\,(a/\ell)]$	(2k)

FIGURE 2 Effect of size of loaded area on maximum interior loading responses.

where

a_2 = radius of semicircle,
b_2 = $[(1.6a_2 + h^2)^{1/2}] - 0.675h$
           if $a_2 < 1.724h$
    = a_2 if $a_2 > 1.724h$, and
other symbols are as defined for Equations 1.

Equation 2a is identical to Equation 3 in "Analytical Tools for Judging Results of Structural Tests of Concrete Pavements" (12). Equation 2b employs the "special theory," which is also used for interior loading. In his 1948 paper, Westergaard (10) presented generalized solutions for maximum stress and deflection produced by elliptical and semielliptical loaded areas placed at a slab edge. Setting the lengths of both the major and minor semiaxes of the ellipse to a or a_2 leads to the corresponding solutions for a circle radius, a, or a semicircle radius, a_2, given by Equations 2c and 2d.

Losberg (18) presented simplified versions of these solutions by introducing "simplifications of the same type as Westergaard (19) himself introduced in his original formula for the case of edge loading" to eliminate the "complicated functional relationship" in which μ appears in these equations. Losberg (18) stated that his simplified equations "are well applicable, for the small μ-values here concerned." These are Equations 2e and 2f. Comparisons made during this study show that Losberg's simplified equations lead to results that are typically about 1 percent greater than those obtained by the general Equations 2c and 2d.

Equation 2g is Westergaard's original equation for edge deflection (5), and Equations 2h and 2i can be obtained from his 1948 paper (10), as indicated previously. The corresponding Losberg formulas (18) are given by Equations 2j and 2k. Setting the radius of the loaded area to zero, these formulas reduce to Equation 2g.

Alternative Westergaard Solutions

It was pointed out earlier, as well as by other investigators (14,18), that in the case of interior deflection and stress, as well as edge deflection, when the "new" formulas are specialized for a circular (or a semicircular) loaded area, they become identical to the corresponding original (5,12) equations. Results from this study show, however, that edge stresses calculated from the "new" formula are considerably different than those computed using the original formula. A number of alternative Westergaard solutions are considered in this section, in order to determine which one, if any, agrees best with finite element results.

In Table 1, five different Westergaard solutions are compared. In all of these, total applied load, P, and applied pressure, p, are matched in the Westergaard and finite element analyses. A previous study (6) confirmed that this is an appropriate representation of the square loaded area used in the finite element solution. The solutions given in Table 1 are code named WESI through WESV and were obtained using WESTER, which incorporates Equations 2.

The range of results in Table 1 is extremely wide. Therefore careful use of Westergaard's theory cannot be overemphasized. The most obvious effect is that the "new" formulas typically lead to stresses 55 percent higher and deflections 8 percent lower than the values obtained using the original formulas (compare WESIV with WESIII).

A comparison of WESIV and WESV indicates that the semicircular load is more severe than the circular load (i.e., leads to higher stresses and deflections), as expected. If both the circular and the semicircular loads are reduced to an equivalent point load acting at the respective center of gravity, this expectation is shown to be justified because the center of gravity of the circle is further toward the interior of the slab than is that of the semicircle.

This argument also leads to the conclusion that the difference in response from a circular and a semicircular load should be fairly small and proportional to the difference in the distance between the respective centers of gravity and the slab edge. The difference between WESIV and WESV stresses is about 1 percent, and deflection difference is about 5 percent. These differences are much more compatible with expected values than is the stress difference obtained using the original equation (compare WESI and WESIII).

