- The concepts of axle-connector (negative offtrack term) and connector-axle distances as they interrelate with overall wheelbase to determine the vehicle-dependent value ΣL^2,
- The correct half-track adjustment of the turning radius to determine the vehicle centerline radius (R), and
- The relative simplicity of the calculations when the formula components have been properly defined.

As a general overview of the formula application, it can be said that the WHI offtrack formula is an accurate and expeditious tool for comparing worst-case vehicle turning performance. Worst case is emphasized because the steady-state values as computed virtually always exceed those for a 90-degree turn. Be aware, however, that the formula may break down for long units on short-radius curves.

Not even mentioned was that the mathematical formulation of the Canadian transient offtrack model now offers the capability to compute maximum offtrack for any given degree of turn. The formulas available in the report by Woodrooffe et al. (6) can be used to adjust the steady-state value when it has been determined. That discussion, however, is a follow-on subject and will not be attempted as part of this presentation.

Offtracking calculations and their interpretations are indeed skills that are "honed" only with frequent use. Further, when improperly used any special-purpose tool will fail to do the job for which it was designed. The purpose of this presentation has been to outline the concepts and procedures required to correctly use the WHI offtracking formula.

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Vehicle Offtracking Models

MICHAEL W. SAYERS

ABSTRACT

When a vehicle turns, the rear wheels track inside the path traced by the front wheels. This behavior is called offtracking and can lead to problems when large trucks operate in confined areas. The methods that have been used by designers to estimate the offtracking of heavy trucks are reviewed, and then a computer method for graphing the complete swept path of an arbitrary vehicle making any type of turn at low speed is described. The method is valid for nearly all truck configurations in use on the highways, including double and triple combinations. The paper includes several example plots, and a computer program that uses this method, developed for the Apple II computer, is described. The program is available free from the Federal Highway Administration.

Motor vehicles typically employ a single steered front axle followed by one or more unsteered rear axles. In low-speed turns, the rear wheels track inside the paths taken by the front wheels, such that the path swept by the vehicle is wider than the vehicle itself. Figure 1 shows how this behavior results in an additional swept width, called offtracking, for the vehicle. Offtracking can pose problems whenever there is not enough space to accommodate both the width of the vehicle and the additional offtracking displacement. Thus engineers laying out geometric designs for intersections, parking areas, and other locations with restricted geometry need to address the potential offtracking requirements of the largest vehicles that will be using the area.

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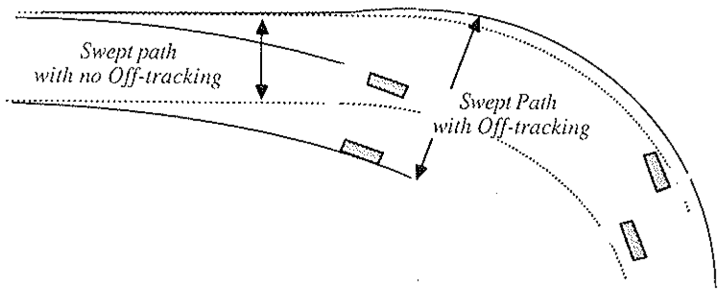

FIGURE 1 Example of the offtracking effect for a single-unit vehicle.

The methods that have been used by designers to estimate the offtracking of heavy trucks are reviewed and their limitations are mentioned. A computer-based method for graphing the complete swept path of an arbitrary vehicle making any type of turn at low speed is then described and demonstrated. The computer method presented is valid for nearly all truck configurations in use on the highways, including double and triple combinations. A computer program that uses this method has been developed for the Apple II computer and is available to the public from FHWA. When equipped with the appropriate plotting hardware, the program produces high-quality scaled drawings of vehicle offtracking. By using this program or an equivalent, the designer can see just how much space will be required by various vehicles to navigate a turn.

Most vehicle models used for offtracking predictions are one dimensional and neglect effects of vehicle width during low-speed turns. The assumptions underlying a one-dimensional "bicycle model" are relevant to the range of applications for which the models are valid and are described first.

BICYCLE MODEL

Description

In this paper are discussed models that assume that all nonsteered wheels that are rigidly connected can be represented by a single "equivalent wheel" located near the centroid of the actual wheel positions. Because highway vehicles are symmetric from right to left, with each wheel on the right side of the vehicle having a corresponding wheel on the left side, the model is based on a single wheel located at the center of the axle. Thus the vehicle is modeled geometrically as a bicycle. Multiple-axled suspensions are similarly modeled as a single effective axle, usually located at the geometric center of the nonsteered axles.

Figure 2 shows how an 18-wheeled tractor-semitrailer combination vehicle would be represented by two linked bicycle models. The bicycle model for the tractor has the front point coinciding with the center of the front axle and the rear point coinciding with a point midway between the two rear axles. The wheelbase, designated L_1, is the distance between these points. Note that the wheelbase parameter is less than the longest wheelbase dimension of the tractor because it does not extend to the second rear axle. Naturally, it is also less than the overall length of the tractor. The wheelbase for the semitrailer, designated L_2, is the distance between the hitch and the center point of the two axles. The front point of the semitrailer does not necessary coincide with the rear point of the tractor unit, and therefore the offset distance, designated λ_1, is also needed. The offset is shown as a positive quantity in the figure because it is in front of the

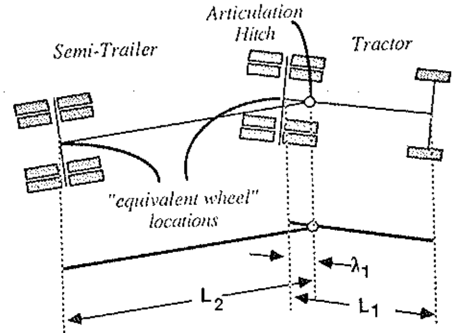

FIGURE 2 Two linked "bicycle" vehicle models.

equivalent wheel position. When the hitch point is located behind the rear wheel, a negative value is used.