Table 2 gives a comparison of WESV, ILLI-SLAB, and H-51 results. The latter is a computerized version of the Pickett and Ray (20) chart for edge loading (21). Stresses exhibit almost perfect agreement even at low (L/ℓ) values. Deflections are more sensitive to slab size effects, as shown in Figure 3. This graph shows that an (L/ℓ) value of about 5.0 is required for the development of Westergaard stresses and about 8.0 is required for Westergaard deflections. The trends shown in Figure 3 are similar to those observed for the interior condition. Note, however, that the requirement for the development of maximum edge stress (L/ℓ = 5.0) is higher than for maximum interior stress (L/ℓ = 3.5). The excellent agreement between ILLI-SLAB and H-51 results and the "new" formula confirms Losberg's observation that "the original formula for edge loading according to Westergaard (5) is, at least from a theoretical viewpoint, completely erroneous."

TABLE 1 Alternative Westergaard Solutions

Run No.	k (psi/in.)	h (in.)	Deflection, δ_e (mils)					Bending Stress, σ_e (psi)				
			WESI	WESII	WESIII	WESIV	WESV	WESI	WESII	WESIII	WESIV	WESV
1	50	12	112.7			104.8	99.7	758	661	638	992	980
2	200	12	56.4			50.8	47.3	638	541	519	831	822
3	500	12	35.7			31.3	28.4	559	462	440	726	720
4	50	16	73.2	Same as WESI		69.0	66.4	468	413	401	615	607
5	200	16	36.6			33.7	31.8	401	346	334	524	517
6	500	16	23.2			20.8	19.4	357	302	289	464	459
7	50	20	52.4			49.8	48.2	320	281	277	422	417
8	200	20	26.2			24.4	23.3	277	238	234	363	359
9	500	20	16.6			15.2	14.3	249	210	206	325	321

Note: WESI = Westergaard's original equations for circular "ordinary" theory, WESII = Westergaard's original equations for semicircular "special" theory, WESIII = Westergaard's original equations for semicircular "ordinary" theory, WESIV = "New" formulas for semicircular "ordinary" theory, and WESV = "New" formulas for circular "ordinary" theory. See Table 2 for other parameters used.

TABLE 2 Comparison with "New" Edge Loading Formulas

Run No.	k (psi/in.)	h (in.)	(L/ℓ)	Deflection, δ_e			Bending Stress, σ_e			
				WESV (mils)	ILLI-SLAB (mils)	ILLI-SLAB WESV	WESV (psi)	ILLI-SLAB (psi)	H-51 (psi)	ILLI-SLAB WESV
1	50	12	4.84	99.7	109.7	1.10	980	974	951	0.99
2	200	12	6.85	47.3	49.2	1.04	822	813	808	0.99
3	500	12	8.61	28.4	29.3	1.03	720	705	711	0.98
4	50	16	3.90	66.4	78.5	1.18	607	593	591	0.98
5	200	16	5.52	31.8	34.1	1.07	517	515	504	1.00
6	500	16	6.94	19.4	20.2	1.04	459	454	452	0.99
7	50	20	3.30	48.2	63.3	1.31	417	395	396	0.95
8	200	20	4.67	23.3	25.8	1.11	359	356	348	0.99
9	500	20	5.87	14.3	15.1	1.06	321	319	317	0.99

Note: For the ILLI-SLAB solution: slab = 25 x 25 ft, (L/ℓ) = 3.30 to 8.61, E = 5 x 10^6 psi, μ = 0.15, P = 50,000 lb, p = 222.2 psi, and A = 15 x 15 in. (edge). For the Westergaard solution (WESV), "New" formulas, circular load, "ordinary" theory: p = 222.2 psi, E = 5 x 10^6 psi, and μ = 0.15. For the H-51 solution: 50 points are used to approximate tire print.
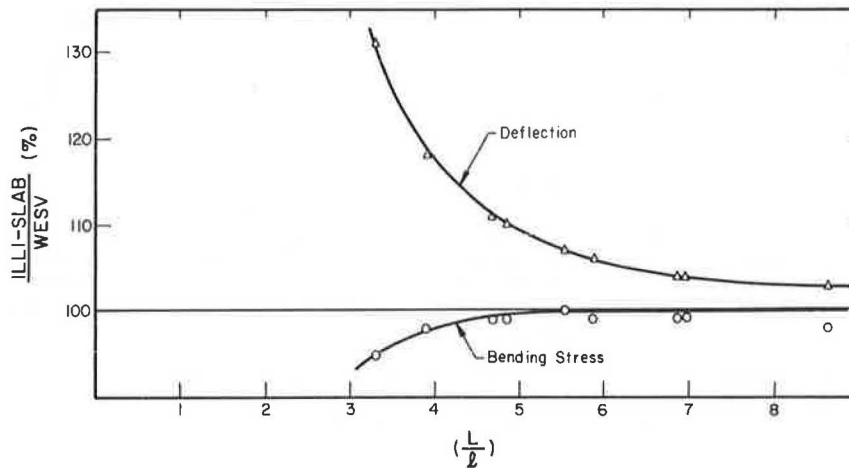