Terms

In this paper, each component (tractor, semitrailer, dolly, and so forth) of a combination vehicle is called a "vehicle unit." The rear point in the bicycle model will be referred to as the "rear axle" for convenience, although it is recognized that it is actually the center of the two or more rear wheels in the actual vehicle being modeled. For a single-unit vehicle, such as a truck, bus, or automobile, the front point always corresponds to the center of the steered front axle. The same is true for the tractor in a combination vehicle. For towed units, the front point always corresponds to the hitch location. Because the front point may represent either the center of a steered axle or a hitch location, it will be referred to simply as "the front point."

Limitations

When turning at low speeds, the unsteered rear wheels of a vehicle follow a path that is determined mainly by two factors: (a) the paths taken by the front wheels and (b) the fixed geometric relationship between the front and rear axle or axles. At higher speeds, the masses of the vehicle components cause forces that resist change in direction. These forces interact with tire forces to determine where the vehicle goes. Usually, the effect of the masses is to force the rear wheels to the outside of the turn, reducing the offtracking. Although it is possible in some cases for the rear axle or axles to actually track outboard at high speeds, the swept width is generally largest at low speeds. Thus models based only on kinematic relations can be used to predict "worst-case" offtracking.

There is a great deal of additional mathematical complexity that is encountered when going from a simple bicycle model to one that includes details of tire mechanics. Fortunately, these effects have only a slight influence in most of the situations that concern pavement layout design. Therefore this paper deals exclusively with methods based on the bicycle model. Before continuing, however, a few exceptional cases will be mentioned for which the bicycle model is not appropriate. The bicycle model is inaccurate in these cases because the tire mechanics act in such a way that identically steered wheels do not perform the same as a single equivalent wheel, as is assumed in the bicycle model.

For an axle with dual tires to follow a curved path, it is necessary for the individual tires to assume nonzero slip angles, generating forces that cancel but nonetheless create a steering moment. This moment acts about the center of the axle and resists any turning. The same effect is created when there are two or more nonsteered axles rigidly connected in a tandem or triple suspension. A result of these tire forces is that significant steering effort is required to navigate a turn in contrast to the zero steering effort that would be adequate if all tires were to roll with no slip.

The steering effort is not of concern as long as the required forces are available. One limitation is the friction of the pavement surface. If the required steering forces at the front axle exceed the friction available, then the front tires will slide and the vehicle will not make the turn as predicted with a no-slip bicycle model. This might happen for a special vehicle with many heavily loaded rear axles that are unsteered and a lightly loaded front axle, but such behavior would be most uncommon for a highway vehicle.

Another case in which a single wheel with zero slip is not a good representation of a group of unsteered wheels is that in which the tires on the wheels are not identical or all the tires in a group are not loaded equally.

When the tires are more or less the same, and equally loaded as intended, the steering moments generated by the nonsteered tires have only a minor influence on the vehicle tracking performance on high-friction surfaces (that is, dry pavement) and can be included in a bicycle model by modifying the wheelbase parameter (1). This effect is typically so slight that a simple geometric averaging is generally acceptable for obtaining the wheelbase parameters of the bicycle model.

On slippery surfaces some of the tires can reach the frictional limits while others do not, resulting in different offtracking performance than would be obtained on a high-friction surface (2). The bicycle model applies only for the case of a high-friction surface.

Although there is nothing to prevent the bicycle model from being used for vehicles with steerable rear axles, steerable rear axles are not treated in this paper. All of the analyses that follow apply only to vehicles with unsteered rear axles, for which an equivalent vehicle unit wheelbase can be assumed.

ANALYSES IN USE

Maximum Offtracking

For a given radius of turn, the maximum offtracking occurs when the vehicle has reached a steady-state condition. The case of steady turning is relatively simple to analyze for the bicycle model. Because the vehicle is a rigid body, there is a center of rotation about which every point in the body rotates. The no-slip condition at the rear means that the circle traced by the rear wheel must be perfectly tangent to the vehicle body, as shown in Figure 3. This condition of tangency means that the Pythagorean theorem can be used to calculate the radius of the circular path at the rear wheel:

$$R_1^2 = R_{in}^2 - L_1^2 \tag{1}$$

Thus the no-slip condition requires that the rear axle must follow a smaller radius than does the front, such that steady-state offtracking is always inboard. Note that the offtracking is maximum at the equivalent rear axle position because it is only at the position of the rear axle that the vehicle frame is exactly tangent to the curve. Either forward or aft of this position, the vehicle must be further from the circular curve (and thus it must lie on a longer radius curve) as can be seen from the figure.

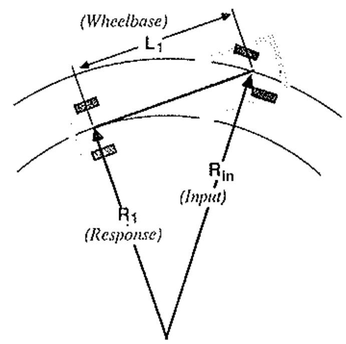

FIGURE 3 Use of the Pythagorean theorem to compute maximum (steady-state) offtracking in a constant-radius turn.

Just as the steady turning of a motor vehicle is determined by the radius of the path followed by the front axle, the turning of a towed trailer is determined by the radius of the articulation point (hitch location). Figure 4 shows the geometry for the case of a tractor-semitrailer vehicle. As before, the Pythagorean theorem applies to the tractor, such that the radius of the effective rear axle (R_1) can be calculated using Equation 1. The Pythagorean theorem

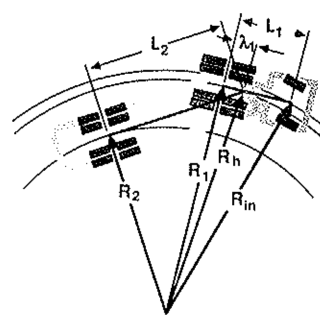

FIGURE 4 Use of the Pythagorean theorem to analyze maximum (steady-state) offtracking of a tractor-semitrailer in a constant-radius turn.

is used consecutively to next calculate the radius of the path traced by the hitch

$$R_h^2 = R_1^2 + \lambda_1^2$$
$$= R_{in}^2 - L_1^2 + \lambda_1^2 \quad (2)$$

and it is used once more to compute the radius of the effective trailer axle:

$$R_2^2 = R_h^2 - L_2^2$$
$$= R_{in}^2 - L_1^2 + \lambda_1^2 - L_2^2 \quad (3)$$

The path taken by the hitch is the input for the trailer, just as the path taken by the front axle is the input for the tractor. This procedure can be extended for more trailing units. Thus complex combination vehicles can be analyzed for steady turning (maximum offtracking) simply by repeated application of the Pythagorean theorem. When this is done, the cumulative offtracking of the rearmost effective axle can always be calculated directly. A simplified version of the general formula was originally recommended by the Western Highway Institute (WHI) (3) and has been adopted as a recommended practice by the Society of Automotive Engineers (SAE) (4). (The hitch offset parameters, designated λ_i in this paper, are neglected because they are usually much smaller than the wheelbase parameters and thus have a negligible effect when squared in Equation 3.)

The steady-turning scenario, represented by the SAE formula, gives only the maximum offtracking that will eventually occur for a given vehicle configuration and input radius. However, for many large trucks the steady turn condition is not reached until the vehicle has turned more than 360 degrees. For tight (small-radius) turns, Equation 3 may not have a solution. For example, typical length parameters for a 60-ft tractor-semitrailer combination vehicle are

L_1 = 16.5 ft (5.0 m),
L_2 = 37.0 ft (11.3 m), and
λ_1 = 0.

Equation 3 will give a zero radius for the rearmost axle when R_{in} = 40.5 ft (12.4 m); for any shorter radius the trailer is forced backward and Equation 3 cannot be used.

In practice, nearly all situations for which offtracking performance is desired are transient. The steady-turning relations were presented here mainly as an introduction to the more generalized analyses of transient turning that follow.

Tractrix Integrator

The transient path followed by the rear axle in a bicycle model is called the general tractrix of the path followed by the front point. The tractrix is defined by the two mathematical constraints that have been illustrated in Figures 3 and 4, namely,

1. The rear axle is always a constant distance from the front axle (wheelbase) and
2. The path traced by the rear axle is at all times tangent to the line connecting the rear axle to the front axle (no-slip condition for the unsteered wheel).

The tractrix integrator is a drafting instrument that can be used to trace the tractrix of a curve (3,5). The instrument consists of a bar supported at one end by a stylus (the front point) and at the other by a single knife-edge wheel (the rear axle). The tractrix integrator is essentially a physical bicycle model.

The distance between the wheel and the stylus or the integrator can be adjusted to model different wheelbases. To use the integrator, a scaled drawing is prepared for the input curve, which would be followed by the front axle of the vehicle of interest. The distance between the stylus and the wheel of the tractrix integrator is adjusted to match the wheelbase of the vehicle according to the scale chosen for the drawing. The wheel of the integrator is coated with wet ink, and the input curve is carefully traced with the stylus. The inked wheel, rolling in line with the bar, draws the tractrix. For a combination vehicle, the instrument would next be adjusted to match the wheelbase of the trailer, and the process would be repeated using the tractrix of the lead unit vehicle as the input for the second unit. Thus the path followed by the rear axle of the trailer is the tractrix of a tractrix.

The tractrix integrator can be used for any tractor-trailer combination and any type of input path. The procedure described for tractor-trailer combinations can be extended to include double and triple combinations by using the tractrix of the previous unit as the input for the following unit. The tractrix integrator gives only the paths that would be taken by the center of the vehicle--the wheels in a bicycle model. To obtain the swept path, the draftsman must manually add the width of the vehicle.

The procedure used for multiple vehicle combinations does not allow for hitch locations that are offset from the equivalent axle locations. Often these offsets are fairly small relative to the wheelbase measurements so this error is negligible.

Exact Solution

General mathematical solutions for the tractrix of both straight-line and circular steering curves have been derived (5) and can be used to show quantitatively just how the radius of the rear axle varies during the turn. Considering the simplicity of the vehicle geometry and path inputs, the relations are striking in their complexity. Because the path of the rear axle of the tractor is not circular, the recursive approach used with the Pythagorean theorem and tractrix integrator cannot be used with the exact solutions. The exact solution is therefore limited to a single-component vehicle, unless great liberties are taken when formulating engineering approximations.

Design Templates

The most popular method for estimating offtracking requirements involves overlaying a template with a scale drawing of the design area. The template shows the swepth path of a specific vehicle in a specific turn--typically a 45-ft radius for the outside wheel, which corresponds to a 41-ft radius for the center of the front axle. For these templates, the vehicle approaches the turn along a straight line, follows the constant radius arc for a specified arc angle, and then departs in a straight line. The arc angles are typically 90 and 180 degrees, although other angles are sometimes also shown. The AASHTO green book includes figures for several design vehicles (6), and similar templates are available from other sources (3). Most of the design templates were prepared graphically using a traxtrix integrator.

NUMERICAL METHOD

None of the methods discussed thus far are completely satisfactory as an everyday design tool. The templates can offer only an approximate indication of the offtracking that a design vehicle would exhibit in a reference turn. The radii used as inputs for the templates may have little in common with the design area. A more immediate problem is that templates are not necessarily available for the vehicle of interest, particularly if it was not previously allowed on public roads. Only the tractrix integrator is capable of providing a representative simulation of an arbitrary vehicle following an arbitrary path. As a drafting instrument, however, it requires scale drawings and a certain amount of skill in its use and interpretation, and it has not proven practical for everyday use. The alternative that follows is basically a computer simulation of the tractrix integrator.

General Approach

A numerical offtracking solution must duplicate the operation of the tractrix integrator. Thus the constraints that define a tractrix need to be translated into mathematical equivalents. Although a generalized mathematical solution to the tractrix problem is not known, it is relatively simple to solve the tractrix equations for very short distances. The general solution method is therefore one of stepping through the trajectories.

Because the computations are intended to be programmed into a computer, it is convenient at this point to consider a flowchart of the simulation, which is shown as Figure 5. The flowchart shows three "loops" through which the flow of the program might be redirected to repeat computational sequences. The main loop, which goes from the bottom diamond box to the box labeled b, indicates that, when all of the calculations have been performed for a specific point along the input path, the vehicle is moved forward slightly and the process is repeated for the new position. The increment (Δs) is usually set to a value of 1 ft (0.3 m). When the vehicle has reached the end of the path, the program finishes as indicated by QUIT at the bottom of the chart.