FIGURE 3 Effect of slab size on maximum edge loading responses.

The Pickett and Ray charts (20) for the edge loading condition on a dense liquid subgrade are based on a pair of integral equations identical to those presented by Westergaard (10). The results from these charts, therefore, agree with the "new" formulas, as indicated by H-51 results in Table 2. It is interesting to note that, although in several design codes reference is made to the original equation, the fact that multiple wheel loads are often considered implies that design charts in these codes have been obtained using the Pickett and Ray charts (i.e., the "new" formulas).

The question of the source of the discrepancy between Westergaard's original and new formulas for edge loading remains unanswered. It is too early to dismiss the original formulas as altogether false and useless. Bergstrom et al. (14) reported that values calculated using these equations are "in relatively close agreement with test results." They furthermore suggested that there are "no reasons to use the new formula for edge loading." On the other hand, Scott (8) suggests that "experimental indications are that the edge stresses experienced in practice are higher than the Westergaard (original) equation indicates." Laboratory model tests by Carlton and Behrmann (22) produced edge stresses 10 to 12 percent lower than the new formula predicts, reinforcing the expectation that in situ values probably lie between the two Westergaard equations.

The theoretical background for the original edge stress equation is also open to debate. The derivation of this equation is not presented in any of Westergaard's papers. Attempts by Losberg (18) to arrive at this formula through integration of the

expressions for the concentrated load (19) did not produce agreement with Westergaard's result. If the probability of a gross theoretical blunder by such a meticulous investigator as Westergaard can be cast aside, for the time being, a possible explanation for this discrepancy is some assumption regarding subgrade support at the slab edge, which is implicit in the original formula but is never explicitly stated.

CORNER LOADING

Of the three fundamental cases of loading investigated by Westergaard, corner loading is undoubtedly the most obscure and debatable. The theoretical background for maximum corner deflection and stress equations is particularly weak. Their semiempirical and approximate nature has led to numerous revisions and modifications in the years since their original publication, in an attempt to reconcile observed slab behavior with theory. These are discussed by Kelley (16) and Pickett (23) and are summarized as follows:

Deflection

$$\delta_c = (P/k\ell^2) [1.1 - 0.88 (a_1/\ell)] \qquad \text{Westergaard } (5) \tag{3a}$$

Stress

$$\sigma_c = (3P/h^2) \qquad \text{Goldbeck } (25), \text{Older } (26) \tag{3b}$$

$$\sigma_c = (3P/h^2) [1 - (a_1/\ell)^{0.6}] \qquad \text{Westergaard } (5) \tag{3c}$$

$$\sigma_c = (3P/h^2) \, [1 - (a/\ell)^{0.6}] \qquad \text{Bradbury } (15) \qquad (3d)$$

$$\sigma_c = (3P/h^2) \, [1 - (a_1/\ell)^{1.2}] \qquad \text{Kelley } (16), \text{ Teller and Sutherland} (13) \qquad (3e)$$

$$\sigma_c = (3.2P/h^2) \, [1 - (a_1/\ell)] \qquad \text{Spangler } (28) \qquad (3f)$$

$$\sigma_c = (4.2P/h^2) \left(1 - \left\{ [(a/\ell)^{1/2}] / [0.925 + 0.22 \, (a/\ell)] \right\} \right) \qquad \text{Pickett } (23) \quad (3g)$$

Distance to point of maximum stress along corner angle bisector

$$X_1 = 2[(a_1\ell)^{1/2}] \qquad \text{Westergaard } (5) \qquad (3h)$$

where

> a = radius of circular load tangent to both edges at corner and
> a_1 = distance to point of action of resultant along corner angle bisector
> = $(2^{1/2})a$. See Equations 1 for other symbols.

In the early 1920s a short and simple piece of analytical work was heralded as "the most important single step in the investigation of the mechanics of road slabs" (24). This was the first attempt to solve the problem of the "corner break" by two prominent engineers of the day working independently, A.T. Goldbeck of the Bureau of Public Roads and Clifford Older of the Illinois Highway Department. By assuming that in the corner region the slab acts as a cantilever of uniform strength (i.e., that in this region the subgrade reaction is negligible compared to the applied load), Equation 3b was proposed for the maximum stress, σ_c, due to a concentrated load, P, acting at the corner of a slab, of thickness h (25,26).

A few years later, Westergaard (5) took up the problem again, trying to account for the effect of a load distributed over some area, the resultant of which could be represented by a point load P acting at a small distance a_1 from the corner, along the bisector of the corner angle. Using a "simple approximate process" involving the use of the principle of minimum potential energy (27) he hoped to achieve an "improved approximation" of corner stress. Thus he first arrived at Equation 3a for corner deflection. He considered this equation "approximately applicable for plausible ranges of a_1 and ℓ" [presumably (a_1/ℓ) is not much greater than 0.1]. From this, he obtained bending moments by integration and concluded that the maximum stress "would be represented with satisfactory accuracy" by Equation 3c. Furthermore, the distance to the point of maximum stress along the corner angle bisector was found to be given "roughly" by Equation 3h.

Equations for the Corner Loading Condition Based on the Finite Element Method

In this section, ILLI-SLAB is used to establish a set of equations that would accurately predict the response of a slab, in full contact with a Winkler foundation, to a single load distributed over a small area at its corner. Equations 3, proposed by previous investigators, suggest that, from a theoretical viewpoint, the parameters involved in the determination of slab response can be lumped into three nondimensional ratios to be investigated, namely, $(\delta_c k \ell^2/P)$ $(\sigma_c h^2/P)$ and (a/ℓ) or (a_1/ℓ) for a circular, or (c/ℓ) for a square load.

The results obtained from several ILLI-SLAB analyses are plotted in a nondimensional fashion in Figures 4 and 5. Other available closed-form solutions are also shown in these figures for comparison. The latter were obtained using a circular load of the same area as the square one in ILLI-SLAB but are
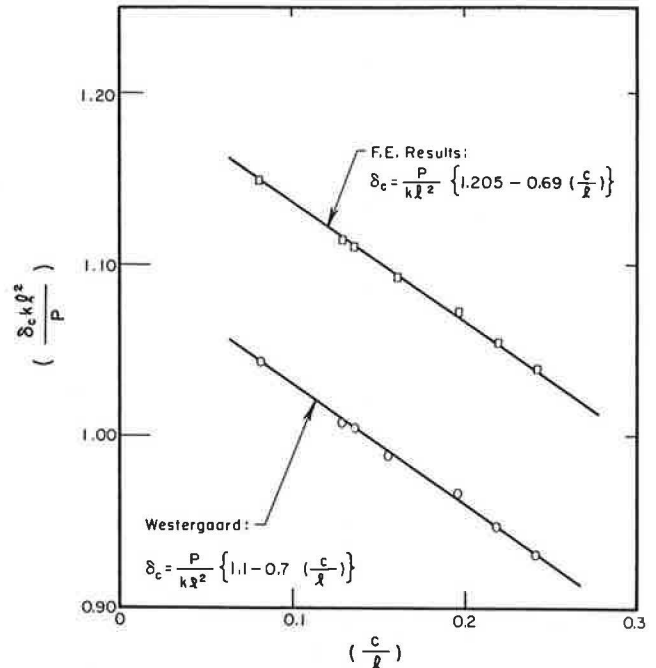
FIGURE 4 Comparison of ILLI-SLAB and Westergaard corner deflections.