There are also two inner loops in which calculations are repeated for each unit in the combination vehicle. The letter n in the decision diamonds indicates the number of vehicle units and would be set to n = 1 for an automobile, n = 2 for a tractor-semitrailer, n = 4 for a doubles combination (tractor, semitrailer, dolly, pup semitrailer), and so forth. Note that for a single-unit vehicle, none of the calculations would be repeated and there would only be the single loop involving the calculation of position as the vehicle stepped through the maneuver.

Coordinates of a Point in a Vehicle Unit

When the position of the rear wheel of a vehicle unit and the heading angle are both known, the position of any point associated with that unit can be calculated on the basis of the position of the point within the vehicle unit. Figure 6 shows the x-y coordinate systems used in this paper to describe points lying on a vehicle unit. There is an absolute coordinate system needed to describe the positions of the vehicle units as they trace a path, designed with capital letters, X,Y. The origin of the system is arbitrary and can be set to any convenient location. (The beginning of the input path is one such convenient location.) In addition, each unit has its

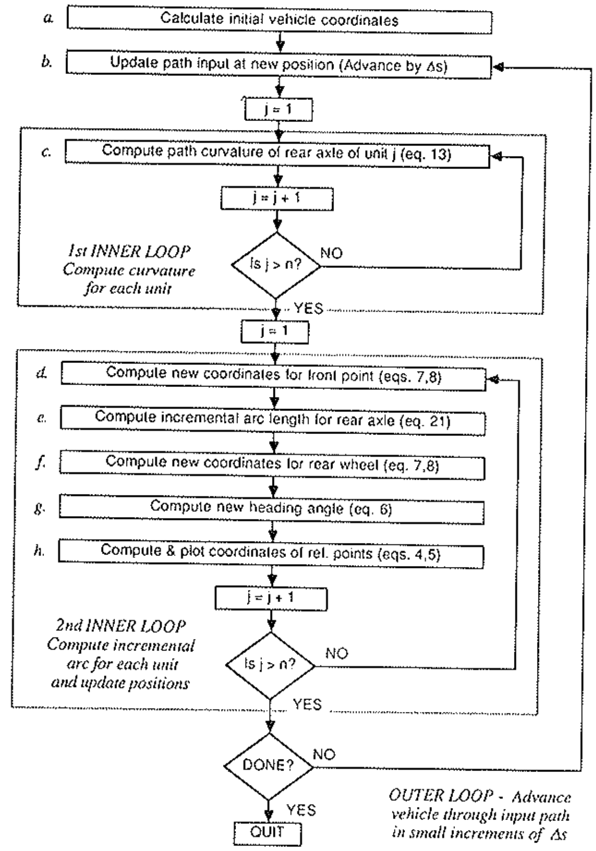

FIGURE 5 Flow chart for offtracking computation method.

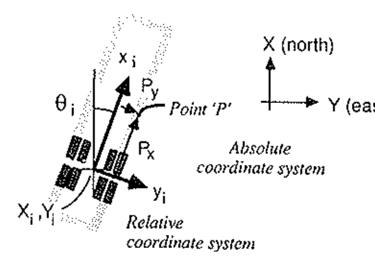

FIGURE 6 Coordinates of arbitrary point P in vehicle unit.

own relative coordinate system, designated by subscript lower-case letters (x_i, y_i). As shown in the figure, the origin of the relative coordinate system (x_i, y_i) has absolute coordinates (X_i, Y_i) and is located at the position of the rear axle of that unit. Note that positive x values lie in front of the axle and that positive y values lie on the right side of the vehicle centerline. The absolute coordinates of the point indicated in the figure are

$$X_p = X_i + P_x \cos \phi - P_y \sin \phi \tag{4}$$

$$Y_p = Y_i + P_x \sin \phi + P_y \cos \phi \tag{5}$$

where X_p and Y_p are the absolute coordinates of the point P, P_x and P_y are the relative coordinates of the point P within the vehicle unit, and X_i and Y_i are the absolute coordinates of the rear wheel of the unit.

Points of interest that would be located using Equations 4 and 5 are the front point (with relative coordinates $P_x = L_i$, $P_y = 0$) and the hitch location (coordinates $P_x = \lambda_i = P_y = 0$). Two other points of interest are (a) the outer-front corner of the leading vehicle unit, which usually defines the outer edge of the swepth path, and (b) the inner-rear wheels on the rearmost unit, which usually define the inner edge of the swept path.

Equations 4 and 5 are used directly in plotting reference points (Step h in Figure 5) and are also used to determine the initial vehicle position (Step a in Figure 5). When used to begin the simulation, X_i and Y_i represent the front point of the unit, and X_p and Y_p are the calculated initial position of the rear axle, using $P_x = -L_i$, and $P_y = 0$.

Characterization of Input Path

Most of the time, designers are interested in the case of the vehicle making a circular turn for some angle of interest (typically 90 degrees) and then exiting the turn in a straight line. Thus the path input is represented by a circular arc and a tangent line. A more general representation would be helpful, however, so that offtracking simulations could deal more realistically with the types of maneuvers that truck drivers actually make. For example, when turning to the right in an intersection, the driver might first turn to the left to make better use of available space.

A generalized input path could be specified as a series of x-y coordinates at closely spaced intervals, but this would reduce flexibility in selecting an appropriate distance increment, and requires an assumption of how the points are connected (straight lines, arcs, polynomial function) in order to derive a solution. In this paper the input path will be characterized as a sequence of arcs. The end point of one arc is also the beginning point of the next, and the arcs are constrained to be tangent where they meet. Using this method, most input paths of interest can be represented with just two arcs--the first a constant-radius turn, and the second a straight line out of the turn. When more complex paths are desired, they can be built up easily from congruent arcs, as shown for three examples in Figure 7. Figure 7a shows the simple case for two arcs, the second of which has zero curvature. Figure 7b shows a more complex type of turn that could be used to model a maneuver in which the driver first turns to the left in order to obtain more room for a right turn. It is composed of four arcs, the fourth of which has zero curvature. Figure 7c shows a lane-change type of path, which could be used to model the maneuver made by a bus pulling into a bus-stop lane and then leaving.