plotted with the corresponding (c/ℓ) values along the ordinate. Though not mathematically rigorous, this facilitates direct comparison of the results. ILLI-SLAB stresses are values of the minor (tensile) principal stress occurring at the top fiber of the slab. The maximum normal stress obtained from finite element analysis (P. Frey, "Development of a Finite Element Based Expression to Predict Maximum Corner Loading Stresses in a Uniformly Supported Rigid Pavement on an Elastic Subgrade," University of Illinois, Urbana, 1983) significantly underestimates the critical stress.

Curves were fitted to ILLI-SLAB data with a special effort made to keep the general form of the equations the same as that of the Westergaard formulas. Thus a straight line may be used to describe corner deflections, δ_c, obtained using the finite element method (Figure 4). This line has the following equation:

$$\delta_c = (P/k\ell^2) \, [1.205 - 0.69 \, (c/\ell)] \qquad (4)$$

where c is the side length of square loaded area.

The similarity to Westergaard's equation indicates that Westergaard's approximation was fairly good. The finite element results obtained are typically about 10 percent higher than those predicted by Westergaard. A small part of this discrepancy is due to the lack of a theoretical solution for a square loaded area, as well as limitations of the finite element solution with respect to mesh fineness and slab size.

In the case of ILLI-SLAB maximum corner stresses, σ_c, curve fitting suggested the following equation:

$$\sigma_c = (3P/h^2) \, [1.0 - (c/\ell)^{0.72}] \qquad (5)$$

The Goldbeck-Older equation is obviously a rough approximation of the theoretical solution, much more so at high values of (c/ℓ). This is expected, in view of the Goldbeck-Older assumption of a concentrated load acting on a cantilever. Assuming that finite element results give a fairly accurate pic-
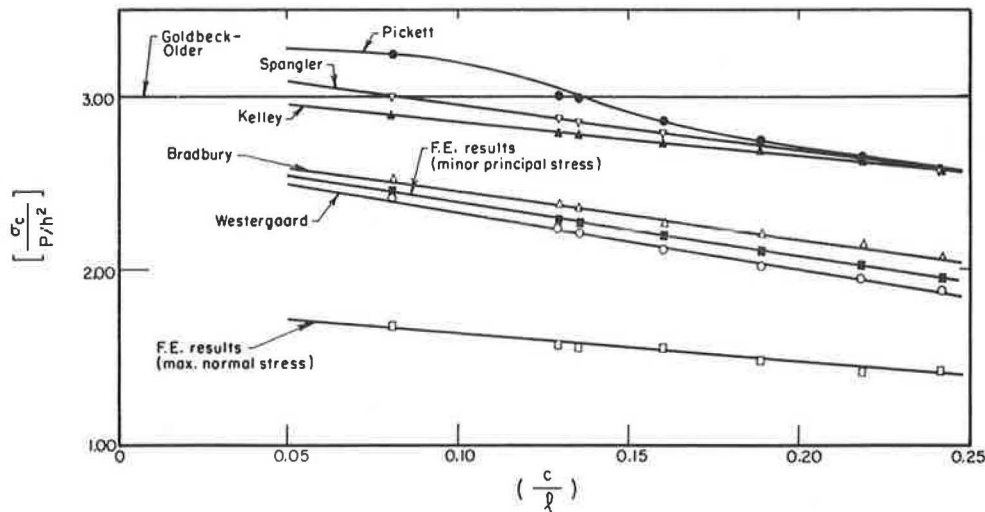
FIGURE 5 Nondimensional maximum bending stress for corner loading.