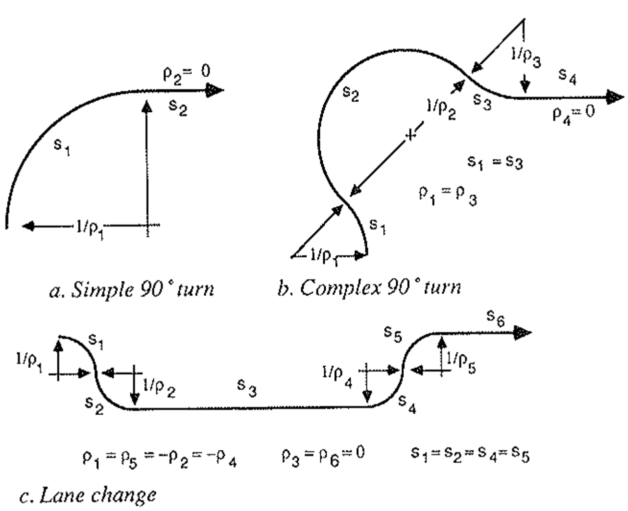

FIGURE 7 Three maneuvers represented as sequences of circular arcs.

Each arc segment in a path is subject to two constraints at the end points to maintain continuity: (a) the end points of consecutive arcs must meet and (b) they must be tangent to each other. As a result of these constraints, each arc can be defined mathematically by two parameters: radius and length. Mathematically, it is more convenient to use curvature--the inverse of radius with units of 1/length--than direct radius (which is infinite for a straight line) or degree of curvature (which has arbitrary units and requires conversion factors). Turns to the right are indicated in this paper as positive curvature, straight lines have zero curvature, and curves to the left have negative curvature. The curvature of an arcs is indicated as ρ, and radius is therefore $1/\rho$.

The second parameter used in the following derivations is arc length, indicated as s. The arc length is used instead of the interior angle because arc length is relevant for straight lines, whereas an interior angle is not.

In addition to the two parameters for each arc, the X-Y coordinates of the first point and the heading angle (ϕ) at the first point can be included for plotting purposes to match the coordinates of the input path to another coordinate system. When these values are specified for the first arc, corresponding coordinates and heading angle can be computed for all subsequent arcs from the conditions of continuity.

Coordinates of a Point on an Arc

To begin the mathematical representation of vehicle offtracking, consider the computation of the x-y coordinates of an arbitrary point on a circular arc. Figure 8 shows a sketch of an arc with curvature ρ and length s. In addition, the coordinates of the beginning point of the arc are given as X_0, Y_0, and the initial heading angle is ϕ_0. At distance s along the arc, the heading angle will be the initial angle plus the angle subtended by the arc. The angle is the product of the arc length and the curvature, and thus

$$\phi = \phi_0 + s\,\rho \tag{6}$$

The coordinates of the end of the arc (for nonzero curvature) can be written as

$$X = X_0 + (1/\rho)\,[\sin(\phi_0 + s\rho) - \sin\phi_0]$$

$$Y = Y_0 - (1/\rho)\,[\cos(\phi_0 + s\rho) - \cos\phi_0]$$

which can be manipulated (using trigonometric identities) to yield

$$X = X_0 + s\,[\mathrm{sinc}\,(s\rho)\cos\phi_0 - \mathrm{sinc}\,(s\rho/2)\sin\phi_0\sin(s\rho/2)] \tag{7}$$

$$Y = Y_0 + s\,[\sin(s\rho/2)\,\mathrm{sinc}\,(s\rho/2)\cos\phi_0 + \mathrm{sinc}\,(s\rho)\sin\phi_0] \tag{8}$$

where the function sine curve (sinc) is defined as

$$\mathrm{sinc}\,(x) \equiv \sin(x)/x \tag{9}$$

Although Equations 7 and 8 are derived for nonzero curvature, they are also valid for straight lines when $\rho = 0$ when they revert to simpler form [if $\rho = 0$, then sinc $(s\rho)$ = sinc $(s\rho/2) = 1$, and sin

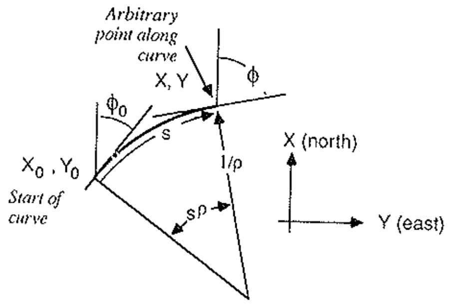

FIGURE 8 Coordinates along a constant-radius curve.

$(s\rho/2) = 0]$. These equations are used in several places in the offtracking simulation to compute new coordinates for various points (Steps b, d, and f in Figure 5).

Rigid Body Rotation

As a rigid body follows a curved path, at any instant it can be characterized by a center of rotation. Figure 3 shows this for a single vehicle unit and also indicates how the center of rotation is calculated for the bicycle model: it is the intersection of the two radial lines (labeled R_1 and R_{in}) that pass through the end points of the bicycle and are normal to the paths followed by those points. Figure 4 shows the special, steady-state case in which both units of a two-unit vehicle have the same center of rotation. In the figure both intersections occur at the same point because this is an illustration of the steady-state case. For transient offtracking, the two centers of rotation change as the vehicle progresses and do not coincide.

For small movements about any given position, the paths of all points on the vehicle are approximately circular, as defined by the instantaneous curvature. Furthermore, the approximation becomes more exact as the distances become smaller. Thus the movements of the axles of the vehicle can be computed using Equations 5-8 if the distance s is small. For most applications, a step interval of several feet is adequate, and an interval of $\Delta s = 1$ ft (0.3 m) is a conservative choice to keep errors negligible.