ture of the theoretical solution, the Westergaard equation represents a considerable improvement over the Goldbeck-Older one. The finite element method gives results that fall between those predicted by Westergaard (5) and those predicted by Bradbury (15). Note that the empirical modifications to the Westergaard formula proposed in the last 60 years--with the exception of Bradbury's--have tended to increase the discrepancy between calculated and theoretical stresses. These modified expressions are much closer to the Goldbeck-Older equation than to the theoretical solution. Scott (8) points out that "experimental indications are that the corner stresses experienced in practice are higher than the Westergaard equation indicates." Note, however, that, in the model tests mentioned previously, measured maximum corner stresses "were only 65 to 75 percent as great as those determined from the Westergaard equation" (22). The very significant limitations of the Winkler subgrade idealization for corner loading are reflected in Figure 5. As a result, discrepancies between measured responses and theory may be expected. In the absence of more conclusive field data, it is prudent to design for a higher corner stress than indicated by Westergaard's formula (Equation 3c).

Location of Maximum Stress

The results from a selected number of ILLI-SLAB runs are given in Table 3, where X_1 as obtained from

Equation 3h is compared with the location of the minor (tensile) principal stress given by ILLI-SLAB. This shows that ILLI-SLAB usually gives a somewhat greater distance than Westergaard. On the other hand, model tests suggest values about 85 percent of Westergaard's (22). Curve fitting through the values of X_1 obtained by extrapolation from ILLI-SLAB, resulted in the following equation:

$$X_1 = 1.80 \, c^{0.32} \, \ell^{0.59} \tag{6}$$

This best-fit equation indicates that the influence of the radius of relative stiffness, ℓ, is much greater than that of the size of the loaded area. Westergaard's equation suggests that these two parameters contribute equally to the determination of X_1.

Slab Size Requirements for Corner Loading Westergaard Responses

The pertinent results from this study are shown in Figure 6, in which ILLI-SLAB deflections and stresses are shown as percentages of the values given by the best-fit equations. The validity of any conclusions drawn from such a comparison is not considered to be greatly affected by the numerical accuracy of the proposed formulas. The patterns observed in Figure 6 are the same as those observed for the other loading conditions. Once again, stresses converge faster, requiring a minimum (L/ℓ) value of about

TABLE 3 Location of Maximum Corner Stress

Run No.	ℓ (in.)	c (in.)	Equivalent a (in.)	Location of σ_c, X_1 (in.)			
					ILLI-SLAB		
				Equation 3h	At Node	By Extrapolation	Proposed Equation
C001	31.07	2.5	1.41	15.74	18.03	18.50	18.33
C002	31.07	5.0	2.82	22.26	25.50	22.93	22.88
C003	31.07	7.5	4.23	27.27	30.41	26.00	26.05
C004	36.95	5.0	2.82	24.28	25.50	25.45	25.34
C005	26.13	5.0	2.82	20.42	20.62	20.67	20.66
C006	22.92	5.0	2.82	19.12	18.03	19.11	19.12
C007	38.56	5.0	2.82	24.80	25.50	26.10	25.99
C011	31.07	5.0	2.82	22.26	25.50	22.93	22.88
CT3	31.07	10.0	6.56	33.96	28.28	28.42	28.56