The method used to compute offtracking can be summarized in two steps, which are repeated as shown by the two inner loops in the flow chart (Figure 5). In the first loop, the curvature at the rear axle is computed for each vehicle unit. In the second step, new positions for each unit are calculated on the basis of an incremental advance of the front axle of the lead vehicle unit.

Computation of Curvature for Vehicle Axles

Figure 9 shows the geometry for an arbitrary vehicle unit (j) hitched to the preceding unit (i) such that

$$j = i + 1$$

Note that two right triangles are formed, with the angles shown defined as

$$\tan \alpha = \lambda_i \rho_i \qquad (10)$$

$$\tan \beta = L_j \rho_j \qquad (11)$$

Also, the angle β can be written as a function of the heading angles of the two units

$$\beta = \theta_i - \theta_j + \alpha \qquad (12)$$

Equations 10-12 can be combined and manipulated to yield

$$\rho_j = [\tan(\theta_i - \theta_j) + \lambda_i \rho_i]/\{L_j [1 - \lambda_i \rho_i \tan(\theta_i - \theta_j)]\} \qquad (13)$$

Even though it is shown for two linked units, Equation 13 also applies to the first vehicle unit where $j = 1$ and $i = 0$. In this case, $\lambda_0 \equiv 0$, and θ_0 and ρ_0 are the current heading angle and curvature of the input path.

Starting with the first vehicle unit, Equation 13 is applied in turn to each vehicle unit to obtain the curvature at the rear axle for that vehicle unit, as shown by the first loop in the flow chart (Figure 5).

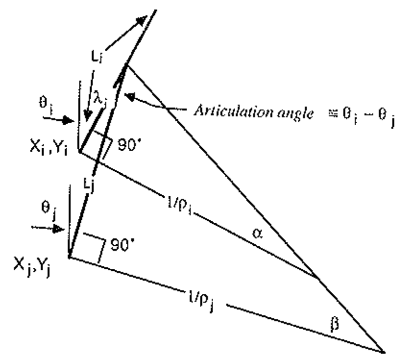

FIGURE 9 Computation of curvature of rear axle.

Updating Vehicle Positions

Equations 5-8 and 13 can be used to compute new coordinates and heading angles for each vehicle unit, if the correct arc distance is known. The incremental distances are not the same for each vehicle unit and will vary during the simulation. Figure 10 shows that the arc length can be defined by a point of intersection of two arcs with known centers and radii.

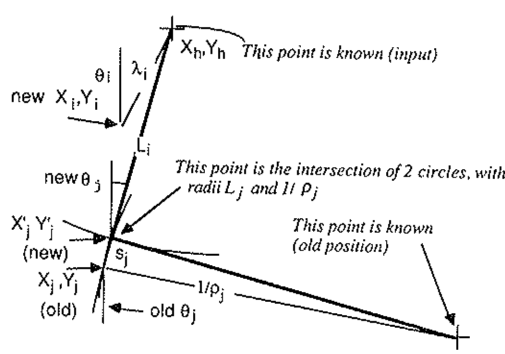

FIGURE 10 Calculation of new position of rear axle.

The coordinates of the new axle position (X_j, Y_j) can be expressed using Equations 7 and 8:

$$X_j' = X_j + s_j[\mathrm{sinc}(s_j\rho_j)\cos\theta_j - \mathrm{sinc}(s_j\rho_j/2)\sin\theta_j \sin(s_j\rho_j/2)] = X_j + S_j\delta_x \qquad (14)$$

$$Y_j' = Y_j + s_j[\sin(s_j\rho_j/2)\mathrm{sinc}(s_j\rho_j/2)\cos\theta_j + \mathrm{sinc}(s_j\rho_j)\sin(\theta_j)] = Y_j + s_j\delta_y \quad (15)$$

where X_j' and Y_j' are the new coordinates for the rear wheel of unit j, and

$$\delta_x = \mathrm{sinc}(s_j\rho_j)\cos\theta_j - \mathrm{sinc}(s_j\rho_j/2)\sin\theta_j\sin(s_j\rho_j/2) \quad (16)$$

$$\delta_y = \sin(s_j\rho_j/2)\,\mathrm{sinc}(s_j\rho_j/2)\cos\theta_j + \mathrm{sinc}(s_j\rho_j)\sin\theta_j \quad (17)$$

The coordinates of the front of the unit are the same as the coordinates of the hitch for the preceding unit. Applying Equations 7 and 8 gives the hitch coordinates:

$$X_h = X_i + \lambda_i\cos\theta_i \quad (18)$$

$$Y_h = Y_i + \lambda_i\sin\theta_i \quad (19)$$

The distance between the front and rear of the unit can be calculated from the coordinates (Equations 14, 15, 18, and 19) and must equal the wheelbase (L_j). The Pythagorean theorem gives

$$L_j^2 = (X_j + s_j\delta_x X_h)^2 + (Y_j + s_j\delta_y - Y_h)^2 \quad (20)$$

Equation 20 can be solved for s_j to yield

$$s_j = (\Delta_x\delta_x + \Delta_y\delta_y - \{(L_j^2 - \Delta_x^2 - \Delta_y^2)(\delta_x^2 + \delta_y^2) + (\Delta_x\delta_x + \Delta_y\delta_y)^2\}^{1/2})/(\delta_x^2 + \delta_y^2) \quad (21)$$

where

$$\Delta_x = X_j - X_h = X_j - X_i - \lambda_i\cos\theta_i \quad (22)$$

$$\Delta_y = Y_j - Y_h = Y_j - Y_i - \lambda_i\sin\theta_i \quad (23)$$

Equation 21 is not a complete mathematical solution for s_j because the terms δ_x and δ_y, which appear in the equation, are themselves functions of s_j (Equations 16 and 17). However, it becomes a good approximation if a close estimate of s_j is used in Equations 16 and 17. Because the increment Δs used in the computation is small enough for all of the paths to be approximately circular, the change in s_j from one increment to the next is small. Thus the value of s_j that was calculated for the previous position can be used in Equations 16 and 17 to compute δ_x and δ_y, and those values are used in Equation 21 to obtain the new value of s_j.

For the first calculation there is no previous value of s_j to use. However, if the multiple units of the vehicle are lined with each other (all articulation angles at the hitches are zero), then all axles must move the same distance. This orientation is assumed for starting purposes, and therefore each variable s_j is initially equal to the increment of the front axle, $s_0 \equiv \Delta s$.

EXAMPLES

Templates for Two Vehicles

Tables 1 and 2 give the parameters that are used to describe two vehicles of interest: a long (60-ft) tractor-semitrailer and a typical (65-ft) doubles combination. Note that a typical doubles combination is composed of four units: tractor, semitrailer, dolly, and second semitrailer. A triples combination would usually be composed of six units containing the four from the doubles plus an additional dolly and semitrailer. Table 2 includes a negative hitch offset (λ_2). This means that the hitch is behind the effective rear axle for the second unit, the first semitrailer. Figures 11 and 12 show traces of the two points that define a swept path in a right turn: the left front corner of the tractor and the midpoint of the right wheels of the rearmost unit. The path lying 4 ft to the left of the input path is also shown. The input for these two figures is a 41-ft-radius turn (at the vehicle center) followed for angles of 90, 180, 270, and 360 degrees. Although longer, the doubles combination sweeps a narrower path than does the tractor-semitrailer combination. Indeed, the tractor-semitrailer is too long to reach a steady-state condition for a 41-ft-radius turn, and Equation 3 (the SAE formula) has no solution for this vehicle.

Figure 13 shows the type of trajectory that would be predicted for the tractor-semitrailer for a continued turn. The figure is based on the same radius input but continues the turn for three complete circles (1080 degrees). Shortly after 360 degrees, the rear wheels of the trailer have tracked so far inboard that the trailer is actually being pushed backwards as the tractor progresses. As it is pushed backwards, it tracks outward in a diverging path un-

TABLE 1 Vehicle Parameters for Long Tractor-Semitrailer Combination (60 ft Overall, 48-ft Trailer)

Bicycle Model Parameters	Reference Points
$n = 2$	Left front corner of tractor (Unit 1)
L_1 = 17.5 ft (5.3 m), λ_1 = 2.1 ft (0.6 m)	P_x = 20.5 ft (6.25 m), P_y = -4.0 ft (1.2 m)
L_2 = 40.0 ft (12.2 m)	Left front wheel of tractor (Unit 1), P_x = 17.5 ft (5.3 m), P_y = -4.0 ft (1.2 m)
	Midpoint between right outside wheels of semitrailer (Unit 2), P_x = 0, P_y = 4.25 ft (1.3 m)

TABLE 2 Vehicle Parameters for Doubles Combination (Cab-Over-Engine Tractor, Two 28-ft Trailers, 65-ft Overall Length)

Bicycle Model Parameters	Reference Points
$n = 4$	Left front corner of tractor (Unit 1)
L_1 = 11.00 ft (3.4 m), λ_1 = 1.8 ft (0.5 m)	P_x = 14.0 ft (4.3 m), P_y = -4.0 ft (1.2 m)
L_2 = 22.8 ft (6.9 m), λ_2 = -2.2 ft (-0.7 m)	Left front wheel of tractor (Unit 1)
L_3 = 6.1 ft (1.9 m), λ_3 = 0	P_x = 11.0 ft (3.4 m), P_y = -4.0 ft (1.2 m)
L_4 = 22.8 ft (6.9 m)	Midpoint between right outside wheels of second semitrailer (Unit 4), P_x = 0, P_y = 4.25 ft (1.3 m)

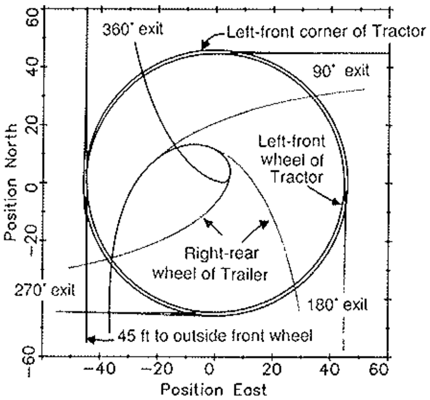

FIGURE 11 Offtracking for a 60-ft tractor-semitrailer vehicle in a 41-ft-radius turn (45 ft to the outside front wheel).

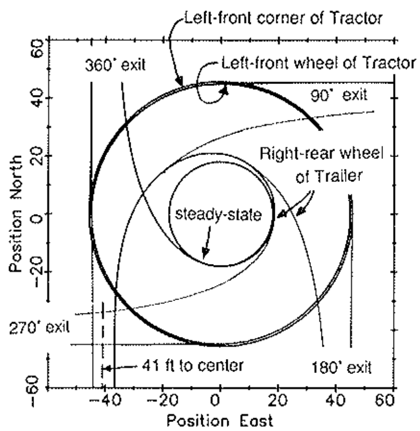

FIGURE 12 Offtracking for a 65-ft doubles combination vehicle in a 41-ft-radius turn (45 ft to the outside front wheel).

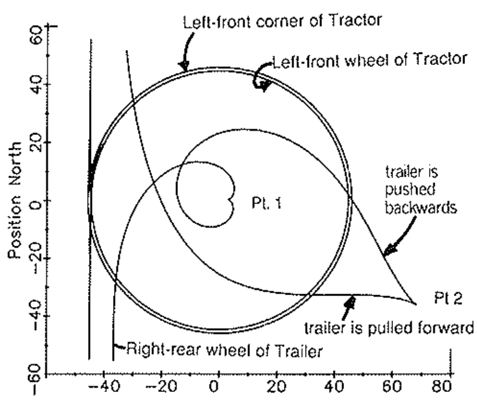

FIGURE 13 Example of a trailer backing up when attempting a continued short-radius turn.

til it is so far outboard that it can go no further. After this point it is once again pulled by the tractor and tracks inboard. The figure demonstrates that the numerical method is stable and versatile, but is also demonstrates a result that could not be obtained with an actual vehicle. To generate the paths shown, the trailer had to pass over the tractor. That is, the articulation angle between tractor and trailer went clear to 180 degrees and continued. Actual tractor-semitrailers are constrained to articulation angles of a little more than 90 degrees, so a real vehicle would have jammed and possibly been damaged if this maneuver had been attempted.