Note: Equivalent a = $(c/\pi^{1/2})$, Equation 3h: $X_1 = 2[(a_1 \ell)^{1/2}]$, $a_1 = (2^{1/2})$ a, and proposed equation: $X_1 = 1.80 \, c^{0.32} \, \ell^{0.59}$
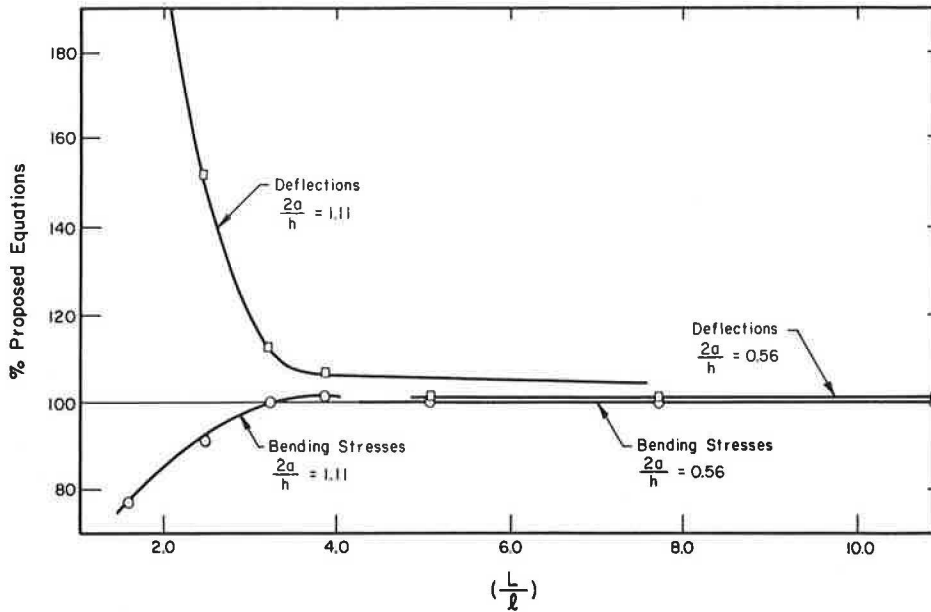
FIGURE 6   Effect of slab size on maximum corner loading responses.

4.0, and are less sensitive to changes in (L/ℓ) than are deflections. The latter are extremely sensitive even to small changes for (L/ℓ), smaller than about 3.0. Infinite slab deflection requires an (L/ℓ) ratio of at least 5.0. Slab size requirements established during this study are summarized in Table 4.

Effect of Size of Loaded Area

Figure 7 shows finite element corner loading responses as a function of (c/ℓ), where c is the side length of the applied square load. Responses are normalized with respect to the values obtained using proposed Equations 4 and 5.

TABLE 4   Slab Size Requirements for Westergaard Responses Based on the Finite Element Method

Load Placement	(L/ℓ) Values for	
	Maximum Deflection	Maximum Bending Stress
Interior	8.0	3.5
Edge	8.0	5.0
Corner	5.0	4.0

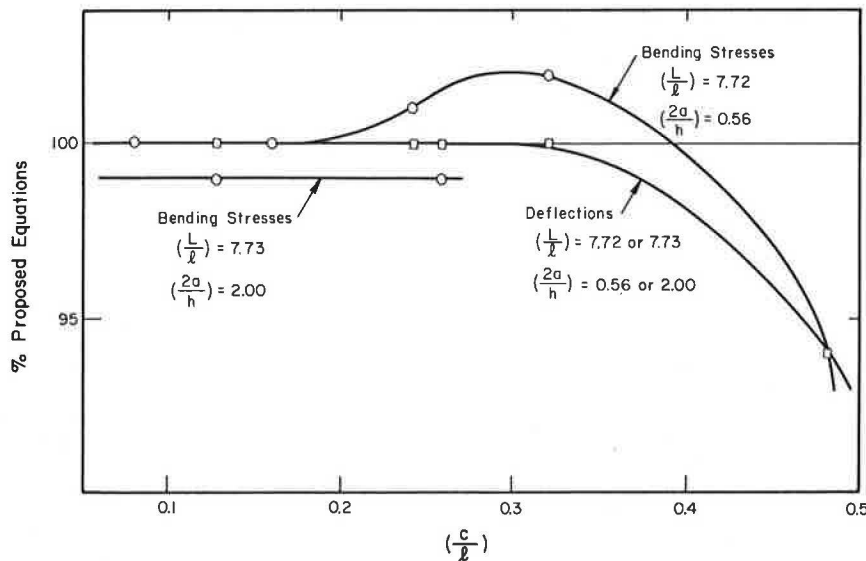Note: L = least slab dimension and ℓ = radius of relative stiffness.