Description of the Apple II Program

A computer program that performs the offtracking computations described in this paper and prepares plots of the paths on an X-Y plotter is available from FHWA. The program was written for the FHWA at the University of Michigan Transportation Research Institute (UMTRI) as part of the project "Impact of Specific Geometric Features on Truck Operations and Safety at Interchanges" (contract DTFH61-83-C-0054). The program runs on an Apple II computer (II+, IIe, IIc) and requires 48k memory and one disk drive. The program is self-contained and relatively user friendly, so that persons who do not have any experience with computers in general (or the Apple II in particular) can use the program without learning much about the operation of an Apple II.

The program allows the user to enter, edit, and save input paths and vehicle descriptions. When a vehicle description and a path description are both chosen, the program simulates the tracking of the selected vehicle as it follows the selected path. The coordinates of the rear axles of each vehicle unit are stored on disk along with the heading angle. Preselected reference points are plotted on the display screen to show the progress of the simulation. Later, the stored data can be used to plot the paths of any arbitrary points in the vehicle.

Scaled hard copies of the vehicle offtracking paths can be obtained in two ways. If a plotter (the Apple X-Y plotter) is available, the program will use it to make scaled ink drawings of the paths traced by any reference points of interest. The plotter can use either paper or transparent material, so it is convenient for making transparent overlay templates. Although the program was developed primarily for use with a plotter, it also allows a dot matrix printer to be used if a "smart" interface card that can control the printer to reproduce the graphic image from the screen is installed in the computer. At the time the program was written there was no software available that allowed detailed graphics covering an entire printed page at once. (The main problem is that a full 8- by 10-in. printer page contains more than 4 million dots and requires more than 400k bytes to store the image.) The hard copy is limited to the information shown on the screen. To make a typical scaled plot, it is necessary to make four hard copies, each showing a different fraction of the entire plot. These can then be taped together, as shown in Figure 14. Thus the trade-off in cost versus performance between a printer and plotter includes both speed and quality. The plotter produces high-quality output in less time than the printer.

The program was developed for a microcomputer instead of a mainframe computer in order to make the program more accessible to state agencies. Most large computers (and many small computers) have special hardware and software for plotting, which means that a program with graphic output will require customization to run on a particular installation. By using an inexpensive and commonly available microcomputer, the problems associated with installation effort and hardware incompatibility are reduced, and they can be completely eliminated by using the supported x-y plotter.

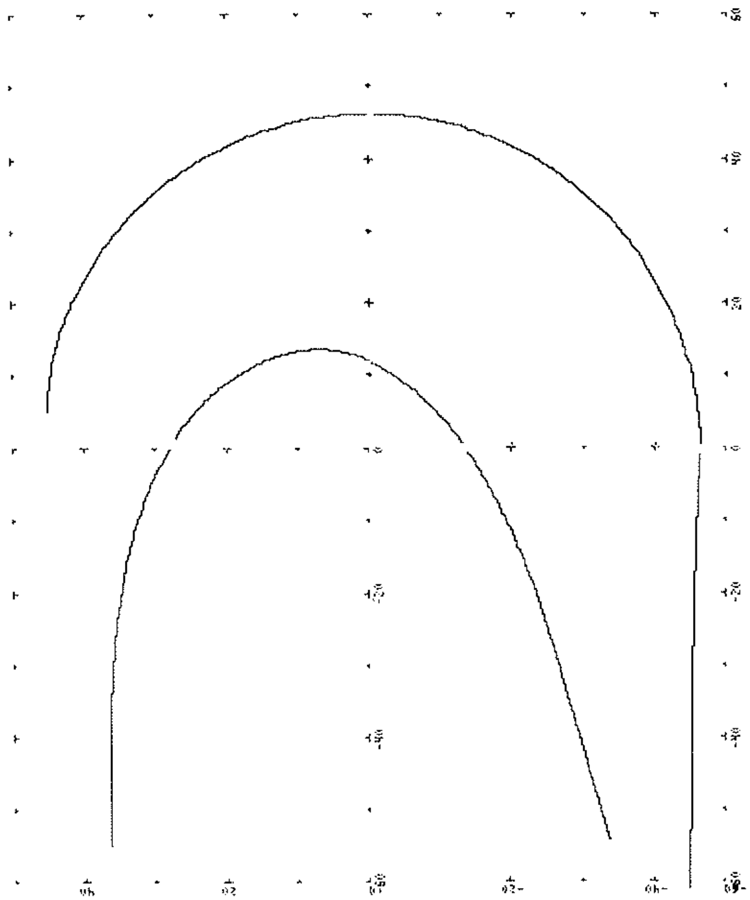

FIGURE 14 Scaled plot created by taping together four images created with a dot-matrix printer and a "smart" interface card.

The main disadvantage of this program is that it is slow to execute. It takes from several minutes to about 20 min to calculate the offtracking paths for a single vehicle maneuver, with the longer times needed for the more complicated vehicles (doubles, triples). It then takes 5 to 15 min to make a plot, as it reads stored data from a disk file. (If the hard copy is made with a dot-matrix printer, the time is multiplied by the number of printed images that must be taped together to obtain the full-scale plot.)

SUMMARY

A review of the methods in use by designers to estimate the offtracking of heavy trucks shows that analytical methods are not available for predicting low-speed offtracking for transient paths. Two graphic methods are used instead: the tractrix integrator (a drafting device) and transparent overlay templates, usually generated with the tractrix integrator. A computer-based method for graphing the complete swept path of an arbitrary vehicle making any type of turn at low speed is described and demonstrated. The computer method is essentially a numerical version of the tractrix integrator, with the improvements that can be obtained using computer graphics equipment. A program that uses this method has been developed for the Apple II computer and is available to the public from the FHWA. When equipped with the appropriate plotting hardware, it produces high-quality scaled drawings of vehicle offtracking. The program can simulate most highway vehicles and handle arbitrarily complex turn geometries. Therefore virtually any geometric design can be evaluated for a particular vehicle of interest.

ACKNOWLEDGMENT

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