FIGURE 7   Effect of size of loaded area on maximum corner responses.

Deflections are not very sensitive to changes in (c/ℓ), but stresses diverge from the "theoretical" values as (c/ℓ) exceeds about 0.2. The trend exhibited by corner stresses is the reverse of that for interior loading. The effect of (c/ℓ) on corner stresses is less pronounced than on interior stresses. Comments made earlier with respect to this effect are also generally applicable to corner loading.

CONCLUSION

Since their first appearance, beginning in the early 1920s, Westergaard equations have often been misquoted or misapplied in subsequent publications. To remedy this situation, a reexamination of these solutions using the finite element method has been presented. This exercise yielded a number of interesting results:

1. Several equations ascribed to Westergaard in the literature are erroneous, usually as a result of a series of typographical errors or misapplications, or both. The correct form of these equations and their limitations have now been conclusively established (Equations 1-3).
2. Westergaard's original equation for edge stress (5) is incorrect. The long-ignored equation given in his 1948 paper (10) should be used instead.
3. Improved expressions for maximum corner loading responses have been developed (Equations 4-6).
4. Slab size requirements for the development of Westergaard responses have also been established (Table 4).

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Establishing Load Transfer in Existing Jointed Concrete Pavements

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ABSTRACT

In this paper are described the results of a research project that had the objective of developing construction procedures for restoring load transfer in existing jointed concrete pavements and of evaluating the effectiveness of the restoration methods. A total of 28 test sections with various load transfer devices were placed. The devices include split pipe, figure eight, vee, double vee, and dowel bars. Patching materials used on the project included three types of fast-setting grouts, three brands of polymer concrete, and plain portland cement concrete. The number and spacing of the devices and dowel bars were also variables in the project. Dowel bars and double vee devices were used on the major portion of the project. Performance evaluations were based on deflection tests conducted with a 20,000-lb axle load. Horizontal joint movement measurements and visual observations were also made. The short-term performance data indicate good results with the dowel bar installations regardless of patching materials. The sections with split pipe, figure eight, and vee devices failed in bond during the first winter cycle. The results with the double vee sections indicate the importance of the patching material to the success or failure of the load transfer system: some sections are performing well and other sections are performing poorly with double vee devices. Horizontal joint movement measurements indicate that neither the dowel bars nor the double vee devices are restricting joint movement.

Many miles of Interstate pavement have been constructed using plain jointed concrete pavements of various thicknesses and joint spacings. The presence of a joint is a discontinuity that causes higher stresses and deflections in the pavement especially in the outside corner area. Many designs of jointed concrete pavement relied on aggregate interlock to provide for the transfer of the load across the joint, thereby reducing stress concentration and deflections under load. Laboratory studies conducted by the Portland Cement Association (PCA) found that the effectiveness of load transfer from aggregate interlock depended on load magnitude, number of repetitions, slab thickness, joint opening, subgrade value, and aggregate angularity (1). It was also found that the effectiveness decreased with cumulative load applications.

The variability of the amount of load transfer available from aggregate interlock created by changes in joint openings points out the need to provide for a more positive means of load transfer. In Georgia, and in many other states, dowel bars are placed in newly constructed pavements. Many older concrete pavements do not have the dowel bars and this absence of a positive means of load transfer is a factor that contributes to the deterioration of these pavement sections. Faulting measurements made in Georgia in 1972 on projects that contained both doweled and nondoweled joints indicated that the presence of dowels reduced the rate of faulting (2).

The distress found in plain jointed concrete pavements in Georgia generally has been caused by the presence of an erodible base or subgrade, infiltration of surface water into the pavement system,